Supplement to Accurate Confidence Intervals in Regression Analyses of Non-Normal Data

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Contents

Abstract

A linear model in which random errors are distributed independently and identically according to an arbitrary continuous distribution is assumed. Second- and third-order accurate confidence intervals for regression parameters are constructed from Charlier differential series expansions of approximately pivotal quantities around Student's t distribution. Simulation verifies that small sample performance of the intervals surpasses that of conventional asymptotic intervals and equals or surpasses that of bootstrap percentile-t and bootstrap percentile-$|t|$ intervals under mild to marked departure from normality.

This supplement gives further details on theorems and algorithms that were reported in the article.

9 Proof of Theorem 1

The moments of V are obtained by expanding V^j as in (11) that and then taking the expectation of V^j with respect to the distribution of $\mathbf{U} = (Z_1 \;\; Z_2)'$. The distribution of $\mathbf{U}$, in turn, is approximated by its Edgeworth expansion. The validity of the Edgeworth expansion can be verified by the approach of Yanagihara (2003, §3).

It follows from the expansion in (11) that

$$\mathrm{E}\left(V^{2j}\right) = \mathrm{E}\left(Z_1^{2j}\right) - \frac{j}{\sqrt{n}}\mathrm{E}\left(Z_1^{2j}Z_2\right) + \frac{j(j+1)}{2n}\mathrm{E}\left(Z_1^{2j}Z_2^2\right) + O\left(n^{-2}\right), \tag{1}$$

because the $O_p(n^{-3/2})$ term has expectation $O(n^{-2})$ and that

$$\mathrm{E}\left(V^{2j+1}\right) = \mathrm{E}\left(Z_1^{2j+1}\right) - \frac{(2j+1)}{2\sqrt{n}}\mathrm{E}\left(Z_1^{2j+1}Z_2\right) + O\left(n^{-3/2}\right), \tag{2}$$

because the $O_p(n^{-1})$ term has expectation $O(n^{-3/2})$. Accordingly, to verify Theorem 1, it is necessary to obtain expressions for $\mathrm{E}(Z_1^{2j})$ with error no larger than $O(n^{-2})$, $\mathrm{E}(Z_1^{2j+1})$ with error no larger than $O(n^{-3/2})$, $\mathrm{E}(Z_1^{2j}Z_2)$ with error no larger than $O(n^{-3/2})$, $\mathrm{E}(Z_1^{2j+1}Z_2)$ with error no larger than $O(n^{-1})$, and $\mathrm{E}(Z_1^{2j}Z_2^2)$ with error no larger than $O(n^{-1})$.

It is shown below that the moments of V depend only on the joint joint moments of Z_1 and Z_2 which are of order four or lower in $\boldsymbol{\varepsilon} = (\varepsilon_1 \;\cdots\; \varepsilon_N)'$ from (1). These moments can be obtained by writing Z_1 and Z_2 as

$$Z_1 = \frac{1}{\sigma\sqrt{q_0}}\sum_{i=1}^{N} b_i\varepsilon_i \text{ and } Z_2 = \frac{1}{\sigma^2\sqrt{n}}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\varepsilon_i\varepsilon_j - \sqrt{n}$$

and then taking expectations, where $\mathbf{A}$ is defined in (4) and $\mathbf{b}$ is defined in (7). The required moments are displayed in Table A1.

Table A1: Joint Moments: $\mathrm{E}(Z_1^i Z_2^j)$

	j		
i	0	1	2
0	1	0	$q_2\kappa_4 + 2$
1	0	$q_1\kappa_3$	Order 5 in $\boldsymbol{\varepsilon}$
2	1	$\frac{q_3}{\sqrt{n}}\kappa_4$	Order 6 in $\boldsymbol{\varepsilon}$
3	$\frac{3q_4}{\sqrt{n}}\kappa_3$	Order 5 in $\boldsymbol{\varepsilon}$	Order 7 in $\boldsymbol{\varepsilon}$
4	$\frac{3q_5}{n}\kappa_4 + 3$	Order 6 in $\boldsymbol{\varepsilon}$	Order 8 in $\boldsymbol{\varepsilon}$

