

Math 591 (2003)

Really Easy Take Home Final

**Instructions:** Do not collaborate with other students. Due December 16 (12pm) in my mailbox. 20 points total.

1. Consider the following perturbed eigenvalue problem for  $u = u(x, y)$

$$\nabla^2 u + \lambda u = 0 \quad , \quad (x, y) \in \Omega_\epsilon \quad (1)$$

$$u = 0 \quad , \quad (x, y) \in \partial\Omega_\epsilon \quad (2)$$

$$\Omega_\epsilon = \left\{ (x, y) : \frac{x^2}{(1 + \epsilon)^2} + y^2 < 1 \right\} \quad (3)$$

where  $\epsilon \ll 1$ . Here the elliptic boundary  $\partial\Omega_\epsilon$  is a perturbed circle. We wish to consider how the smallest eigenvalue changes as the domain perturbs. In the above,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

- a) (6pts) Under the transformation

$$X = \frac{x}{1 + \epsilon} \quad , \quad Y = y \quad , \quad U(X, Y) = u(x, y)$$

derive the exact partial differential equation and associated boundary condition  $U(X, Y)$  must solve. In the  $(X, Y)$  coordinates the new domain is a circle.

- b) (6pts) Letting  $(X, Y) = (r \cos \theta, r \sin \theta)$  and using the expansions

$$\begin{aligned} U &= U_0(r, \theta) + \epsilon U_1(r, \theta) + \dots \\ \lambda &= \lambda_0 + \epsilon \lambda_1 + \dots \end{aligned}$$

derive problems for  $U_0$  and  $U_1$ .

- c) (6pts) Determine a formula for  $\lambda_1$  in terms of polar integrals and Bessel functions  $J_0, J_1$ .  
d) (2pts) Evaluate your result in c) using the identities on the back. Here it's a bit more orderly to integrate in  $\theta$  first.

**Bessel function identities:**

$$\begin{aligned}\frac{d}{dz}J_0(z) &= -J_1(z) \\ \frac{d^2}{dz^2}J_0(z) &= -J_0(z) + \frac{1}{z}J_1(z) \\ \int rJ_0(\xi r)^2 dr &= \frac{1}{2}r^2 \left( J_0(\xi r)^2 + J_1(\xi r)^2 \right) \\ \int J_0(\xi r)^2 J_1(\xi r) &= -\frac{1}{2\xi} J_0(\xi r)^2\end{aligned}$$