Systems Review

QUESTION ONE

For each of the following matrices A find two independent solutions of

$$\vec{x}' = A\vec{x}$$

Note that answers are not unique.

a) $A = \begin{bmatrix} -7 & -6 \\ 4 & 7 \end{bmatrix}$ b) $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$

c) $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$ d) $A = \begin{bmatrix} -8 & 4 \\ -10 & 4 \end{bmatrix}$

e) $A = \begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}$ f) $A = \begin{bmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix}$

ANSWER OUTLINE

a) $P(\lambda) = \det(A - \lambda I) = \lambda^2 - 25 = 0 \qquad \lambda = \pm 5$

$\lambda_1 = 5 \qquad (A - \lambda_1 I)\vec{\xi}_1 = \vec{0} \qquad \vec{\xi}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2 = -5 \qquad (A - \lambda_2 I)\vec{\xi}_2 = \vec{0} \qquad \vec{\xi}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$$\vec{x}_1(t) = e^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \vec{x}_2(t) = e^{-5t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

b) $P(\lambda) = (\lambda+1)(\lambda-2) \qquad \lambda_1 = -1, \ \lambda_2 = 2$

$$\vec{x}_1(t) = e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \vec{x}_2(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

c) $P(\lambda) = \det(A - \lambda I) = \lambda^2 + 9 \qquad \lambda = \pm 3i$ (Complex)

$(A - 3iI)\vec{\xi} = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}\vec{\xi} = \vec{0}$

$\vec{\xi} = \vec{a} + i\vec{b} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{x}(t) = e^{\lambda t}\vec{\xi} = e^{3it}(\vec{a} + \vec{b}i)$

Take real/imaginary parts ($e^{i\theta} = \cos\theta + i\sin\theta$)

$$\vec{x}_1(t) = \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} \qquad \vec{x}_2(t) = \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

d) $P(\lambda) = \lambda^2 + 4\lambda + 8 = 0 \Rightarrow \lambda = -2 \pm 2i$ (Complex)

$(A - \lambda I)\vec{\xi} = \begin{bmatrix} -6-2i & 4 \\ * & * \end{bmatrix}\vec{\xi} = \vec{0}$

$\vec{\xi} = \begin{pmatrix} 2 \\ 3+i \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{a} + i\vec{b}$

$\vec{x}(t) = e^{\lambda t}\vec{\xi}$

$\vec{x}(t) = e^{-2t}(\cos 2t + i\sin 2t)(\vec{a} + i\vec{b})$

Take real and imaginary parts

$$\vec{x}_1 = e^{-2t}\begin{pmatrix} 2\cos 2t \\ 3\cos 2t - \sin 2t \end{pmatrix}$$

$$\vec{x}_2 = e^{-2t}\begin{pmatrix} 2\sin(2t) \\ 3\sin(2t) + \cos(2t) \end{pmatrix}$$

e) $P(\lambda) = \det(A-\lambda I) = (\lambda-3)^2$ $\qquad \lambda = 3$ repeated

$$(A-\lambda I)\vec{\eta}_0 = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}\vec{\eta}_0 = \vec{0} \qquad \vec{\eta}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Next find a soln (not unique) of

$$(A-\lambda I)\vec{\eta}_1 = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}\vec{\eta}_1 = \vec{\eta}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

One such soln $\vec{\eta}_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$

$$\vec{x}_1 = e^{\lambda t}\vec{\eta}_0 = e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{x}_2 = e^{\lambda t}(\vec{\eta}_0 t + \vec{\eta}_1) = e^{3t}\begin{pmatrix} t \\ -t+\frac{1}{2} \end{pmatrix}$$

f) $P(\lambda) = \det(A-\lambda I) = \lambda^2 - 3\lambda + \frac{9}{4}$ $\qquad \lambda = \frac{3}{2}$ repeated

$$(A-\lambda I)\vec{\eta}_0 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\vec{\eta}_0 = \vec{0} \qquad \vec{\eta}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then $(A-\lambda I)\vec{\eta}_1 = \vec{\eta}_0$ is

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ * & * \end{bmatrix}\vec{\eta}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{\eta}_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\vec{x}_1 = e^{\lambda t}\vec{\eta}_0 = e^{\frac{3}{2}t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{x}_2 = e^{\lambda t}(\vec{\eta}_0 t + \vec{\eta}_1) = e^{\frac{3}{2}t}\begin{pmatrix} t \\ t+2 \end{pmatrix}$$

<u>QUESTION TWO.</u> Find a particular soln for

$$\vec{x}' = A\vec{x} + \vec{f}(t) \qquad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \qquad \vec{f} = \begin{pmatrix} e^{3t} \\ \frac{1}{2}e^{3t} \end{pmatrix}$$

<u>Answer</u> One can find two independent solns of the homogeneous problem

$$\vec{x}_1 = \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix} \qquad \vec{x}_2 = \begin{pmatrix} -e^{-3t} \\ e^{-3t} \end{pmatrix}$$

Use variation of parameters to find $\vec{x}_p(t)$

$$\vec{x}_p(t) = \Phi(t)\vec{v}(t) \qquad \vec{v}(t) = \int^t \Phi^{-1}(s)\vec{f}(s)\,ds$$