9.1 Expression for $\mathrm{E}(Z_1^j)$

Employing the moments of Z_1 from Table A1, it is readily shown that the Edgeworth expansion for the density of Z_1 is as follows:

$$\varphi_{Z_1}(z) = \varphi_Z(z)\left[1 + \frac{q_4\kappa_3 H_3(z)}{2\sqrt{n}} + \frac{q_5\kappa_4 H_4(z)}{8n} + \frac{q_4^2\kappa_3^2 H_6(z)}{8n} + O\left(n^{-\frac{3}{2}}\right)\right], \tag{3}$$

where $\varphi_{Z_1}(z)$ is the density of Z_1, $\varphi_Z(z)$ is the density of the standard normal distribution, and $H_i(z)$ is the Hermite polynomial of order i. Accordingly, the j^{th} moment of Z_1 can be expressed as

$$\mathrm{E}(Z_1^j) = \mathrm{E}_Z\left\{Z^j\left[1 + \frac{q_4\kappa_3 H_3(Z)}{2\sqrt{n}} + \frac{q_5\kappa_4 H_4(Z)}{8n} + \frac{q_4^2\kappa_3^2 H_6(Z)}{8n}\right]\right\} + R(j,n),$$

where $\mathrm{E}_Z(\cdot)$ denotes expectation with respect to the standard normal distribution, $R(j,n) = O(n^{-2})$ if j is even, and $R(j,n) = O(n^{-3/2})$ if j is odd. If $Z \sim \mathrm{N}(0,1)$, then

$$\mathrm{E}(Z^{2j}) = \frac{(2j)!}{j!2^j} \text{ and } \mathrm{E}(Z^{2j+1}) = 0.$$

It follows that

$$\mathrm{E}\left(Z_1^{2j}\right) = \frac{(2j)!}{j!2^j}\left[1 + \frac{j(j-1)(j-2)q_4^2\kappa_3^2}{n} + \frac{j(j-1)q_5\kappa_4}{2n}\right] + O\left(n^{-2}\right) \text{ and}$$

$$\mathrm{E}\left(Z_1^{2j+1}\right) = \frac{j(2j+1)!q_4\kappa_3}{2^j j!\sqrt{n}} + O\left(n^{-\frac{3}{2}}\right).$$

9.2 Expression for $\mathrm{E}(Z_1^j Z_2)$

Define $Z_{2\cdot 1}$ as

$$Z_{2\cdot 1} \stackrel{\text{def}}{=} Z_2 - q_1\kappa_3 Z_1.$$

It follows from the moments in Table A1 that

$$\mathrm{E}(Z_{2\cdot 1}) = 0, \quad \mathrm{Cov}(Z_1, Z_{2\cdot 1}) = 0, \text{ and } \mathrm{Var}(Z_{2\cdot 1}) = q_2\kappa_4 + 2 - q_1^2\kappa_3^2.$$

Note that

$$\mathrm{E}(Z_1^j Z_2) = \mathrm{E}\left[Z_1^j\left(Z_{2\cdot 1} + q_1\kappa_3 Z_1\right)\right] = \mathrm{E}(Z_1^j Z_{2\cdot 1}) + q_1\kappa_3\mathrm{E}\left(Z_1^{j+1}\right). \tag{4}$$

Accordingly, an expression for $\mathrm{E}(Z_1^j Z_2)$ can be obtained by evaluating $\mathrm{E}(Z_1^j Z_{2\cdot 1})$ and using the result in §9.1.

Employing the moments of Z_1 and Z_2 from Table A1, it can be shown that the Edgeworth expansion for the joint density of Z_1 and $Z_{2\cdot 1}$ is as follows:

$$\varphi_{\mathbf{Z}^*}(\mathbf{u};\boldsymbol{\Sigma}) = \varphi_{\mathbf{Z}}(\mathbf{u};\boldsymbol{\Sigma})\left[1 + \frac{1}{6}\mathbf{H}_3(\mathbf{u};\boldsymbol{\Sigma})\mathrm{E}(\mathbf{Z}^*\otimes\mathbf{Z}^*\otimes\mathbf{Z}^*) + O\left(n^{-1}\right)\right],$$

where $\mathbf{Z}^* = (Z_1\ Z_{2\cdot 1})'$, $\varphi_{\mathbf{Z}^*}(\mathbf{u};\boldsymbol{\Sigma})$ is the density function of $\mathbf{Z}^*$ evaluated at $\mathbf{u} = (u_1\ u_2)'$, $\varphi_{\mathbf{Z}}(\mathbf{u};\boldsymbol{\Sigma})$ is the density function for a two-dimensional random vector with distribution

$$\mathbf{Z} \sim \mathrm{N}(\mathbf{0},\boldsymbol{\Sigma}), \text{ where } \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & q_2\kappa_4 + 2 - q_1^2\kappa_3^2 \end{pmatrix},$$

and $\mathbf{H}_3(\mathbf{u};\boldsymbol{\Sigma})$ is the third order bivariate Hermite polynomial. Specifically,

