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Computation of Confidence Intervals for Poisson and Binomial Parameters

R. J. Boik, April 25, 2005

1 Poisson Confidence Intervals

Suppose that $x_i \overset{\text{iid}}{\sim} \text{Poisson}(\theta)$ for $i = 1, \ldots, n$. Define T and $\overline{X}$ as

$$T \overset{\text{def}}{=} \sum_{i=1}^{n} X_i \text{ and } \overline{X} = \frac{1}{n} T.$$

Furthermore, denote a realization of T by t.

1.1 Invert LR Test to Obtain Exact One-Sided Intervals

1.1.1 Lower Confidence Limits

The LR test rejects $\text{H}_0\colon \theta \leq \theta_0$ in favor of $\text{H}_0\colon \theta > \theta_0$ if $T \geq c$, where c is the smallest value that satisfies $P_{\theta_0}(T \geq c) \leq \alpha$. The p-value is $P_{\theta_0}(T \geq t)$. Also H_0 is not rejected if p-value $> \alpha$. Accordingly,

$$\text{H}_0 \text{ accepted } \Longrightarrow P_{\theta_0}(T \geq t) > \alpha$$

$$\Longrightarrow F_Y(n\theta_0, t, 1) > \alpha, \text{ where } Y \sim \text{Gamma}(t, 1)$$

$$\Longrightarrow \frac{1}{n} F_Y^{-1}(\alpha, t, 1) < \theta_0$$

$$\Longrightarrow \frac{1}{2n} \chi^2_{\alpha, 2t} < \theta_0.$$

Caution, $\chi^2_{\alpha,\nu}$ is the 100α percentile of the χ^2 distribution with ν degrees of freedom.

1.1.2 Upper Confidence Limits

The LR test rejects $\text{H}_0\colon \theta \geq \theta_0$ in favor of $\text{H}_0\colon \theta < \theta_0$ if $T \leq c$, where c is the largest value that satisfies $P_{\theta_0}(T \leq c) \leq \alpha$. The p-value is $P_{\theta_0}(T \leq t)$. Also, H_0 is not rejected if p-value $> \alpha$. Accordingly,

$$\begin{aligned}
\text{H}_0 \text{ accepted } &\Longrightarrow P_{\theta_0}(T \le t) > \alpha \\
&\Longrightarrow 1 - P_{\theta_0}(T \ge t+1) > \alpha \\
&\Longrightarrow 1 - \alpha > P_{\theta_0}(T \ge t+1) \\
&\Longrightarrow 1 - \alpha > F_Y(n\theta_0, t+1, 1), \text{ where } Y \sim \text{Gamma}(t+1, 1) \\
&\Longrightarrow \theta_0 < \frac{1}{n} F_Y^{-1}(1-\alpha, t+1, 1) \\
&\Longrightarrow \theta_0 < \frac{1}{2n} \chi^2_{1-\alpha, 2(t+1)}.
\end{aligned}$$

Caution, $\chi^2_{1-\alpha,\nu}$ is the $100(1-\alpha)$ percentile of the χ^2 distribution with ν degrees of freedom.

Two-sided $100(1-\alpha)\%$ intervals can be obtained by computing upper and lower $100(1-\alpha/2)\%$ one-sided intervals.

1.2 Invert LR Test to Obtain Approximate Two-Sided Intervals

The large sample LR test rejects $\text{H}_0\colon \theta = \theta_0$ in favor of $\text{H}_0\colon \theta \ne \theta_0$ if

$$-2\ln(\lambda) \ge \chi^2_{1-\alpha,1}, \text{ where } \lambda = e^{n(\bar{x}-\theta_0)} \left(\frac{\theta_0}{\bar{x}}\right)^{n\bar{x}}.$$

Accordingly, θ is in the $100(1-\alpha)\%$ confidence interval if

$$-2n(\bar{x} - \theta) - 2\bar{x} \ln\left(\frac{\theta_0}{\bar{x}}\right) - \chi^2_{1-\alpha,1} \le 0.$$

The upper and lower confidence limits for fixed $\bar{x} = t/n$ are obtained by finding the two values of θ that satisfy

$$-2n(\bar{x} - \theta) - 2\bar{x} \ln\left(\frac{\theta}{\bar{x}}\right) - \chi^2_{1-\alpha,1} = 0.$$

