Math 455 Dynamical Systems
Supplementary Lecture Notes

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Contents

1 Liapunov Functions

Liapunov functions $V(x)$ can be used to estimated basins of attraction of fixed points, show that a system has no periodic orbits or show a system has a globally attracting fixed point.

Definition 1 *Let U be an open subset of $\mathbb{R}^n$ containing a fixed point $\bar{x}$ of*

$$\dot{x} = f(x) \quad , \quad x \in \mathbb{R}^n \tag{1}$$

where $f \in C^1(\mathbb{R}^n)$. A function $V : U \to \mathbb{R}$ is a Liapunov function on U if

- *i)* $V \in C^1(U)$
- *ii)* $V(\bar{x}) = 0$
- *iii)* $V(x) > 0 \quad , \quad \forall x \in U, x \neq \bar{x}$
- *iv)* $\dot{V} = \frac{dV}{dt} < 0 \quad , \quad \forall x \in U, x \neq \bar{x}$.

Note that i)-iii) can simultaneously be true only if the system (1) has a unique fixed point $\bar{x} \in U$.

When V satisfies property iii), the function V is said to be (strictly) *positive definite* on U. If this condition is weakened to $V(x) \geq 0, \forall x \in U$, V is said to be *semi positive definite* on U. Also, if condition iv) is relaxed to

$$\dot{V} = \frac{dV}{dt} \leq 0 \quad , \quad \forall x \in U$$

then V is said to be a *weak Liapunov function*.

Theorem 1 *Let $V(x)$ be a Liapunov function on U. If $x(t)$ is a solution of (1) with $x(0) = x_0 \in U$ then $x(t) \to \bar{x}$ as $t \to \infty$.*

We do not give a formal proof here but note that

$$V(x(t)) - V(x_0) = \int_0^t \dot{V}(t)dt < 0$$

implies that as t increases, $x(t)$ travels toward lower valued level sets of V. Since $V = 0$ is the lowest valued level set in U, $x(t)$ ulimately approaches the fixed point $\bar{x}$.

If $U = \mathbb{R}^n$ then $x(t) \to \bar{x}$ for all initial conditions x_0 so that the fixed point $\bar{x}$ is *globally attracting*. If in addition $\bar{x}$ is stable then it would be *globally asymptotically stable*. In either of these instances the system (1) cannot have a periodic orbit. This is how one uses Liapunov functions to show that systems do not have periodic orbits.

If U is a strict subset of $\mathbb{R}^n$ the Liapunov function yields information about the basin of attraction $B(\bar{x})$ of the fixed point. In particular, the Theorem implies $U \subset B(\bar{x})$. Typically in this case U is a proper subset of $B(\bar{x})$.

2 Flow, Limit sets, Invariance

In this section we make definitions required to state and prove the Poincare-Bendixson Theorem stated in the next section. Toward that end, recall that $\phi(t, x_0)$ is a flow function of

$$\dot{x} = f(x) \quad , \quad x(0) = x_0 \quad , \quad x \in \mathbb{R}^n$$

if

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= f(\phi) \\ \phi(0, x_0) &= x_0 \end{aligned}$$

Definition 2 *p is an ω limit point of x_0 if there exists a sequence of times $\{t_i\}$, $t_i \to \infty$ such that*

$$\phi(t_i, x_0) \to p \quad as \quad i \to \infty$$

p is an α limit point of x_0 if the same holds except that $t_i \to -\infty$.

From this definition we define the ω limit set of x_0 as:

$$\omega(x_0) = \{p : p \, is \; a \; limit \; point \; of \; x_0\}$$

with an analagous definition for the α limit set of x_0. Roughly speaking, $\omega(x_0)$ is the set of points that a trajectory through x_0 gets near as $t \to \infty$. Conversely, $\alpha(x_0)$ is the set of points the same trajectory approaches as $t \to -\infty$. In some texts ω limit sets are defined in different but logically equivalent manners. For instance, if overbar $\bar{A}$ is used to denote the closure of the set A then

$$\omega(x_0) = \bigcap_{T \geq 0} \overline{\{\phi(t, x_0) t \geq T\}}$$

Such definitions are more amenable to generalizations of ω limits sets to include abstract topological spaces.