$$\mathbf{H}_3(\mathbf{u};\boldsymbol{\Sigma}) = \left(\mathbf{u}'\boldsymbol{\Sigma}^{-1}\otimes\mathbf{u}'\boldsymbol{\Sigma}^{-1}\otimes\mathbf{u}'\boldsymbol{\Sigma}^{-1}\right)$$

$$-\left[\left(\mathrm{vec}\,\boldsymbol{\Sigma}^{-1}\right)'\otimes\mathbf{u}'\boldsymbol{\Sigma}^{-1}\right]\left[\mathbf{I}_6 + \left(\mathbf{I}_2\otimes\mathbf{I}_{(2,2)}\right) + \mathbf{I}_{(2,4)}\right],$$

where $\mathbf{I}_{(a,b)}$ is the commutation matrix (MacRae, 1974). This matrix is identical to $\mathbf{K}_{b,a}$ in Magnus and Neudecker (1979, 1999 §3.7).

Denote the expectation of a function of U_1 and U_2, say $r(U_1,U_2)$, with respect to the normal density $\varphi_{\mathbf{Z}}(\mathbf{u};\boldsymbol{\Sigma})$ by $\mathrm{E}_{\mathbf{Z}}\left[r(U_1,U_2)\right]$. Then, the required expectation is

$$\mathrm{E}(Z_1^j Z_{2\cdot 1}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_1^j u_2\varphi_{\mathbf{Z}^*}(\mathbf{u};\boldsymbol{\Sigma})\,du_1du_2$$

$$= \begin{cases} O\left(n^{-1}\right) & \text{if } j \text{ is odd,} \\ \frac{1}{6}\mathrm{E}_{\mathbf{Z}}\left[U_1^j U_2\mathbf{H}_3(\mathbf{U};\boldsymbol{\Sigma})\mathrm{E}(\mathbf{Z}^*\otimes\mathbf{Z}^*\otimes\mathbf{Z}^*)\right] + O\left(n^{-\frac{3}{2}}\right) & \text{if } j \text{ is even.} \end{cases}$$

Denote the variance of $Z_{2\cdot 1}$ by ξ. That is, $\xi = q_2\kappa_4 + 2 - q_1^2\kappa_3^2$. Writing $\mathbf{H}_3(\mathbf{U};\boldsymbol{\Sigma})\mathrm{E}(\mathbf{Z}^* \otimes \mathbf{Z}^* \otimes \mathbf{Z}^*)$ in scalar terms reveals that

$$\mathrm{E}(Z_1^{2j} Z_{2\cdot 1}) =$$

$$\frac{1}{6}\mathrm{E}_{\mathbf{Z}}\Bigg\{U_1^{2j}U_2\left[U_1^3\mathrm{E}(Z_1^3) + \frac{3U_1^2U_2}{\xi}\mathrm{E}(Z_1^2Z_{2\cdot 1}) + \frac{3U_1U_2^2}{\xi^2}\mathrm{E}(Z_1Z_{2\cdot 1}^2) + \frac{U_2^3}{\xi^3}\mathrm{E}(Z_{2\cdot 1}^3)\right]$$

$$-3U_1^{2j}U_2\left[U_1E(Z_1^3) + \frac{U_2}{\xi}\mathrm{E}(Z_1^2Z_{2\cdot 1}) + \frac{U_1}{\xi}\mathrm{E}(Z_1Z_{2\cdot 1}^2) + \frac{U_2}{\xi^2}\mathrm{E}(Z_{2\cdot 1}^3)\right]\Bigg\} + O\left(n^{-\frac{3}{2}}\right)$$

$$= \frac{1}{2}\mathrm{E}_{\mathbf{Z}}\left(U_1^{2(j+1)} - U_1^{2j}\right)\mathrm{E}(Z_1^2Z_{2\cdot 1}) + O\left(n^{-\frac{3}{2}}\right).$$

Using the results from §9.1 together with (4), it follows that

$$\mathrm{E}\left(Z_1^{2j}Z_{2\cdot 1}\right) = \frac{j(2j)!}{2^j j!\sqrt{n}}\left(q_3\kappa_4 - 3q_1q_4\kappa_3^2\right) + O\left(n^{-\frac{3}{2}}\right),$$

$$\mathrm{E}\left(Z_1^{2j}Z_2\right) = \frac{j(2j)!}{2^j j!\sqrt{n}}\left(q_3\kappa_4 + 2(j-1)q_1q_4\kappa_3^2\right) + O\left(n^{-\frac{3}{2}}\right), \text{ and}$$

$$\mathrm{E}\left(Z_1^{2j+1}Z_2\right) = \frac{q_1(2j+1)!}{2^j j!}\kappa_3 + O\left(n^{-1}\right).$$