1.3 Invert Score Test to Obtain Approximate One- or Two-Sided Intervals

The score function is

$$S(\theta, T) = \frac{n(\overline{X} - \theta)}{\theta}.$$

The mean and variance are

$$\mathrm{E}(S) = 0 \text{ and } \mathrm{Var}(S) = \frac{n^2\theta/n}{\theta^2} = \frac{n}{\theta}.$$

It follows from the CLT that

$$Z \stackrel{\text{def}}{=} \frac{S - \mathrm{E}(S)}{\sqrt{\mathrm{Var}(S)}} = \frac{\sqrt{n}(\overline{X} - \theta)}{\sqrt{\theta}} \xrightarrow{\text{dist}} \mathrm{N}(0,1).$$

Accordingly, $\mathrm{H}_0\colon \theta \le \theta_0$ is <u>not</u> rejected in favor of $\mathrm{H}_0\colon \theta > \theta_0$ by the score test if

$$\frac{\sqrt{n}(\overline{x} - \theta_0)}{\sqrt{\theta_0}} < z_{1-\alpha},$$

where $z_{1-\alpha}$ is the $100(1-\alpha)$ percentile of the standard normal distribution. Furthermore,

$$\frac{\sqrt{n}(\overline{x} - \theta_0)}{\sqrt{\theta_0}} < z_{1-\alpha} \Longrightarrow \frac{n(\overline{x} - \theta_0)^2}{\theta_0} < z^2_{1-\alpha}$$

$$\Longrightarrow \bar{x} + \frac{z^2_{1-\alpha}}{2n} - \frac{z_{1-\alpha}}{\sqrt{n}}\sqrt{\bar{x} + z^2_{1-\alpha}/(4n)} < \theta_0.$$

Similarly, the upper endpoint for a one-sided interval is

$$\theta_0 < \bar{x} + \frac{z^2_{1-\alpha}}{2n} + \frac{z_{1-\alpha}}{\sqrt{n}}\sqrt{\bar{x} + z^2_{1-\alpha}/(4n)} = \bar{x} + \frac{z^2_{\alpha}}{2n} - \frac{z_{\alpha}}{\sqrt{n}}\sqrt{\bar{x} + z^2_{\alpha}/(4n)}$$

and the endpoints for a two-sided $100(1-\alpha)\%$ large sample interval are

$$\bar{x} + \frac{z^2_{1-\alpha/2}}{2n} \pm \frac{z_{1-\alpha/2}}{\sqrt{n}}\sqrt{\bar{x} + z^2_{1-\alpha/2}/(4n)}.$$

2 Binomial Confidence Intervals

Suppose that $x_i \stackrel{\text{iid}}{\sim} \mathrm{Bern}(\theta)$ for $i = 1, \ldots, n$. Define T and $\overline{X}$ as

$$T \stackrel{\text{def}}{=} \sum_{i=1}^{n} X_i \text{ and } \overline{X} = \frac{1}{n}T.$$

Furthermore, denote a realization of T by t.

2.1 Invert LR Test to Obtain Exact One-Sided Intervals

2.1.1 Lower Confidence Limits

The LR test rejects $\mathrm{H}_0\colon \theta \le \theta_0$ in favor of $\mathrm{H}_0\colon \theta > \theta_0$ if $T \ge c$, where c is the smallest value that satisfies $P_{\theta_0}(T \ge c) \le \alpha$. The p-value is $P_{\theta_0}(T \ge t)$. Also H_0 is not rejected if p-value $> \alpha$. Accordingly,

$$
\begin{aligned}
\mathrm{H}_0 \text{ accepted } &\implies P_{\theta_0}(T \geq t) > \alpha \\
&\implies 1 - P_{\theta_0}(T \leq t-1) > \alpha \\
&\implies P_{\theta_0}(T \leq t-1) < 1 - \alpha \\
&\implies F_Y(1-\theta_0, n-t+1, t) < 1-\alpha, \text{ where } Y \sim \mathrm{Beta}(n-t+1, t) \\
&\implies 1 - \theta_0 < F_Y^{-1}(1-\alpha, n-t+1, t) \\
&\implies 1 - F_Y^{-1}(1-\alpha, n-t+1, t) < \theta_0 \\
&\implies \left(1 + \frac{n-t+1}{t\, F_{\alpha, 2t, 2(n-t+1)}}\right)^{-1} < \theta_0,
\end{aligned}
$$

where $F_{\alpha,2t,2(n-t+1)}$ is the $100(1-\alpha)$ percentile of the F distribution with $2t$ and $2(n-t+1)$ degrees of freedom.

