Sturm Liouville Eigenvalue Problems

Sturm Liouville Differential Operator

$$Lu \equiv -(p(x)u')' + q(x)u \qquad x \in (a,b)$$

Boundary Conditions (homogeneous)

$$B_1 u \equiv \alpha_1 u(a) + \beta_1 u'(a) = 0$$

$$B_2 u \equiv \alpha_2 u(b) + \beta_2 u'(b) = 0$$

Boundary conditions are said to be homogeneous if $B_k u = 0$ otherwise they are nonhomogeneous as in $B_k u = 1$.

Dirichlet Boundary condition has $\beta_k = 0$ as in

$$B_1 u = u(a) = 0$$

is a Dirichlet B.C. at $x = a$.

Neumann Boundary condition has $\alpha_k = 0$ as in

$$B_2 u = u'(b) = 0$$

If a Boundary condition is neither Dirichlet or Neumann it is said to be a Mixed B.C.

Sturm Liouville Problem (SLP) Defn

(1) $Lu = -(p(x)u')' + q(x)u = \lambda u$

(2) $B_1 u = \alpha_1 u(a) + \beta_1 u'(a) = 0$

(3) $B_2 u = \alpha_2 u(b) + \beta_2 u'(b) = 0$

Regardless of the value of λ one soln is

$$u(x) \equiv 0 \qquad \forall x \in [a, b]$$

The problem is regular only if all hold:

(i) $[a, b]$ bounded interval

(ii) $p, p', q \in C[a, b]$

(iii) $p \neq 0$ on $[a, b]$

If it is not regular it is a singular (SLP)

<u>Defn</u> Any value λ_k for which (1)-(3) has a nontrivial soln $y_k(x) \not\equiv 0$ is said to be an <u>eigenvalue</u>
$y_k(x)$ is the associated <u>eigenfunction</u>

Nomenclature Examples

$$-(xu')' + x^2 u = \lambda u \qquad p = x, \quad q = x^2$$
$$u(0) = 0 \qquad \text{Dirichlet at } x=0$$
$$u(1) = 0 \qquad \text{Dirichlet at } x=1$$

$$u'' + u = \lambda u \qquad p = 1, \quad q = 1$$
$$u'(0) = 0 \qquad \text{Neumann at } x=0$$
$$u(1) + 2u'(1) = 0 \qquad \text{Mixed at } x=1$$

Generalizations for SL operators

$$Lu = x^2 u'' + 2xu' = -(x^2 u')'$$

is a Sturm Liouville operator where $p(x) = x^2$

$$Lu = u'' + u' \neq -(pu'' + p'u')$$

for any $p(x)$ hence is not an SL operator. Why? Would have $p(x) = 1$ but $p'(x) = 0$ doesn't match the coefficient of $u'(x)$.

Any operator eqn

$$Ly = a_1 y'' + a_2 y' + (a_3 + \lambda) y = 0$$

under the transformations/defns

$$p = \exp\left(\int^x \frac{a_2}{a_1} dx\right) \qquad q = \frac{a_3}{a_1} p \qquad r = \frac{p}{a_1}$$

becomes

$$(py')' + (q - \lambda r) y = 0$$

which is identical to our theory when $r = 1$.

Operator formulation.

If one defines a domain for the differential operator L as:

$$D \equiv \{ u \in C^2[a,b] : B_1(u) = B_2(u) = 0 \}$$

then L maps from D into $C[a,b]$

$$L : D \to C[a,b]$$

Here $u \in C^2[a,b]$ in D so u'' makes sense in the defn of

$$Lu = -(pu')' + qu$$

With this construct established the Sturm Liouville Eigenvalue problem is simply

$$\boxed{Lu = \lambda u \qquad u \in D}$$

which is much like a matrix e-val problem

$$Ax = \lambda x \qquad x \in \mathbb{R}^n$$

Theorem (Complete L^2 basis - SLP)

The regular (SLP)

$$Lu = \lambda u \qquad u \in D$$

has infinitely many eigenvalues λ_K with associated eigenfns $u_K(x)$ have the following properties

(a) $\lambda_K \in \mathbb{R}$ $\qquad k = 1, 2, \ldots$

(b) $\lim_{K \to \infty} |\lambda_K| = \infty$

(c) $\lambda_j \neq \lambda_K \Rightarrow \langle u_K, u_j \rangle = 0$

(d) for every $f(x) \in L^2[a,b]$ there are c_K such that

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x)$$

in the L^2 sense.

Basically the eigenvalues are real become unbounded and e-fns of distinct e-values are orthogonal } plain english version

Property (d) is called a "completeness" property.