9.3 Expression for $\mathrm{E}(Z_1^{2j}Z_2^2)$

It follows from the first order joint normal distribution of Z_1 and $Z_{2\cdot 1}$ that

$$\mathrm{E}(Z_1^{2j}Z_{2\cdot 1}^2) = \frac{(2j)!}{j!2^j}\left(q_2\kappa_4 + 2 - q_1^2\kappa_3^2\right) + O\left(n^{-1}\right).$$

Accordingly,

$$\mathrm{E}\left(Z_1^{2j}Z_2^2\right) = \frac{(2j)!}{2^j j!}\left(q_2\kappa_4 + 2 + 2jq_1^2\kappa_3^2\right) + O\left(n^{-1}\right).$$

Theorem 1 is verified by substituting the above expression, the expressions for $\mathrm{E}(Z_1^{2j}Z_2)$ and $\mathrm{E}(Z_1^{2j+1}Z_2)$ in §9.2, and the expressions for $\mathrm{E}(Z_1^{2j})$ and $\mathrm{E}(Z_1^{2j+1})$ in §9.1 into equations (1) and (2).

10 Proof of Theorem 3

Define Z_3 as

$$Z_3 \stackrel{\text{def}}{=} \sqrt{n}\left(\frac{1}{nq_6\sigma^3}\sum_{i=1}^N e_i^3 - \kappa_3\right),$$

where e_i is the i^{th} residual. It follows from (14) that $\mathrm{E}(Z_3) = 0$. Furthermore, $\sqrt{n}(\widehat{\kappa}_3 - \kappa_3)$ can be expanded as

$$\sqrt{n}(\widehat{\kappa}_3 - \kappa_3) = Z_3 - \frac{3}{2}Z_2\kappa_3 + O_p\left(n^{-\frac{1}{2}}\right),$$

where Z_2 is defined in (11). By expanding V^j as in (11) and using $\widehat{\kappa}_3 = \kappa_3 + O_p(n^{-1/2})$, the quantity $\mathrm{E}\left[V^j\sqrt{n}(\widehat{\kappa}_3 - \kappa_3)\right]$ can be written as

$$\mathrm{E}\left[V^j\sqrt{n}(\widehat{\kappa}_3 - \kappa_3)\right] = \begin{cases} O\left(n^{-\frac{1}{2}}\right) & \text{if } j \text{ is even,} \\ \mathrm{E}\left[Z_1^jZ_3 - \frac{3}{2}Z_1^jZ_2\kappa_3\right] + O\left(n^{-1}\right) & \text{if } j \text{ is odd.} \end{cases}$$

Using the same methods as described in §9, it is readily shown that $\mathrm{Cov}(Z_1, Z_3) = q_8\kappa_4$, where q_8 is defined in (7). Accordingly, if $Z_{3\cdot1}$ is defined as $Z_{3\cdot1} \stackrel{\text{def}}{=} Z_3 - q_8\kappa_4 Z_1$, then $\mathrm{E}(Z_{3\cdot1}) = 0$, $\mathrm{Cov}(Z_1, Z_{3\cdot1}) = 0$, and

$$\mathrm{E}(Z_1^{2j+1} Z_3) = \mathrm{E}\left[Z_1^{2j+1}\left(Z_{3\cdot1} + q_8\kappa_4 Z_1\right)\right] = \mathrm{E}\left(q_8\kappa_4 Z_1^{j+1}\right) + O\left(n^{-1}\right).$$

Employing the expression for $\mathrm{E}({Z_1}^j)$ from §9.1, it follows that

$$\begin{aligned}\mathrm{E}\left[V^{2j+1}\sqrt{n}(\widehat{\kappa}_3 - \kappa_3)\right] &= \mathrm{E}\left[Z_1^{2(j+1)}\left(q_8\kappa_4 - \frac{3q_1}{2}\kappa_3^2\right)\right] + O\left(n^{-1}\right) \\ &= \frac{(2j+1)!}{2^j j!}\left(q_8\kappa_4 - \frac{3q_1}{2}\kappa_3^2\right) + O\left(n^{-1}\right).\end{aligned}$$

11 Exponential Approximation to Polynomials

The various Cornish-Fisher polynomial transformations can be written as

$$h(x) \stackrel{\text{def}}{=} x + \sum_{j=0}^{k} \widehat{m}_j x^j, \text{ where } \widehat{m}_j = \begin{cases} O_p\left(n^{-\frac{1}{2}}\right) & \text{if } j = 0 \text{ or } 2, \text{ and} \\ O_p\left(n^{-1}\right) & \text{otherwise.}\end{cases}$$