2.1.2 Upper Confidence Limits

The LR test rejects $\mathrm{H}_0\colon \theta \geq \theta_0$ in favor of $\mathrm{H}_0\colon \theta < \theta_0$ if $T \leq c$, where c is the largest value that satisfies $P_{\theta_0}(T \leq c) \leq \alpha$. The p-value is $P_{\theta_0}(T \leq t)$. Also, H_0 is not rejected if p-value $> \alpha$. Accordingly,

$$
\begin{aligned}
\mathrm{H}_0 \text{ accepted } &\implies P_{\theta_0}(T \leq t) > \alpha \\
&\implies F_Y(1-\theta_0, n-t, t+1) > \alpha, \text{ where } Y \sim \mathrm{Beta}(n-t, t+1) \\
&\implies 1 - \theta_0 > F_Y^{-1}(\alpha, n-t, t+1) \\
&\implies \theta_0 < 1 - F_Y^{-1}(\alpha, n-t, t+1) \\
&\implies \theta_0 < \left(1 + \frac{n-t}{(t+1)F_{1-\alpha, 2(t+1), 2(n-t)}}\right)^{-1},
\end{aligned}
$$

where $F_{1-\alpha,2t,2(n-t)}$ is the 100α percentile of the F distribution with $2t$ and $2(n-t)$ degrees of freedom.

Two-sided $100(1-\alpha)\%$ intervals can be obtained by computing upper and lower $100(1-\alpha/2)\%$ one-sided intervals.

2.2 Invert LR Test to Obtain Approximate Two-Sided Intervals

The large sample LR test rejects $\mathrm{H}_0\colon \theta = \theta_0$ in favor of $\mathrm{H}_0\colon \theta \neq \theta_0$ if

$$-2\ln(\lambda) \geq \chi^2_{1-\alpha,1}, \text{ where } \lambda = \left(\frac{n\theta_0}{t}\right)^t \left(\frac{n(1-\theta)}{n-t}\right)^{n-t}.$$

Accordingly, θ is in the $100(1-\alpha)\%$ confidence interval if

$$-2t\ln\left(\frac{n\theta}{t}\right) - 2(n-t)\ln\left(\frac{n(1-\theta)}{n-t}\right) \leq \chi^2_{1-\alpha,1}.$$

The upper and lower confidence limits for fixed t are obtained by finding the two values of θ that satisfy

$$-2t\ln\left(\frac{n\theta}{t}\right) - 2(n-t)\ln\left(\frac{n(1-\theta)}{n-t}\right) - \chi^2_{1-\alpha,1} = 0.$$

2.3 Invert Score Test to Obtain Approximate One- or Two-Sided Intervals

The score function is

$$S(\theta, T) = \frac{n(\overline{X} - \theta)}{\theta(1-\theta)}.$$

The mean and variance are

$$\mathrm{E}(S) = 0 \text{ and } \mathrm{Var}(S) = \frac{n^2\theta(1-\theta)/n}{\theta^2(1-\theta)^2} = \frac{n}{\theta(1-\theta)}.$$

It follows from the CLT that

$$Z \stackrel{\text{def}}{=} \frac{S - \mathrm{E}(S)}{\sqrt{\mathrm{Var}(S)}} = \frac{\sqrt{n}(\overline{X} - \theta)}{\sqrt{\theta(1-\theta)}} \xrightarrow{\text{dist}} \mathrm{N}(0,1).$$

Accordingly, $\mathrm{H}_0\colon \theta \leq \theta_0$ is <u>not</u> rejected in favor of $\mathrm{H}_0\colon \theta > \theta_0$ by the score test if

$$\frac{\sqrt{n}(\overline{x} - \theta_0)}{\sqrt{\theta_0(1-\theta_0)}} < z_{1-\alpha},$$

where $z_{1-\alpha}$ is the $100(1-\alpha)$ percentile of the standard normal distribution. Furthermore,

$$\frac{\sqrt{n}(\overline{x} - \theta_0)}{\sqrt{\theta_0(1-\theta_0)}} < z_{1-\alpha} \Longrightarrow \frac{n(\overline{x} - \theta_0)^2}{\theta_0(1-\theta_0)} < z^2_{1-\alpha}$$

$$\Longrightarrow \frac{2n\bar{x} + z^2_{1-\alpha}}{2(n + z^2_{1-\alpha})} + \left(\frac{z_{1-\alpha}}{2(n + z^2_{1-\alpha})}\right)\sqrt{z^2_{1-\alpha} + 4n\bar{x}(1-\bar{x})} < \theta_0.$$