Example 1 *If $\bar{x}$ is a globally attracting fixed point of $\dot{x} = f(x)$ then $\omega(x_0) = \bar{x}$ for all initial conditions x_0. Since $\| x(t) \| \to \infty$ as $t \to -\infty$, the α limit set is empty, i.e., $\alpha(x_0) = \{\}, \forall x_0 \neq \bar{x}$.*

Example 2 *Suppose γ is a homoclinic orbit of $\dot{x} = f(x), x \in \mathbb{R}^2$ and is homoclinic to the saddle $\bar{x}$. Then,*

$$\lim_{t \to \pm\infty} \phi(t, x_0) = \bar{x} \quad , \quad \forall x_0 \in \gamma$$

implies

$$\bar{x} = \omega(x_0) = \alpha(x_0) \quad , \quad \forall x_0 \in \gamma$$

Definition 3 *A set M is positively invariant if $\phi(t, M) \subset M$ for all $t \geq 0$. If in addition M is closed and bounded (compact) then M is called a trapping region.*

An easy way to determine if M is a trapping region is to show that the vector field $f(x)$ points into the region. If the (smooth) boundary of M is ∂M and N is an <u>inward</u> normal vector then the dot product condition

$$f(x) \cdot N > 0 \quad , \quad \forall x \in \partial M \tag{2}$$

proves M is a trapping region. As a separate example, if $V(x)$ is a Liapunov function on $U \subset \mathbb{R}^n$ then U is positively invariant.

To complete this section we state (without proof) several properties of limit sets.

Theorem 2 *Let $\phi(t, x_0)$ be the flow for $\dot{x} = f(x), x \in \mathbb{R}^n$.*

Existence	*The ω limit set of a bounded orbit is nonempty.*
Closure	$\omega(x_0)$ *is a closed set*
Invariance	$y \in \omega(x_0) \Rightarrow \phi(t, y) \in \omega(x_0)$, $\forall t \geq 0$
Connectedness	*If $\phi(t, x_0)$ is a bounded orbit then $\omega(x_0)$ is a connected set*
Transitivity	*If $z \in \omega(y)$ and $y \in \omega(x_0)$ then $z \in \omega(x_0)$.*

3 Poincare-Bendixson Theorem

The Poincare-Bendixson Theorem has several variants - all of which only apply to planar systems. A more general form is:

Theorem 3 *Suppose that M is a trapping region for the planar system*

$$\dot{x} = f(x) \quad , \quad x(0) = x_0 \in \mathbb{R}^2$$

which contains at most a finite number of isolated fixed points $\bar{x}_i, i = 1, 2, \ldots n$. Then, for any $x_0 \in M$ one of the following is true:

- *(a)* $\omega(x_0)$ *is a fixed point*
- *(b)* $\omega(x_0)$ *is a periodic orbit contained in* M
- *(c)* $\omega(x_0)$ *consists of a finite number of fixed points* $p_i, i = 1, 2, \ldots m$ *and orbits* γ_{ij} *such that* $\omega(\gamma_{ij}) = p_i$ *and* $\alpha(\gamma_{ij}) = p_j$.

The latter conclusion (c) is a statement that the ω limit sets of x_0 may either be homoclinic orbits, collections of homoclinic orbits or heteroclinic cycles.

A more common variant of this Theorem is:

Theorem 4 *(Poincare-Bendixson) Let M be a trapping region for*

$$\dot{x} = f(x) \; .$$

which contains no fixed points. Then there exists a periodic orbit contained entirely within M.