The proposed exponential approximation to the polynomial is

$$\widehat{h}(x) \stackrel{\text{def}}{=} \widehat{m}_0 + x e^{\widehat{m}_1} + \exp\left\{-\frac{\widehat{d}}{2}x^2\right\}\sum_{j=2}^{k}\widehat{m}_j x^j,$$

where $\widehat{d}$ is chosen to be the smallest non-negative value such that $\widehat{h}$ is a monotonic function of x. This appendix describes an iterative algorithm for minimizing $\widehat{d}$ subject to $\widehat{d} \geq 0$ and $\widehat{h}^{(1)}(x) \geq 0$ for all x, where

$$\widehat{h}^{(1)}(x) \stackrel{\text{def}}{=} \frac{\partial\,\widehat{h}(x)}{\partial\,x} = e^{\widehat{m}_1} + \exp\left\{-\frac{\widehat{d}}{2}x^2\right\}\sum_{j=2}^{k}\left(jx^{j-1} - \widehat{d}x^{j+1}\right)\widehat{m}_j.$$

In general, a closed-form solution for $\widehat{d}$ does not exist. Nonetheless, the magnitude of $\widehat{d}$ is known. it can be shown that if $\widehat{d}$ is the smallest nonnegative number such that $\widehat{h}^{(1)}(x) \geq 0$ for all x, then

$$\exp\left\{-\frac{\widehat{d}}{2}x^2\right\} = \begin{cases} 1 + O_p\left(n^{-1}\right) & \text{if } k = 2 \text{ and} \\ 1 + O_p\left(n^{-\frac{2}{k-1}}\right) & \text{if } k \geq 3,\end{cases}$$

where k is the degree of the polynomial transformation $h(x)$ and x is any constant.

Denote the minimizer of $\widehat{h}^{(1)}(x)$ with respect to x by $x_{\min}$. It is apparent that $x_{\min}$ satisfies $\widehat{h}^{(2)}(x_{\min}) = 0$, where

$$\widehat{h}^{(2)}(x) \stackrel{\text{def}}{=} \exp\left\{\frac{\widehat{d}}{2}x^2\right\}\frac{\partial^2\widehat{h}(x)}{(\partial\,x)^2} = \sum_{j=2}^{k}\left[j(j-1)x^{j-2} - (2j+1)\widehat{d}x^j + \widehat{d}^2x^{j+2}\right]\widehat{m}_j.$$

Specifically, $x_{\min}$ is the real solution to the $k+2$ degree polynomial $\widehat{h}^{(2)}(x) = 0$ for which $\widehat{h}^{(1)}(x)$ is smallest. The goal can be restated as minimizing $\widehat{d}$ subject to $\widehat{d} \geq 0$ and $\widehat{h}^{(1)}(x_{\min}) = 0$. Using a modified Lagrange multiplier approach, the value of $\widehat{d}$ after iteration i is

$$\widehat{d}_{i+1} = \widehat{d}_i - \delta_i, \text{ where } \delta_i = \min\left\{\frac{\widehat{d}_i}{2},\ \widehat{h}^{(1)}(x_{\min})\left[\frac{\partial\,\widehat{h}^{(1)}(x_{\min})}{\partial\,\widehat{d}}\Big|_{\widehat{d}=\widehat{d}_i}\right]^{-1}\right\}.$$

The computation of $\partial\widehat{h}^{(1)}(x_{\min})/\partial\widehat{d}$ depends on $\partial x_{\min}/\partial\widehat{d}$. The latter derivative can be obtained with the aid of the implicit function theorem. Specifically, $\partial x_{\min}/\partial\widehat{d}$ can be obtained by solving $\partial\widehat{h}^{(2)}(x_{\min})/\partial\widehat{d} = 0$. The solution is

$$\frac{\partial x_{\min}}{\partial\widehat{d}} = \frac{\sum_{j=2}^{k}\left[(2j+1)x_{\min}^{j} - 2\widehat{d}x_{\min}^{j+2}\right]\widehat{m}_j}{\sum_{j=2}^{k}\left[j(j-1)(j-2)x_{\min}^{j-3} - \widehat{d}j(2j+1)x_{\min}^{j-1} + \widehat{d}^2(j+2)x_{\min}^{j+1}\right]\widehat{m}_j}.$$

The derivative of $\widehat{h}^{(1)}(x_{\min})$ with respect to $\widehat{d}$ is

$$\frac{\partial\widehat{h}^{(1)}(x_{\min})}{\partial\widehat{d}} =$$

$$\sum_{j=2}^{k}\left\{\left[j(j-1) - \widehat{d}(2j+1)x_{\min}^2 + \widehat{d}^2x_{\min}^4\right]\frac{\partial x_{\min}}{\partial\widehat{d}} - \frac{j+2}{2}x_{\min}^3 + \frac{\widehat{d}}{2}x_{\min}^5\right\}x_{\min}^{j-2}\widehat{m}_j.$$