Similarly, the upper endpoint for a one-sided interval is

$$\theta_0 < \frac{2n\bar{x} + z_\alpha^2}{2(n + z_\alpha^2)} + \left(\frac{z_\alpha}{2(n + z_\alpha^2)}\right)\sqrt{z_\alpha^2 + 4n\bar{x}(1 - \bar{x})},$$

and the endpoints for a two-sided $100(1-\alpha)\%$ large sample interval are

$$\frac{2n\bar{x} + z_{1-\alpha/2}^2}{2(n + z_{1-\alpha/2}^2)} \pm \left(\frac{z_{1-\alpha/2}}{2(n + z_{1-\alpha/2}^2)}\right)\sqrt{z_{1-\alpha}^2 + 4n\bar{x}(1 - \bar{x})}.$$

These intervals were known as early as 1927 (Wilson, *JASA*, **22**, 209–212). Note that the mid-point of the two-sided 95% interval is

$$M = \frac{2n\bar{x} + 1.96^2}{2(n + 1.96^2)} \approx \frac{t+2}{n+4},$$

which would be the value of $\bar{x}$ if two additional successes and two additional failures had been observed. This is the basis of the plus four procedure described in Moore & McCabe. The usual large sample two sided interval is

$$\frac{t}{n} \pm \left(\frac{z_{1-\alpha/2}}{\sqrt{n}}\right)\sqrt{\frac{t}{n}\left(1 - \frac{t}{n}\right)}.$$

The plus four interval is an approximation to the Wilson interval and is computed as

$$\frac{t+2}{n+4} \pm \left(\frac{z_{1-\alpha/2}}{\sqrt{n+4}}\right)\sqrt{\frac{t+2}{n+4}\left(1 - \frac{t+2}{n+4}\right)}.$$

3 Coverage of Intervals for the Binomial Parameter

3.1 Exact Two-Sided Intervals

For fixed θ_0, the two one-sided tests reject $\mathrm{H}_0\colon \theta = \theta_0$ if $T \le c_1$ or $T \ge c_2$, where c_1 and c_2 satisfy

$$P_{\theta_0}(T \le c_1) \le \frac{\alpha}{2} \text{ and } P_{\theta_0}(T \ge c_2) \le \frac{\alpha}{2}.$$

The value of c_1 can be found as follows:

$$P_{\theta_0}(T \le c_1) \le \frac{\alpha}{2} \Longrightarrow F_T(c_1; n, \theta_0) \le \frac{\alpha}{2}$$

$$\Longrightarrow c_1 \le F_T^{-1}(\alpha/2; n, \theta_0)$$

$$\Longrightarrow c_1 = F_T^{-1}(\alpha/2; n, \theta_0) - 1,$$

because the inverse cdf is defined as

$$F_T^{-1}(\alpha; n, \theta_0) = \inf y \text{ subject to } F_T(y; n, \theta) \ge \alpha.$$

For all $\theta \in (0,1)$, except for a set with measure zero, it is necessary to subtract one from $F_T^{-1}(\alpha/2; n, \theta)$ to satisfy $F_T(c_1; n, \theta) \le \alpha/2$. Similarly,

$$P_{\theta_0}(T \ge c_2) \le \frac{\alpha}{2} \Longrightarrow 1 - P_{\theta_0}(T \le c_2 - 1) \le \frac{\alpha}{2}$$

$$\Longrightarrow F_T(c_2 - 1; n, \theta_0) \ge 1 - \frac{\alpha}{2}$$

$$\Longrightarrow c_2 - 1 \ge F_T^{-1}(1 - \alpha/2; n, \theta_0)$$

$$\Longrightarrow c_2 = F_T^{-1}(1 - \alpha/2; n, \theta_0) + 1.$$

The coverage of the confidence interval is the probability that $\mathrm{H}_0\colon \theta = \theta_0$ is <u>not</u> rejected. That is

$$\text{Coverage}(\theta_0) = P_{\theta_0}(c_1 < T < c_2)$$

$$= P_{\theta_0}(c_1 < T \le c_2 - 1) = F_T(c_2 - 1; n, \theta_0) - F_T(c_1; n, \theta_0).$$

3.2 Conventional Z Intervals

The conventional large sample interval for θ is

$$\frac{T}{n} - \frac{z_{1-\alpha/2}}{\sqrt{n}} \sqrt{\frac{T}{n}\left(1 - \frac{t}{n}\right)} < \theta < \frac{T}{n} + \frac{z_{1-\alpha/2}}{\sqrt{n}} \sqrt{\frac{T}{n}\left(1 - \frac{t}{n}\right)}.$$