A typical application of this latter variant is to prove the existence of a periodic orbit which contains a unique (unstable) fixed point. From index theory, such a fixed point cannot be a saddle. Thus, as a first step one tries to locate a candidate unstable hyperbolic fixed point (typically unstable spiral) to construct a trapping region around. Then, one surrounds $\bar{x}$ with a curve ∂M whose interior contains the sole fixed point $\bar{x}$. The curve ∂M should be chosen so that condition (2) is satisfied. Then, one removes a small neighbourhood $N_\epsilon(\bar{x})$ of $\bar{x}$ from the interior creating an “annular” region M. Since $\bar{x}$ is unstable, the flow on the boundary of $N_\epsilon(\bar{x})$ must be towards the interior of M. Thus, all the conditions of the Theorem are satisfied proving the existence of a periodic orbit. Note, however, the Theorem does not guarantee such a periodic orbit is unique. It is possible that there are many within M.

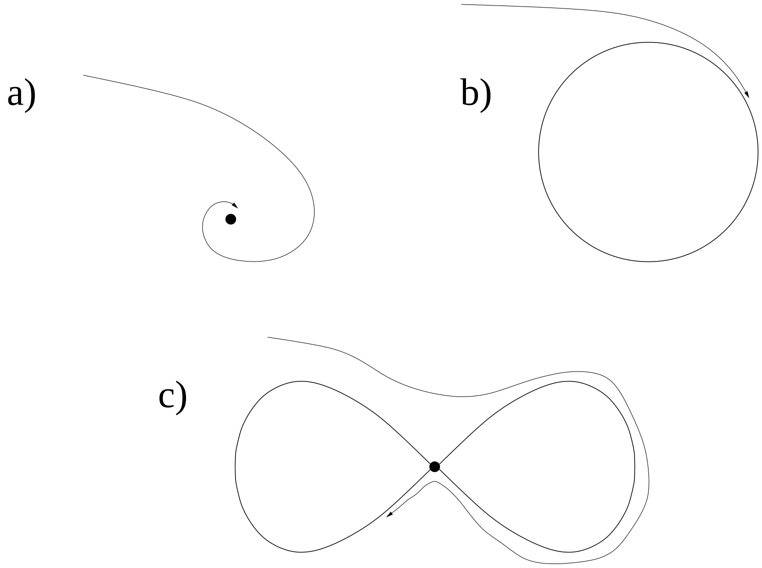

Figure 1: Illustrates conclusions a)-c) of Poincare-Bendixson Theorem in text

4 Bifurcation Theory - Introduction

For planar systems

$$\dot{x} = f(x, \mu) \quad , \quad x \in \mathbb{R}^2 \quad , \quad \mu \in \mathbb{R}^n$$

many types of bifurcations can occur. Such bifurcations may include

a) Fixed points (or equilibria)

b) Fixed points and periodic orbits

c) periodic orbits

d) homoclinic and heteroclinic orbits

For instance, as μ is varied two or more fixed points can "collide" and disappear. Generically such a bifurcation is called a saddle-node bifurcation of fixed points. Since planar systems admit periodic solutions, it is also possible that two periodic orbits could collide and disappear. This would be an example of a saddle-node bifurcation of periodic orbits. Given the possible combinations of colliding fixed points, periodic orbits, heteroclinic orbits and homoclinic orbits the number of kinds of bifurcations is numerous.

A separate issue involves the number of parameters needed to characterize the bifurcation. As a trivial example consider:

$$\dot{x} = f(x, \mu) = \mu_1 + \mu_2 x + x^2 \quad , \quad x \in \mathbb{R} \, . \tag{3}$$

Such quadratic systems on $\mathbb{R}$ have saddle-node bifurcations when the minima (or maxima) of f intersects the x-axis. From basic calculus this minimal value is

$$\eta = \mu_1 - \frac{1}{4}\mu_2^2 \, .$$

Thus, only the value of η is needed to characterize the saddle-node bifurcation, i.e., $\eta = 0$. So, even though the system has two parameters, only one is needed to describe the bifurcation. In this case the saddle-node bifurcation is said to be codimension 1. Without getting into technical definitions, the *codimension* of a bifurcation is the minimum number of parameters needed to describe the particular kind of bifurcation being examined.

