

Separable Differential Equations

A first order differential equation is said to be separable if it can be written in the form

$$y' = f(x)g(y)$$

for some functions f and g . Examples

$$y' = xy^2$$

$$f(x) = x$$

$$g(y) = y^2$$

$$y' = xy$$

$$f(x) = x$$

$$g(y) = y$$

$$y' = \frac{x^2}{1+y^2}$$

$$f(x) = x^2$$

$$g(y) = \frac{1}{1+y^2}$$

The general solution is found through the following integration

$$\frac{dy}{g(y)} = f(x)g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

The solution of the initial value problem (IVP)

$$y' = f(x)g(y), \quad y(x_0) = y_0$$

is found from the definite integral

$$\int_{y_0}^y \frac{dz}{g(z)} = \int_{x_0}^x f(t)dt$$

EXAMPLE

Find the general solution of $y' = xy^2$

$$\int \frac{dy}{y^2} = \int x dx$$

$$-\frac{1}{y} = \frac{1}{2}x^2 + c \quad c \in \mathbb{R} \text{ constant}$$

$$y(x) = -\frac{2}{2c + x^2}$$

EXAMPLE

A falling object with initial velocity v_0 and coefficient of friction k has a velocity $v(t)$ which satisfies

$$\frac{dv}{dt} = g - kv \quad v(0) = v_0$$

Find $v(t)$. Here $g = 32 \text{ ft/sec}^2$ is grav. const.

$$\int_{v_0}^v \frac{dz}{g - kz} = \int_0^t dt = t$$

$$-\frac{1}{k} \log(g - kz) \Big|_{v_0}^v = t$$

$$\log\left(\frac{g - kv}{g - kv_0}\right) = -kt$$

$$v(t) = \frac{g}{k} - \left(\frac{g}{k} - v_0\right)e^{-kt}$$

Note $v(t) \rightarrow v_T = \frac{g}{k}$ (terminal velocity)

EXAMPLE Some solutions are implicitly defined

$$y' = \frac{x}{3y^2 + 1}$$

yields $y^3 + y = x + c$

Orthogonal Curves and Separable Equations

Suppose a family of curves are defined by the algebraic equation

$$(1) \quad f(x, y) = c$$

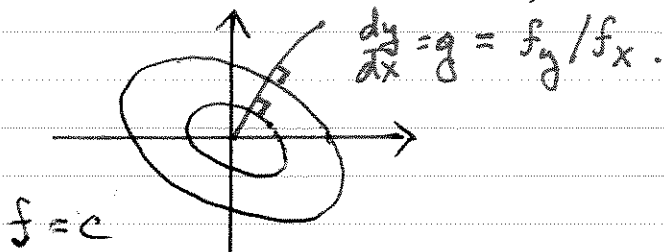
Using implicit differentiation one finds

$$(2) \quad \frac{dy}{dx} = -\frac{f_x}{f_y}$$

Curves that are everywhere orthogonal (\perp) to those defined by (1) must have a slope equal to the negative reciprocal of that in equation (2):

$$(3) \quad \frac{dy}{dx} = \frac{f_y}{f_x} = g(x, y)$$

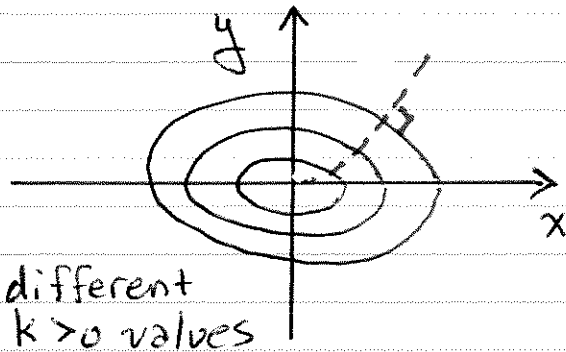
If (3) can be solved, orthogonal curves can be found.



EXAMPLE Let $a > 0$ and consider the family of curves (ellipses)

$$(1) \quad \frac{x^2}{a^2} + y^2 = k^2$$

$$f(x, y) = k^2$$



Show left are the curves and orthogonal curves (---)

The slope of the curves in (1) are found by implicit differentiation

$$\frac{2}{a^2}x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{a^2 y} = m$$

Thus orthogonal curves have slope $-\frac{1}{m}$ and

$$\frac{dy}{dx} = \frac{a^2 y}{x}$$

which is separable and has solution

$$y(x) = c x^{a^2} \quad \left. \vphantom{y(x)} \right\} \text{orthog curves.}$$

Bernoulli Equation

$$(1) \quad y' + p(x)y = q(x)y^n$$

can be transformed into a linear first order equation in

$$(2) \quad u = y^{1-n}$$

In particular

$$\frac{1}{(1-n)} \frac{du}{dx} = y^{-n} \frac{dy}{dx}$$

Then noting (1) may be written

$$y^{-n} y' + p(x) y^{1-n} = q(x)$$

$$\frac{1}{(1-n)} u' + p(x) u = q(x)$$

$$(3) \quad u' + (1-n)p(x)u = (1-n)q(x)$$

If one solves (3) for $u(x)$, $y(x)$ can be retrieved from (2).

Historical Note: Jacob Bernoulli (1690) showed the soln of a certain Bernoulli eqn solved the isochrone curve problem. An isochrone curve has a shape such that a mass falling along it under gravity (no fric) will end at its base in the same time independent of starting point.

EXAMPLE

$$y' - \frac{2}{x}y = -x^2 y^2$$

Divide by ~~y~~²

$$y^{-2}y' - \frac{2}{x}y^{-1} = -x^2$$

then since $(y^{-1})' = y^{-2}y'$ we have

$$-u' - \frac{2}{x}u = -x^2 \quad u = y^{-1}$$

$$(1) \quad u' + \frac{2}{x}u = +x^2 \quad u = y^{-1}$$

Integrating factor for the linear eqn (1) is

$$\mu(x) = \exp\left(\int \frac{2}{x} dx\right) = \exp(2 \ln x) = x^2$$

Multiply through by $\mu(x) = x^2$ yields

$$(x^2 u)' = x^4$$

$$x^2 u = \frac{1}{5}x^5 + C$$

$$u = \frac{1}{5}x^3 + \frac{C}{x^2}$$

$$\frac{1}{y} = \frac{1}{5}x^3 + \frac{C}{x^2}$$

Thus

$$y(x) = \frac{x^2}{\frac{1}{5}x^3 + C}$$

is the general solution.