12 Inversion of Exponential Approximations

This appendix describes a Modified Newton algorithm for finding the value of x that satisfies $\widehat{h}(x) = x_0$, where $\widehat{h}(x)$ is defined in §11 and x_0 is a non-zero constant. The solution is unique because $\widehat{h}(x)$ is a monotonic function of x by construction. After iteration i of the algorithm, the value of x is

$$x_{i+1} = x_i - \delta_i,\ \text{where}\ \delta_i = \begin{cases} \dfrac{x_i}{2} & \text{if}\ \dfrac{\widehat{h}(x_i) - x_0}{x_i\,\widehat{h}^{(1)}(x_i)} > \dfrac{1}{2}, \\ \dfrac{\widehat{h}(x_i) - x_0}{\widehat{h}^{(1)}(x_i)} & \text{otherwise,} \end{cases}$$

and $\widehat{h}^{(1)}(x_i)$ is defined in §11.

13 Additional Displays

Figures 4 and 5 display the coverage probabilities of eight one-sided nominal 95% interval estimators under conditions (a) and (d). Sub-plots in each figure display coverage for one method over the three log-normal distributions. Solid (dashed) line segments reflect coverage of one-sided lower (upper) intervals. The interval estimators are numbered as in Table 2. Bootstrap intervals were based on 1,000 bootstrap samples. Coverage of nominal 95% two-sided symmetric intervals under conditions (a) and (d) is displayed in Figure 6.

14 References

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Yanagihara, H. (2003). Asymptotic expansion of the null distribution of test statistic for linear hypothesis in nonnormal linear model. *Journal of Multivariate Analysis*, **84**, 222–246.