When considering only bifurcations of fixed points it is useful to note that planar systems inherit all bifurcations that the system

$$\dot{x} = f(x, \mu) \quad , \quad x, \mu \in \mathbb{R} \tag{4}$$

can have. For instance if (6) has a branch of equilibria $\bar{x}(\mu)$ then the planar system

$$\begin{aligned} \dot{x} &= f(x, \mu) && (5) \\ \dot{y} &= -y && (6) \end{aligned}$$

will have a branch $(\bar{x}, \bar{y}) = (\bar{x}(\mu), 0)$.

4.1 Fixed Point Location

In this section, we restrict our attention to bifurcations involving fixed points only. By a branch $\bar{x}(\mu)$ of fixed points of

$$\dot{x} = f(x, \mu) \quad , \quad x = (x_1, x_2) \in \mathrm{I\!R}^2 \quad , \quad \mu \in \mathrm{I\!R} \tag{7}$$

we mean a vector valued function $\bar{x} : \mathrm{I\!R} \to \mathrm{I\!R}^2$ with an associated interval (a, b) such that

$$f(\bar{x}(\mu), \mu) = 0 \quad , \quad \forall \mu \in (a, b) \; .$$

For the most part we shall only consider isolated branches of fixed points, recalling that

Definition 4 *A fixed point $\bar{x}$ of (7) is isolated if there exists a neighbourhood $N_\epsilon(\bar{x})$ of $\bar{x}$ such that*

$$x \in N_\epsilon(\bar{x}), x \neq \bar{x} \Rightarrow f(x) \neq 0$$

Now suppose that f is a smooth function with a smooth branch of fixed points and define

$$g(\mu) = f(\bar{x}(\mu), \mu) \quad , \quad g : \mathrm{I\!R} \to \mathrm{I\!R}^2$$

Since f is a planar vector field, $g = (g_1, g_2)$ has two components each of which depends on μ. A Taylor series expansion of g about $\mu = \mu^*$ yields

$$g(\mu) = g(\mu^*) + g'(\mu^*)(\mu - \mu^*) + \frac{1}{2} g''(\mu^*)(\mu - \mu^*)^2 + \cdots$$

Since $g(\mu) = 0$, $\forall \mu \in (a, b)$, the coefficients of the Taylor series must vanish. In particular, $g'(\mu^*)$ must vanish from which we deduce

$$g'(\mu^*) = Df(\bar{x}(\mu^*), \mu^*) \frac{d\bar{x}}{d\mu} + \frac{\partial f}{\partial \mu}(\bar{x}(\mu^*), \mu^*) = 0$$

where $Df(\bar{x}(\mu^*), \mu^*)$ is the Jacobian evaluate on the branch at μ^*. Providing the Jacobian is nonsingular,

$$\frac{d\bar{x}}{d\mu} = -\, Df(\bar{x}(\mu^*), \mu^*)^{-1} \, \frac{\partial f}{\partial \mu}(\bar{x}(\mu^*), \mu^*) \tag{8}$$

This equation determines a vector $\frac{d\bar{x}}{d\mu}$ tangent to the branch as is illustrated in Figure 2. The existence of such a branch with a well defined derivative defined in (8) is proveable with the Implicit Function Theorem.

Theorem 5 *(Implicit Function Theorem) Let $f = f(x, \mu)$, $f := \mathrm{I\!R}^2 \times \mathrm{I\!R} \to \mathrm{I\!R}^2$ be $C^1(U)$ where U is some open set in $\mathrm{I\!R}^3$. Suppose there exists a pair $(x^*, \mu^*) \in U$ such*

- *a)* $f(x^*, \mu^*) = 0$
- *b)* $Df(x^*, \mu^*)$ *is invertible.*

then there exists a function $\bar{x} : \mathrm{I\!R} \to \mathrm{I\!R}^2$ and an interval $I = (a, b)$ containing μ^* such that