Proof (a) $\lambda_k \in \mathbb{R}$ in text

Proof (b) Efns with distinct evals are $\perp$

Let $\lambda_1 \neq \lambda_2$ be distinct eigenvalues

(1) $$L y_1 = \lambda_1 y_1 \qquad x \in (a,b)$$

(2) $$L y_2 = \lambda_2 y_2$$

hence

(3) $$\int_a^b (y_2 L y_1 - y_1 L y_2)\, dx = (\lambda_1 - \lambda_2) \underbrace{\int_a^b y_1 y_2 \, dx}_{\langle y_1, y_2 \rangle}$$

Given $Ly = -(py')' + qy$ eqn (3) becomes

(4) $$(\lambda_1 - \lambda_2)\langle y_1, y_2 \rangle = \int_a^b \underbrace{-y_2 (p y_1')' + y_1 (p y_2')'}_{\text{exact differential}} \, dx$$

Using the identity

$$\frac{d}{dx}\left[p(y_1 y_2' - y_2 y_1')\right] = -y_2 (p y_1')' + y_1 (p y_2')'$$

Thus (4) becomes

$$(\lambda_1 - \lambda_2)\langle y_1, y_2 \rangle = p(y_1 y_2' - y_2 y_1') \Big|_{x=a}^{x=b}$$

If $y_k(a) = y_k(b) = 0$ or $y_k'(a) = y_k'(b) = 0$ the RHS clearly vanishes. Harder to show for Mixed B.C. but

$$(\lambda_1 - \lambda_2)\langle y_1, y_2 \rangle = 0$$

EXAMPLE Find all eigenvalues and eigenfns of the following (SLP)

$$(1) \qquad y'' + \lambda y = 0 \qquad y(0) = 0 \qquad y(\pi) = 0$$

Depending on if $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ we have three different general solutions

$$(2) \quad y(x) = c_1 \sinh \mu x + c_2 \cosh \mu x \qquad \lambda = -\mu^2 < 0$$

$$(3) \quad y(x) = c_1 x + c_2 \qquad \lambda = 0$$

$$(4) \quad y(x) = c_1 \sin \mu x + c_2 \cos \mu x \qquad \lambda = \mu^2 > 0$$

We show that for $\lambda \leq 0$ only trivial solns exist ($c_1 = c_2 = 0$) hence no eigenvalues.
For $\lambda > 0$ we show trivial solutions unless $\lambda = \lambda_n = n^2$ (eigenvalues).

CASE $\lambda < 0$ Using general soln (2) above

$$y(0) = c_2 \cosh \mu = 0$$

$$y(\pi) = c_1 \sinh \mu\pi + c_2 \cosh \mu\pi = 0$$

Since both $\sinh z$ and $\cosh z$ are positive the only solution of the above eqns are

$$c_1 = 0 \qquad c_2 = 0$$

hence the (SLP) has no negative e-values.

CASE $\lambda = 0$ Use general soln (3)

$$y(0) = c_2 = 0$$

$$y(\pi) = c_1 \pi + c_2 = 0$$

whose only soln is $c_1 = c_2 = 0$. Thus $\lambda = 0$ is not an e-value of the (SLP)

CASE $\lambda > 0$ Use general soln (4)

$$y(0) = c_2 = 0$$

$$y(\pi) = c_1 \sin \mu\pi + c_2 \cos \mu\pi = 0$$

Since $c_2 = 0$ the latter reduces to

$$\sin \mu\pi = 0 \qquad \mu_n = n$$

which is true if $\mu = 1, 2, 3, \ldots$ i.e, $\mu = \mu_n = n$

Eigenvalues and functions for $\lambda > 0$

$$\lambda = \lambda_n = \mu_n^2 = n^2$$

with associated eigenfunctions

$$y_n(x) = \sin(nx)$$

$$\lambda_n = n^2$$

Necessarily $\{y_n\}$ is an orthogonal set on $[0, \pi]$!

EXAMPLE **Purely Neumann B.C.**

(1) $$y'' + \lambda y = 0 \qquad y'(0) = y'(\pi) = 0$$

CASE $\lambda = 0$ General solution $y(x) = c_1 x + c_2$
Since $y'(x) = c_1$ must have $c_1 = 0$

$$\lambda_0 = 0 \qquad y_0(x) \equiv 1$$

where the function $y_0(x) \equiv 1$ is an efn for $\lambda_0 = 0$.

CASE $\lambda = \mu^2 > 0$ General solution

$$y(x) = c_1 \sin(\mu x) + c_2 \cos(\mu x)$$

$$y'(x) = c_1 \mu \cos(\mu x) - c_2 \mu \sin(\mu x)$$

Boundary conditions can be written in matrix form

$$\begin{array}{l} y'(0) = 0 \\ y'(\pi) = 0 \end{array} \qquad \underbrace{\begin{bmatrix} \mu & 0 \\ \mu\cos(\pi\mu) & -\mu\sin(\pi\mu) \end{bmatrix}}_{A \in \mathbb{R}^{2\times 2}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The system $A\vec{c} = \vec{0}$ has nontrivial solutions only if

$$\det A = -\mu^2 \sin(\pi\mu) = 0 \quad \Big\} \text{ eval eqn}$$

from which we deduce $\mu = \mu_n = n$ (integer).