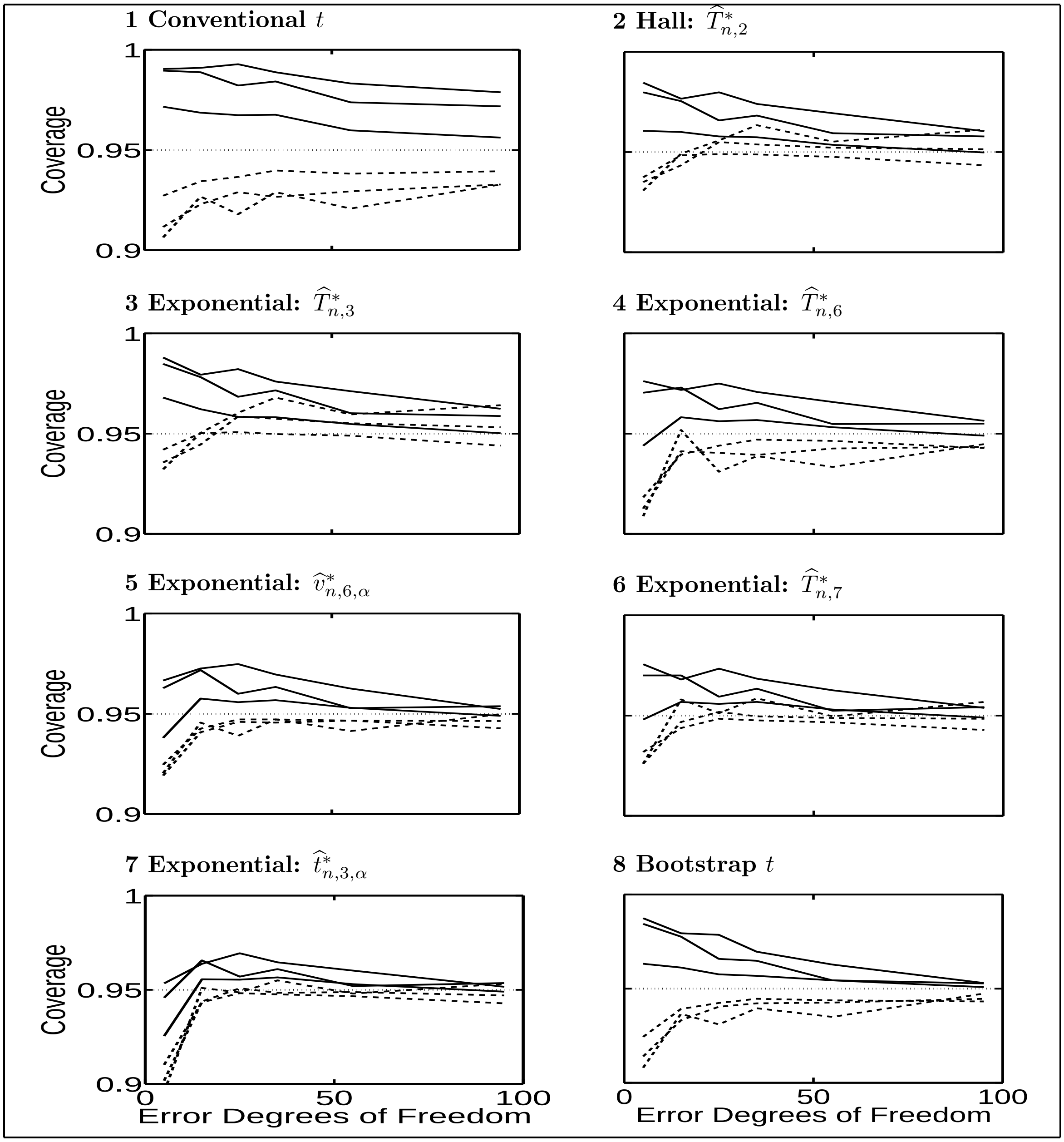

Figure 4: Coverage of One-Sided Intervals Under Condition (a) in Table 3

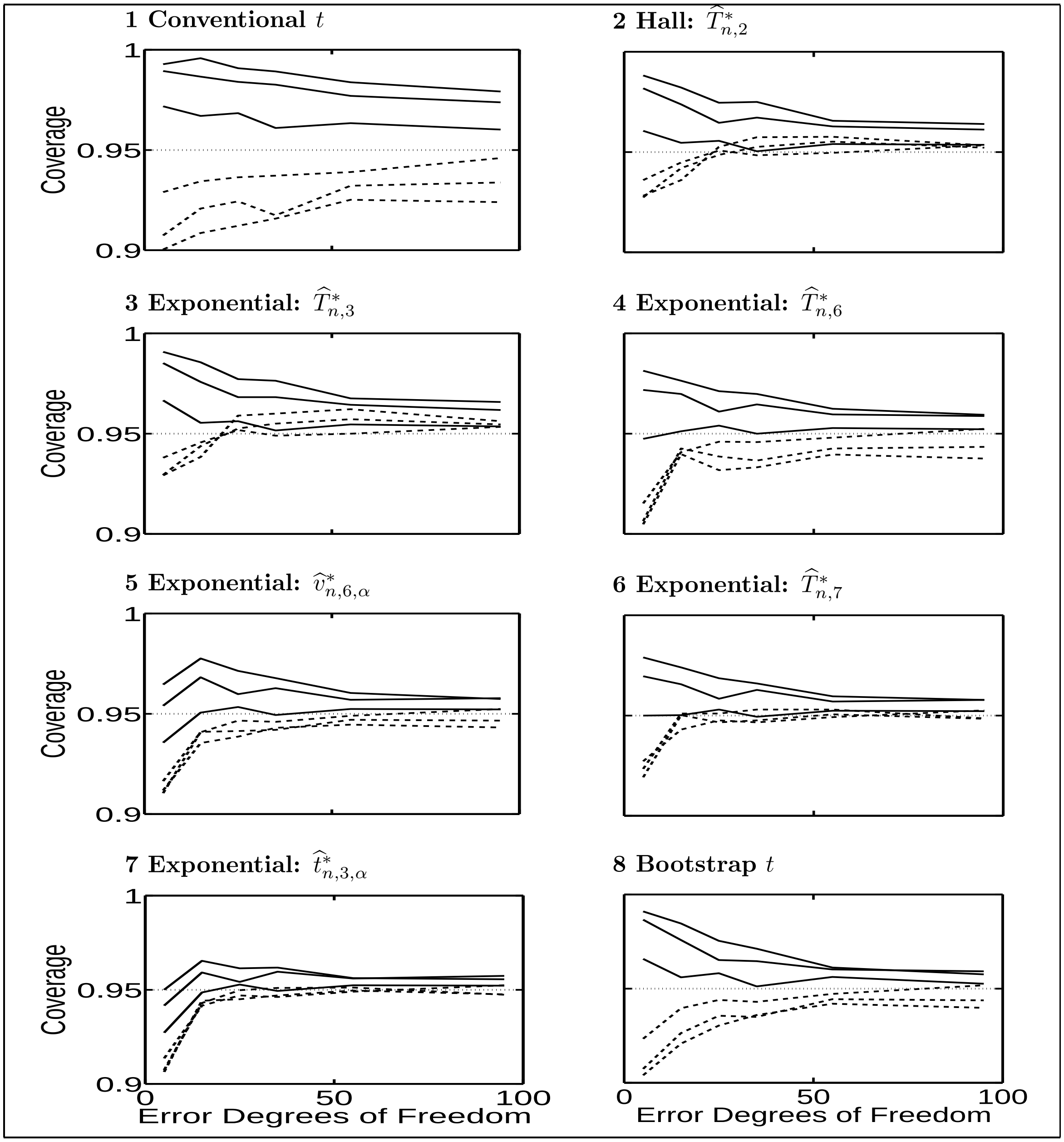