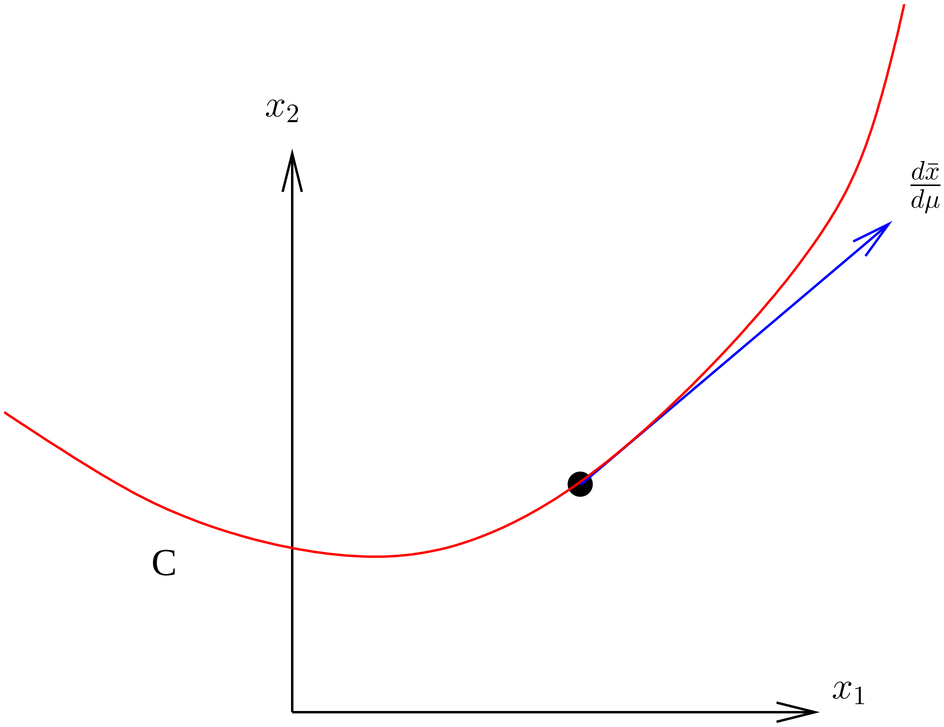

Figure 2: Illustrates the vector $\frac{d\bar{x}}{d\mu}$ tangent to the branch of fixed points. Note that $\bar{x} = (\bar{x}_1, \bar{x}_2)$ has two componets so that in the x_1x_2-plane the branch is the curve C

i) $\bar{x}(\mu^*) = x^*$

ii) $f(\bar{x}(\mu), \mu) = 0 \quad , \quad \forall \mu \in I$

iii) $\bar{x}(\mu) \in C^1(\mathbb{R})$

Moreover, the derivative of $\bar{x}(\mu)$ is given by (8).

If the Jacobian Df has a zero eigenvalue it is not invertible and $\frac{d\bar{x}}{d\mu}$ cannot be uniquely determined in such a fashion. Thus, if $det(Df) = 0$ at $\mu = \mu^*$ the fixed point $x^* = \bar{x}(\mu^*)$ is nonhyperbolic and a potential bifurcation point at which multiple branches of fixed points can coalesce.

Example 3 *Consider the planar system $\dot{x} = f(x)$ where*

$$f(x, \mu) = \begin{pmatrix} -\mu x_1 + x_2 \\ x_2 - x_1^3 \end{pmatrix}$$

It is easy to show that for $\mu > 0$ there are three disjoint branches of fixed points:

$$\bar{x}_+(\mu) = \begin{pmatrix} \sqrt{\mu} \\ -\mu^{3/2} \end{pmatrix} \quad , \quad \bar{x}_-(\mu) = \begin{pmatrix} -\sqrt{\mu} \\ +\mu^{3/2} \end{pmatrix} \quad , \quad \bar{x}_0(\mu) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For this problem

$$Df(x, \mu) = \begin{bmatrix} -\mu & 1 \\ -3x_1^2 & 1 \end{bmatrix} \quad , \quad \frac{\partial f}{\partial \mu}(x, \mu) = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix}$$

The three branches coalesce at $\mu = \mu^* = 0$ at which the Jacobian has zero determinant. The explicit formulae for the branches can be differentiated in μ to determine $\frac{d\bar{x}}{d\mu}$.