Here Φ = fund matrix. Wronskian $W = \det \Phi = 3 \Rightarrow$

$$\Phi = \begin{bmatrix} 2e^{3t} & -e^{-3t} \\ e^{3t} & e^{-3t} \end{bmatrix} \qquad \Phi^{-1} = \begin{bmatrix} \frac{1}{3}e^{-3t} & \frac{1}{3}e^{-3t} \\ -\frac{1}{3}e^{3t} & \frac{2}{3}e^{3t} \end{bmatrix}$$

Then

$$\vec{v}(t) = \int^t \Phi^{-1} f\,ds = \int^t \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} ds = \begin{pmatrix} t/2 \\ 0 \end{pmatrix}$$

Having found $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t)\vec{v}(t) = \begin{bmatrix} 2e^{3t} & -e^{-3t} \\ e^{3t} & e^{-3t} \end{bmatrix}\begin{bmatrix} t/2 \\ 0 \end{bmatrix}$$

$$\vec{x}_p(t) = e^{3t}\begin{pmatrix} t \\ t/2 \end{pmatrix}$$

QUESTION THREE Find a particular soln of

$$\vec{x}' = A(t)\,\vec{x} + \vec{f}(t) \qquad \vec{f} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

given the fundamental matrix is

$$\mathbb{X}(t) = \begin{bmatrix} 2t & t^2 \\ t & t^2 \end{bmatrix}$$

ANSWER (Don't need to know $A(t)$)

Apply variation of parameters method

$$\vec{x}_p = \mathbb{X}(t)\,\vec{v}(t) \qquad \vec{v}(t) = \int^t \mathbb{X}^{-1}(s)\,\vec{f}(s)\,ds$$

The Wronskian $W(t) = \det \mathbb{X}(t) = t^3$ hence

$$\mathbb{X}^{-1} = \frac{1}{t^3}\begin{bmatrix} t^2 & -t^2 \\ -t & 2t \end{bmatrix} = \begin{bmatrix} t^{-1} & -t^{-1} \\ -t^{-2} & 2t^{-2} \end{bmatrix}$$

$$\mathbb{X}^{-1}\vec{f} = \begin{pmatrix} 0 \\ t^{-2} \end{pmatrix}$$

Hence

$$\vec{v}(t) = \int^t \begin{pmatrix} 0 \\ s^{-2} \end{pmatrix} ds = \begin{pmatrix} 0 \\ -\frac{1}{t} \end{pmatrix}$$

Then

$$\vec{x}_p = \mathbb{X}\vec{v} = \begin{bmatrix} 2t & t^2 \\ t & t^2 \end{bmatrix}\begin{bmatrix} 0 \\ -\frac{1}{t} \end{bmatrix} = \begin{pmatrix} -t \\ -t \end{pmatrix}$$

QUESTION FOUR Solve the initial value problem

$$\vec{x}' = A\,\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}_*$$

given two independent solutions are

$$\vec{x}_1(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix} \qquad \vec{x}_2(t) = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$$

Soln The general solution is

$$\vec{x}(t) = \mathbb{X}(t)\,\vec{c} \qquad \mathbb{X}(t) = \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}$$

Given I Cond determines $\vec{c}$

$$\vec{x}(0) = \mathbb{X}(0)\,\vec{c} \qquad \mathbb{X}(0) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Thus ($\det \mathbb{X}(0) = 1$)

$$\vec{c} = \mathbb{X}(0)^{-1}\vec{x}(0) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence

$$\vec{x}(t) = \mathbb{X}(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$$

is the unique soln of the IVP

QUESTION FIVE Solve the initial value problem

$$\vec{x}' = A\vec{x} + \vec{f}(t) \qquad \vec{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

given two independent homogeneous solns $\vec{x}_1(t)$, $\vec{x}_2(t)$ and the particular soln:

$$\vec{x}_1(t) = \begin{pmatrix} \cos t \\ \cos t \end{pmatrix} \quad \vec{x}_2(t) = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix} \quad \vec{x}_p(t) = \begin{pmatrix} t+1 \\ 2e^t \end{pmatrix}$$

Soln

$$\vec{x}(t) = \mathbb{X}(t)\vec{c} + \vec{x}_p(t) \qquad \mathbb{X}(t) = \left[\vec{x}_1 \,\middle|\, \vec{x}_2\right]$$

$$\vec{x}(0) = \mathbb{X}(0)\vec{c} + \vec{x}_p(0)$$

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\vec{c} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Thus

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\vec{c} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\vec{c} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\vec{c} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Intial value problem soln

$$\vec{x}(t) = \begin{bmatrix} \cos t & e^t \\ \cos t & 2e^t \end{bmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} t+1 \\ 2e^t \end{pmatrix} = \begin{pmatrix} 2\cos t + t + 1 \\ 2\cos t + 2e^t \end{pmatrix}$$

QUESTION SIX If $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ above, what is $\vec{f}(t)$?

$$\vec{f}(t) = \vec{x}_p' - A\vec{x}_p = \begin{pmatrix} 1 \\ 2e^t \end{pmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{pmatrix} t+1 \\ 2e^t \end{pmatrix} = \text{etc.}$$