Fast-Slow Dynamics - An introduction

Many models in mathematical physiology result in a system of ordinary differential equations:

$$\frac{dx}{dt} = f(x,y) \quad , \quad x \in \mathbb{R}^n \tag{1}$$

$$\frac{dy}{dt} = \varepsilon g(x,y) \quad , \quad y \in \mathbb{R}^m \tag{2}$$

where ε is a small parameter, i.e., $\varepsilon \ll 1$. Posed this way, $x = (x_1, x_2, \ldots, x_n)$ are fast variables, $y = (y_1, y_2, \ldots, y_m)$ are slow variables and t is the fast time. Some authors let $\tau = \varepsilon t$ and cast the system (1)-(2) in the form

$$\varepsilon \frac{dx}{d\tau} = f(x,y) \tag{3}$$

$$\frac{dy}{d\tau} = g(x,y) \tag{4}$$

Here x and y are still fast and slow variables, respectively, but τ is a slow time. Regardless of how time is scaled, so long as $f(x,y) \neq 0$,

$$\left|\frac{dx}{dt}\right| \gg \left|\frac{dy}{dt}\right|$$

Thus, it is the relative rates which make x "fast".

The fast subsystem (FS) corresponding to (1)-(2) is

$$\frac{dx}{dt} = f(x,y) \tag{5}$$

$$\frac{dy}{dt} = 0 \tag{6}$$

whereas the slow subsystem (SS) is

$$0 = f(x,y) \tag{7}$$

$$\frac{dy}{d\tau} = g(x,y) \tag{8}$$

The dynamics of the original system are then often explained in terms of the respective fast and slow subsystems.

To do this, one must first understand the dynamics of the fast subsystem. Since y are fixed in the (FS), these are generally regarded as bifurcation parameters. Given some region D of y, the (FS) may possess stable attractors - generally of two types: a) stable equilibria b) stable periodic (limit cycle) solutions. If for any given $y \in D$ there are two stable attractors the system is said to be bistable. For the classic FitzHugh-Nagumo model

$$\frac{dx}{dt} = f(x) - y \qquad f(x) = x(a-x)(x-1) \tag{9}$$

$$\frac{dy}{dt} = \varepsilon(x - x_0 - \gamma y) \tag{10}$$

where (a, x_0, γ) are parameters, $m = 1$ and $n = 1$, the (FS) is bistable for some interval D of y.

Regardless of what value y is fixed at the (FS), the solution of the (FS) will rapidly approach one of the attractors. Which one depends on what the initial conditions for x are. As is often the case, one or more equilibria of (FS) are stable for some range of y. Thus, for these y the solution of the (FS) rapidly approaches these stable equilibria. Note that for these equilibria, $f(x, y) = 0$ and that this is one of the equations in the (SS).

Since, asymptotically, the (FS) describes the dynamics of (1)-(2) only for times of $O(1)$ one expects the solution of (1)-(2) to behave like the solutions of (FS) until they get near these stable equilibria. Once trajectories of (1)-(2) have reached these equilibria, one then expects solutions to be approximated by the (SS). Noting again that $f(x, y) = 0$ for the slow subsystem, the (SS) really describes motion on the surface $f = 0$ in (x, y)-space.

In dynamical system theory, one identifies a region D of y for which a particular branch $S(D)$ of the (FS) equlilibria are stable. This "branch" is a stable manifold for the fast flow to which solutions of (1)-(2) are (at least locally) attracted. Specifically,

$$S(D) \equiv \{(x, y) : f(x, y) = 0, (x, y) \ \textit{stable in}(FS), y \in D\} \tag{11}$$

is a stable manifold (or union of). The FitzHugh-Nagumo model above has two disjoint stable manifolds $S(D_1)$ and $S(D_2)$ and $D = D_1 \cap D_2$ is the region of bistability.

To understand solutions of (1)-(2) one must therefore determine the bifurcation structure for (FS), its attractors, bifurcation points etc. and then examine the (SS) to decide if transitions between slow stable manifolds occur. The situation is complicated somewhat when the (FS) has stable periodic solutions coexisting with stable equilibria (as is the case in some types of bursting). In that case the appropriate slow subsystem near the (FS) periodic solutions is not given by (7)-(8). Usually, the method of averaging is used to compute an appropriate slow flow near the "manifold" of period solutions.