Alternately, equation (8) can be used. For instance, for $\bar{x}_+$*,*

$$Df(\bar{x}_+, \mu) = \begin{bmatrix} -\mu & 1 \\ -3\mu & 1 \end{bmatrix} \quad , \quad \frac{\partial f}{\partial \mu}(x, \mu) = \begin{pmatrix} -\sqrt{\mu} \\ 0 \end{pmatrix}$$

from which

$$\frac{d\bar{x}_+}{d\mu} = -\frac{1}{2\mu} \begin{bmatrix} 1 & -1 \\ 3\mu & -\mu \end{bmatrix} \begin{pmatrix} -\sqrt{\mu} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\mu^{-1/2} \\ \frac{3}{2}\mu^{1/2} \end{pmatrix}$$

None of this analysis says anything about the stability of the fixed points, however.

4.2 Fixed Point Stability

Once the location of the fixed points as a function of the parameter μ has been determined, their stability can be determined from standard linear analysis. We recall that for a planar system $\dot{x} = f(x, \mu)$ the eigenvalues λ of the Jacobian Df are roots of the characteristic polynomial

$$\lambda^2 - Tr(Df)\lambda + det(Df) = 0.$$

The stability of a hyperbolic fixed point $\bar{x}$ is then determined the signs of $Tr(Df(\bar{x}))$ and $det(Df(\bar{x}))$. In short, the location of the point

$$Q(\mu) = (Tr(Df(\bar{x}(\mu))), det(Df(\bar{x}(\mu)))) \tag{9}$$

As μ varies the location of $\bar{x}(\mu)$ changes and the position of $Q(\mu)$ in the $(Tr(Df), det(Df))$-plane determines its stability.

In a generic Saddle-Node (SN) bifurcation a a saddle $\bar{x}_2(\mu)$ collides with a node $\bar{x}_1(\mu)$ at $\mu = \mu^*$. This collision in the $(Tr(Df), det(Df))$-plane is nicely illustrated in Figure 3.

incomplete

4.3 Hopf Bifurcations

Consider the planar system

$$\frac{dx_1}{dt} = f_1(x_1, x_2; \mu) \quad , \tag{10}$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2; \mu) \quad , \tag{11}$$

where μ is a parameter. Alternately, we have the notations:

$$\frac{dx}{dt} = f(x) = \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2; \mu) \\ f_2(x_1, x_2; \mu) \end{pmatrix}$$

Further, let $\bar{x}(\mu) = (\bar{x}_1(\mu), \bar{x}_2(\mu))$ be the equilibria. The Jacobian of the vector field $f(x)$ at $\bar{x}$ is