Figure 5: Coverage of One-Sided Intervals Under Condition (d) in Table 3

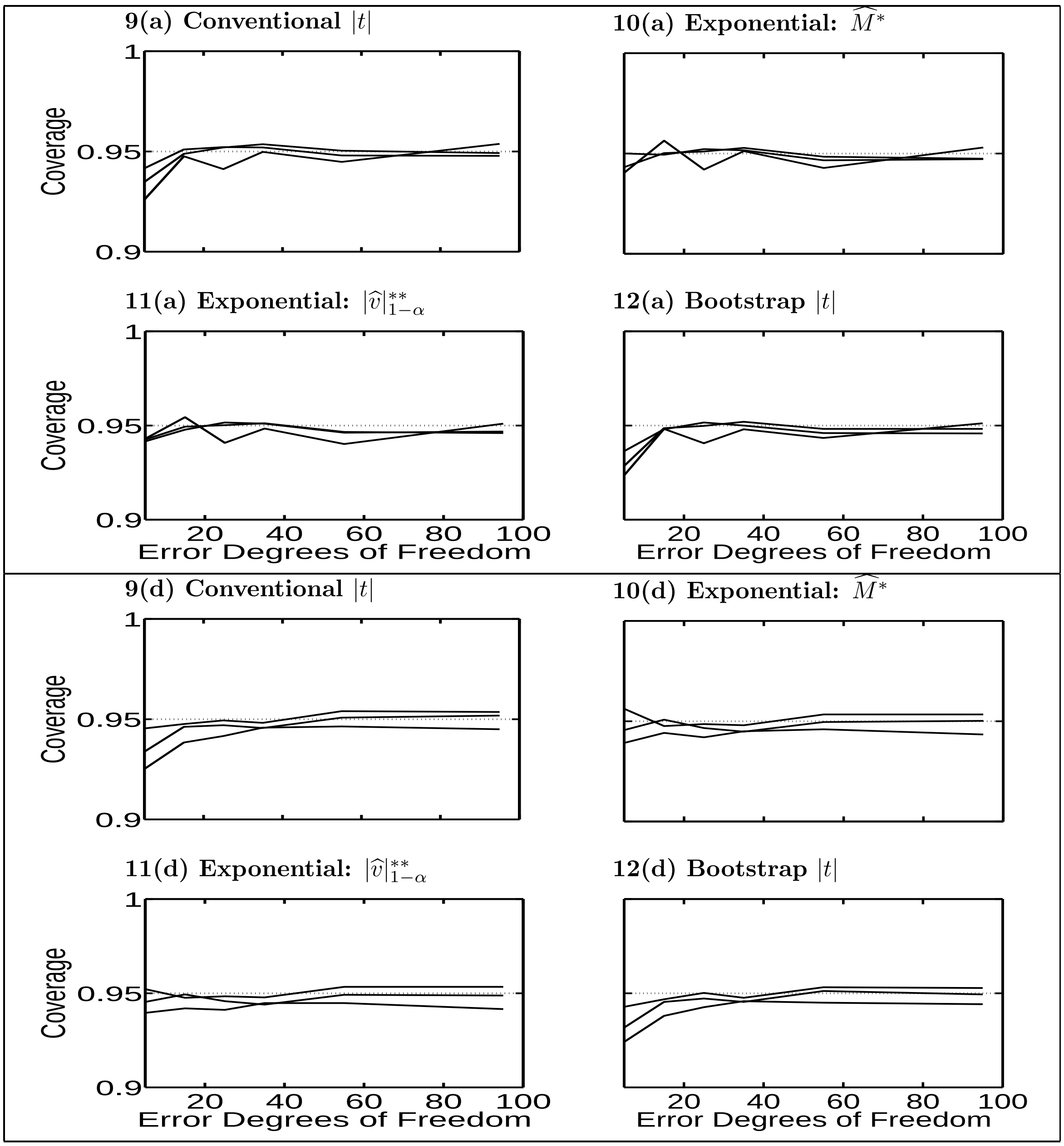

Figure 6: Coverage of Two-Sided Symmetric Intervals Under Conditions (a) and (d) in Table 3