Also, since $y'(0) = \mu c_1 = 0 \Rightarrow c_1 = 0$ the eval/efn pairs are

$$\lambda_n = \mu_n^2 = n^2 \qquad\qquad y_n(x) = \cos(nx)$$

Collectively

$$\lambda_0 = 0 \qquad\qquad y_0(x) \equiv 1$$

$$\lambda_n = n^2 \qquad\qquad y_n(x) = \cos(nx)$$

form a (complete) orthogonal basis for $L^2[0,\pi]$ in the sense that $\forall f \in L^2[a,b]$ there exist c_k such that

$$f(x) = c_0 y_0(x) + \sum_{n=1}^{\infty} c_n y_n(x)$$

in the L^2-sense.

Detail $\lambda = -\mu^2 < 0$ not possible since

$$y(x) = c_1 \cosh \mu x + c_2 \sinh \mu x$$

leads to $A\vec{c} = \vec{0}$ where $\vec{c} = (c_1, c_2)^T$ and

$$A = \begin{bmatrix} 0 & \mu \\ \mu \sinh \mu\pi & \mu \cosh \mu\pi \end{bmatrix}$$

and

$$\det A = -\mu^2 \sinh \mu\pi \neq 0 \qquad \forall \mu.$$

so no negative eigenvalues.

EXAMPLE Eigenvalues of

(1) $$y'' + \lambda y = 0$$

(2) $$y(0) - y'(0) = 0 \qquad y(\pi) - y'(\pi) = 0$$

Unlike previous problems $\lambda < 0$ is possible

CASE $\lambda = -\mu^2 < 0$ $\quad \mu \neq 0$

$$y(x) = c_1 \cosh \mu x + c_2 \sinh \mu x \qquad \text{gen soln}$$

$$y'(x) = c_1 \mu \sinh \mu x + c_2 \mu \cosh \mu x$$

If we use these to evaluate B.C.

$$BC_1 = c_1 + \mu c_2 = 0$$

$$BC_2 = (\cosh \mu\pi - \mu \sinh \mu\pi) c_1 + (\mu \cosh \mu\pi - \sinh \mu\pi) c_2 = 0$$

Written in matrix form this system is

$$A\vec{c} = \begin{bmatrix} 1 & \mu \\ \cosh \mu\pi - \mu \sinh \mu\pi & \mu \cosh \mu\pi - \sinh \mu \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{0}$$

Nontrivial solutions exist only if $\det A = 0$
Here

$$\det A = (\mu^2 - 1) \sinh \mu = 0$$

has a soln for $\mu = \pm 1$. In either case $\lambda = -\mu^2 = -1 < 0$ is a negative eval.

For $\mu = -1$, $BC_1 = 0 \Rightarrow c_1 = c_2$ so that

$$y(x) = c_1 (\cosh \mu x + \sinh \mu x) \qquad \mu = -1$$

Using properties of $\cosh x$, $\sinh x$ this simplifies to

$$\lambda = -1 \qquad\qquad y(x) = c_1 e^{x}$$

as an eval/efn pair.

<u>CASE $\lambda = 0$</u>

$$y(x) = c_1 x + c_2 \qquad \text{gen soln}$$

$$y'(x) = c_1$$

In a similar way the BC yield a system of eqns for c_1 and c_2

BC_1 : $\quad -c_1 + c_2 = 0$

BC_2 : $\quad (\pi - 1) c_1 + c_2 = 0$

In matrix form is $A\vec{c} = 0$ where $\vec{c} = (c_1, c_2)^T$

$$A = \begin{bmatrix} -1 & 1 \\ (\pi - 1) & 1 \end{bmatrix}$$

Since $\det A = -\pi \neq 0$ all solns are trivial, i.e. have $c_1 = c_2 = 0$

CASE $\lambda > 0$ $\quad \lambda = +\mu^2 \quad \mu \neq 0$

$$y(x) = c_1 \cos\mu x + c_2 \sin\mu x$$

$$y'(x) = -c_1 \mu \sin\mu x + c_2 \mu \cos\mu x$$

In a similar fashion the BC yield a system of eqns for c_K

$$A\vec{c} = \vec{0} \qquad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

where (after calculations) we find

$$A = \begin{bmatrix} 1 & -\mu \\ \cos(\mu\pi) + \mu\sin(\mu\pi) & (\sin\mu\pi) - \mu\cos\mu\pi \end{bmatrix}$$

Nontrivial solns are possible only if

$$\det A = (1+\mu^2)\sin(\mu\pi) = 0$$

This eigenvalue eqn $\Rightarrow$ $\mu_n = n$, $n = 1, 2, 3.$
And from A

$$c_1 - \mu_n c_2 = 0$$

so that $c_1 = \mu_n c_2$ hence

$$\lambda_n = n^2 \qquad y_n(x) = (n\cos nx + \sin nx)$$

Expansions of $f(x) \in L^2[0,\pi]$ using these orthogonal functions have the form

$$f(x) = c_0 e^x + \sum_{n=1}^{\infty} c_n \{ n\cos nx + \sin nx \}$$

Summary of evals/efns

$\lambda_0 = -1 \qquad \phi_0(x) = e^x$

$\lambda_n = n^2 \qquad \phi_n(x) = n\cos nx + \sin nx$

where $\phi_0(x), \phi_1(x), \phi_2(x), \ldots$ are orthogonal under the inner product

$$\langle f, g \rangle = \int_0^{\pi} f(x) g(x)\, dx .$$