$$Df(\bar{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

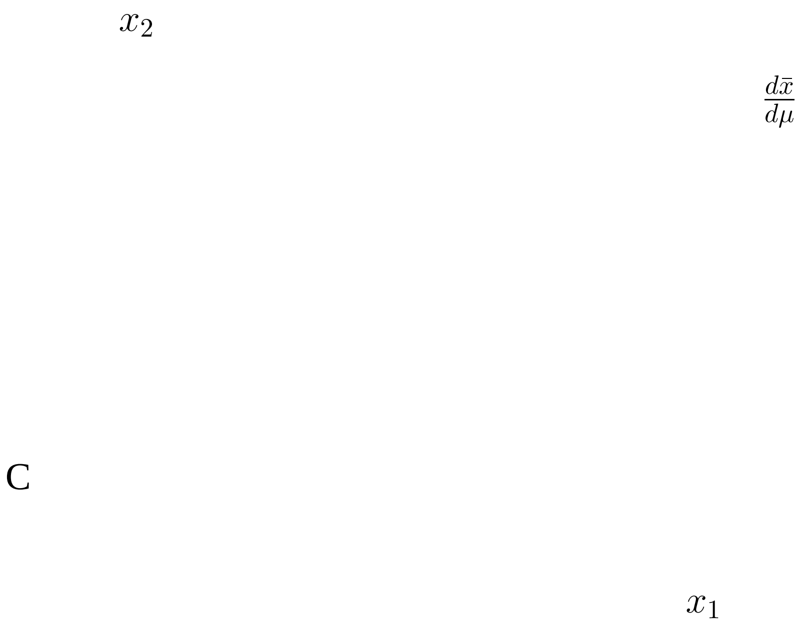

Figure 3: Illustrates a Saddle-Node bifurcation in the $(Tr(Df), det(Df))$-plane.

The eigenvalues of $Df(\bar{x})$ are functions of the parameter μ. In terms of the trace $TrDf$ and determinant $detDf$, the eigenvalues of the Jacobian are:

$$\lambda_{\pm}(\mu) = \frac{TrDf \pm \sqrt{(TrDf)^2 - 4detDf}}{2}$$

In this summary we consider the special case where at some parameter value $\mu = \mu^*$

$$TrDf(\bar{x}(\mu^*)) = 0 \tag{12}$$

$$detDf(\bar{x}(\mu^*)) > 0 \tag{13}$$

When these two conditions are satisfied, the eigenvalues of the Jacobian are purely imaginary. If, in addition to (12)-(13) being satisfied, the *transversality condition*

$$\frac{d}{d\mu}\left\{\mathbb{R}e\left(\lambda_+(\mu)\right)\right\}|_{\mu=\mu^*} \neq 0 \tag{14}$$

is satisfied, then a *Hopf* bifurcation occurs at the bifurcation point $(\bar{x}(\mu^*)), \mu^*)$ (here, $\mathbb{R}e(z)$ is the real part of z). At such a Hopf bifurcation for some μ near μ^*, small amplitude oscillations (limit cycles) exist. The amplitude of these oscillations approaches zero as μ approaches μ^*. Though Hopf theory guarantees the existence of such periodic orbits for $\mu \simeq \mu^*$, it does not guarantee the existence of the oscillations for μ further away from μ^*. Often, however, the periodic orbits persist and grow in amplitude as $|\mu - \mu^*|$ increases.

At $\mu = \mu^*$ the linearized system (linearization of (10)-(11) about $x^* = \bar{x}(\mu^*)$)

$$\frac{dz}{dt} = Df(x^*)z. \quad , \quad z = (z_1, z_2) \in \mathbb{R}^2 \tag{15}$$

has a center at $z = 0$. Therefore, solutions $z(t)$ have the form

$$z(t) = c_1\vec{\zeta}_1 \cos \omega t + c_2\vec{\zeta}_2 \sin \omega t$$

for some real constants c_k and constant vectors $\vec{\zeta}_k$, $k = 1, 2$. Given the assumed conditions (12)-(13), $\lambda_{\pm} = \pm\omega i$ where $i^2 = -1$ and

$$\omega = \sqrt{detDf} \tag{16}$$

By Hopf theory, if (12)-(14), are satisfied then for every μ with $|\mu - \mu^*|$ sufficiently small, there exists a T-periodic orbit (limit cycle) $x_p(t; \mu)$ which satisfy (10)-(11). The period $T = T(\mu)$ and Hopf theory also guarantees

$$\lim_{|\mu-\mu^*|\to 0} T(\mu) = \frac{2\pi}{\omega} \tag{17}$$

In other words, for μ very nearly equal μ^*, the period of the (emergent) periodic orbits of (10)-(11) nearly equals the period of the concentric periodic orbits of the linearized system (15).