The interval captures θ if

$$\left|\frac{\sqrt{n}(\hat{\theta}-\theta)}{\sqrt{\hat{\theta}(1-\hat{\theta})}}\right| < z_{1-\alpha/2},$$

where $\hat{\theta} = T/n$. Equivalently, the interval captures θ if

$$\frac{n(T-n\theta)^2}{T(n-T)} < z^2_{1-\alpha/2}.$$

Solving for T yields the coverage:

$$\text{Coverage}(\theta) = P_\theta(c_1 < T < c_2) = F_T(c_2 - \epsilon; n, \theta) - F_T(c_1; n, \theta), \text{ where}$$

$$c_1 = \frac{n\theta + \frac{z^2_{1-\alpha/2}}{2} - \frac{z_{1-\alpha/2}}{2}\sqrt{z^2_{1-\alpha/2} + 4n\theta(1-\theta)}}{1 + \frac{z^2_{1-\alpha/2}}{n}},$$

$$c_2 = \frac{n\theta + \frac{z^2_{1-\alpha/2}}{2} + \frac{z_{1-\alpha/2}}{2}\sqrt{z^2_{1-\alpha/2} + 4n\theta(1-\theta)}}{1 + \frac{z^2_{1-\alpha/2}}{n}},$$

and ϵ is a small number that can be set to zero.

3.3 Plus 4 Intervals

These intervals are identical to the conventional intervals except that T is replaced by $T+2$ and n is replaced by $n+4$. The coverage is

$$\text{Coverage}(\theta) = P_\theta(c_1 - 2 < T < c_2 - 2) = F_T(c_2 - 2 - \epsilon; n, \theta) - F_T(c_1 - 2; n, \theta),$$

where

$$c_1 = \frac{(n+4)\theta + \frac{z^2_{1-\alpha/2}}{2} - \frac{z_{1-\alpha/2}}{2}\sqrt{z^2_{1-\alpha/2} + 4(n+4)\theta(1-\theta)}}{1 + \frac{z^2_{1-\alpha/2}}{(n+4)}},$$

$$c_2 = \frac{(n+4)\theta + \frac{z^2_{1-\alpha/2}}{2} + \frac{z_{1-\alpha/2}}{2}\sqrt{z^2_{1-\alpha/2} + 4(n+4)\theta(1-\theta)}}{1 + \frac{z^2_{1-\alpha/2}}{(n+4)}},$$

and ϵ is a small number that can be set to zero.

3.4 Score Intervals

The score test rejects $\mathrm{H}_0 \colon \theta = \theta_0$ if

$$\left| \frac{\sqrt{n}\,(T/n - \theta_0)}{\sqrt{\theta_0(1-\theta_0)}} \right| < z_{1-\alpha/2}$$

or, equivalently, if

$$\frac{(T - n\theta_0)^2}{n\theta_0(1-\theta_0)} < z^2_{1-\alpha/2}.$$

Solving for T yields the coverage probability:

$$\text{Coverage}(\theta) = P_\theta(c_1 < T < c_2) = F_T(c_2 - \epsilon; n, \theta) - F_T(c_1; n, \theta), \text{ where}$$

$$c_1 = n\theta - z_{1-\alpha/2}\sqrt{n\theta(1-\theta)},$$

$$c_2 = n\theta + z_{1-\alpha/2}\sqrt{n\theta(1-\theta)},$$

and ϵ is a small number that can be set to zero.

3.5 Large Sample LRT Intervals

The LR test fails to reject $\mathrm{H}_0 \colon \theta = \theta_0$ if

$$-2T \ln\left(\frac{n\theta_0}{T}\right) - 2(n-T)\ln\left(\frac{n(1-\theta_0)}{n-T}\right) < \chi^2_{1-\alpha,1}.$$

The coverage at θ_0 is the probability that H_0 is not rejected. This probability is

$$\text{Coverage}(\theta) = P_\theta(c_1 < T < c_2) = F_T(c_2 - \epsilon; n, \theta) - F_T(c_1; n, \theta),$$

where c_1 and c_2 satisfy $c_1 < c_2$ and

$$-2c_i \ln\left(\frac{n\theta}{c_i}\right) - 2(n-c_i)\ln\left(\frac{n(1-\theta)}{n-c_i}\right) - \chi^2_{1-\alpha,1} = 0$$

for $i = 1, 2$.