For some systems, the Jacobian has the very special form:

$$Df(x^*) = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} \quad , \quad x^* = \bar{x}(\mu^*) \tag{18}$$

where $\mu^* = 0$. If the system does not have a Jacobian of this form, one can transform it to one that does by shifting and scaling transformations. For instance,

$$X_i = \frac{x_i - \bar{x}_i^*}{\delta_i} \quad , \quad i = 1, 2 \quad , \quad \eta = \mu - \mu^*$$

for appropriately defined constants δ_1 and δ_2 will transform the original system for x to one in X:

$$\frac{dX}{dt} = F(X, \eta)$$

whose Jacobian $DF(X^*)$ at $\eta = 0$ will have the form (18).

For systems whose Jacobian is of the form (18), a third-order Taylor Series expansion of (10)-(11) about x^* yields a system of the form:

$$\frac{dz_1}{dt} = (d\mu + a(z_1^2 + z_2^2))z_1 - (\omega + c\mu + b(z_1^2 + z_2^2))z_2 \tag{19}$$

$$\frac{dz_1}{dt} = (\omega + c\mu + b(z_1^2 + z_2^2))z_1 + (d\mu + a(z_1^2 + z_2^2))z_2 \tag{20}$$

which when expressed in polar coordinates is

$$\frac{dr}{dt} = (d\mu + ar^2)r \tag{21}$$

$$\frac{d\theta}{dt} = (\omega + c\mu + br^2) \tag{22}$$

for constants a, b, c, d, ω, $z_1 = r\cos\theta, z_2 = r\sin\theta$. Note the equation for $r(t)$ is not coupled to the equation for θ. Furthermore, this third-order system possesses periodic orbits along the locus [1]

$$\mu = -ar^2/d \quad , \quad d \neq 0$$

---

[1] noting that $\frac{d\theta}{dt} \neq 0$ near the Hopf point at $\mu = 0$

Note that the shape of this curve in the (μ, r) bifurcations diagram is quadratic. Thus, according to Hopf theory, the envelope of periodic orbits eminating from the bifurcation point (μ^*, x^*) form a parabolid in (μ, x_1, x_2)-space (at least locally). Some authors use the term *quadratic tangency* to describe this.

It can be shown that

$$d = \frac{d}{d\mu} \{\mathbb{R}e\,(\lambda_+(\mu))\}\,|_{\mu=\mu^*}$$

so that the existence of periodic orbits local to the bifurcation point depends on $d \neq 0$. This is just the transversality condition (14).

Computing the other coefficients a and c can be a substantial undertaking. After some transformations like those discussed in relation to (18) the original system can sometimes be transformed into a new system of the form

$$\begin{aligned} \frac{dx}{dt} &= \mu x - \omega y + f(x,y) \qquad , &(23)\\ \frac{dy}{dt} &= +\omega x + \mu y + g(x,y) \qquad , &(24) \end{aligned}$$

where

$$f(0,0) = g(0,0) = f_x(0,0) = f_y(0,0) = g_x(0,0) = g_y(0,0) = 0\ .$$

When this is possible the system (23)-(24) has a Hopf Bifurcation at $\mu = \mu^* = 0$ and the coefficient a is given by (Discussed in Guggenheimer and Holmes, 1983)

$$a = \frac{1}{16}\left(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right) + \frac{1}{16\omega}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right]$$

evaluated at $(x, y) = (0, 0)$.

One reason one wants to determine this coefficient is that the sign of d and a together are needed to determine the stability (and direction) of the eminating periodic solutions. Such information can be ascertained from equation (21). For example, were $a > 0$ and $d < 0$, (21) would have a sole unstable (positive) fixed point corresponding to an unstable periodic orbit. If the eminating orbits are stable the bifurcation is said to be *supercritical*. When unstable orbits eminate, the Hopf bifurcation is said to be *subcritical*.

For an exact statement of the Hopf Theorem (1947) see Guggenheimer and Holmes' (1983) Book: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, pg 151.