SUPPLEMENT TO AN IMPLICIT FUNCTION APPROACH TO CONSTRAINED OPTIMIZATION WITH APPLICATIONS TO ASYMPTOTIC EXPANSIONS

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October 31, 2008

Contents

1 Introduction

This supplement gives derivatives and other details concerning the Edgeworth expansion of estimators in the exploratory factor analysis model.

2 Derivatives of the Quartic Rotation Criterion in Factor Analysis

The first four derivatives of the quartic criteria in (25) are the following:

$$\begin{aligned}
\mathbf{D}^{(1)}_{Q;\boldsymbol{\gamma}_\lambda} &= 4\left(\boldsymbol{\gamma}'_\lambda \otimes \mathbf{I}_{mp}\right)\mathbf{N}_{mp}\mathbf{W}\left(\boldsymbol{\gamma}_\lambda \otimes \boldsymbol{\gamma}_\lambda\right), \\
\mathbf{D}^{(2)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda} &= 4\,\mathrm{dvec}\left[\mathbf{N}_{mp}\mathbf{W}\left(\boldsymbol{\gamma}_\lambda \otimes \boldsymbol{\gamma}_\lambda\right), mp, mp\right] \\
&\quad +8\left(\boldsymbol{\gamma}'_\lambda \otimes \mathbf{I}_{mp}\right)\mathbf{W}^*\left(\boldsymbol{\gamma}_\lambda \otimes \mathbf{I}_{mp}\right), \\
\mathbf{D}^{(3)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda,\boldsymbol{\gamma}'_\lambda} &= 16\,\mathrm{dvec}\left\{\left(\mathbf{N}_{mp}\otimes\mathbf{I}_{mp}\right)\mathrm{vec}\left[\mathbf{W}^*\left(\boldsymbol{\gamma}_\lambda \otimes \mathbf{I}_{mp}\right)\right], mp, m^2p^2\right\} \\
&\quad +8\left(\boldsymbol{\gamma}'_\lambda \otimes \mathbf{I}_{mp}\right)\mathbf{W}^*,\ \text{and} \\
\mathbf{D}^{(4)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda,\boldsymbol{\gamma}'_\lambda,\boldsymbol{\gamma}'_\lambda} &= 8\,\mathrm{dvec}\Big\{\left[\left(\mathbf{I}_{mp}\otimes 2\mathbf{N}_{mp}\otimes\mathbf{I}_{mp}\right) + \left(\mathbf{I}_{(m^2p^2,mp)}\otimes\mathbf{I}_{mp}\right)\right] \\
&\quad \times \mathrm{vec}\left(\mathbf{W}^*\right), mp, m^3p^3\Big\},
\end{aligned}$$

where $\mathbf{W}^* = \mathbf{N}_{mp}\mathbf{W}\mathbf{N}_{mp}$ and $\mathbf{N}_{mp}$ is defined in Corollary 1.1. All higher-order derivatives are zero.

3 Derivatives of the Constraint Function in Factor Analysis

The covariance structure, from (28), is

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\theta}) = (\boldsymbol{\Sigma})^{\frac{1}{2}}_D\boldsymbol{\Delta}(\boldsymbol{\Sigma})^{\frac{1}{2}}_D,\ \text{where}$$

$$\boldsymbol{\Delta} = \boldsymbol{\Xi}\boldsymbol{\Phi}\boldsymbol{\Xi}' + \boldsymbol{\Psi},\quad \boldsymbol{\Xi} = \boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda,\quad \boldsymbol{\Psi} = \mathbf{I}_p - \boldsymbol{\Lambda}^2,\quad \boldsymbol{\theta} = \begin{pmatrix}\boldsymbol{\gamma}'_\lambda & \boldsymbol{\phi}' & \boldsymbol{\lambda}' & \boldsymbol{\sigma}_d{}'\end{pmatrix}',$$

$\boldsymbol{\gamma}_\lambda = \mathrm{vec}\,\boldsymbol{\Gamma}_\lambda$, $\boldsymbol{\lambda}$ is the p-vector that contains the diagonal components of $\boldsymbol{\Lambda}$, $\boldsymbol{\phi}$ contains the distinct correlation coefficients in $\boldsymbol{\Phi}$, $\boldsymbol{\sigma}_d$ contains the p diagonal components of $(\boldsymbol{\Sigma})^{\frac{1}{2}}_D$, and $\mathbf{1}_p$ is a p-vector of ones. The vectors $\boldsymbol{\lambda}$, $\boldsymbol{\phi}$, and $\boldsymbol{\sigma}_d$ are related to the matrices $\boldsymbol{\Lambda}$, $\boldsymbol{\Phi}$, and $(\boldsymbol{\Sigma})^{\frac{1}{2}}_D$ as follows:

$$\mathbf{L}'_{p,21}\,\mathrm{vec}\,\boldsymbol{\Lambda} = \boldsymbol{\lambda},\quad \mathrm{vec}\,\boldsymbol{\Lambda} = \mathbf{L}_{p,21}\boldsymbol{\lambda},$$

$$\frac{1}{2}\mathbf{L}'_m\,\mathrm{vec}\,\boldsymbol{\Phi} = \boldsymbol{\phi},\quad \mathrm{vec}\,\boldsymbol{\Phi} = \mathbf{L}_m\boldsymbol{\phi} + \mathrm{vec}\,\mathbf{I}_m,$$

$$\mathbf{L}'_{p,21}\,\mathrm{vec}\,(\boldsymbol{\Sigma})^{\frac{1}{2}}_D = \boldsymbol{\sigma}_d,\ \text{and}\ \mathrm{vec}\,(\boldsymbol{\Sigma})^{\frac{1}{2}}_D = \mathbf{L}_{p,21}\boldsymbol{\sigma}_d,\ \text{where}$$

$$\mathbf{L}_m \overset{\text{def}}{=} \sum_{i=1}^{m}\sum_{j=i+1}^{m} 2\mathbf{N}_m\left(\mathbf{e}_i^m \otimes \mathbf{e}_j^m\right)\mathbf{e}_h^{d\prime}, \quad h = \frac{(2m-i)(i-1)}{2} + j - i, \quad d = \frac{m(m-1)}{2}$$

and $\mathbf{L}_{p,21}$ is defined in (3).

Partition $\boldsymbol{\theta}$ in

$$\boldsymbol{\theta} = \begin{pmatrix}\boldsymbol{\theta}_1\\ \boldsymbol{\theta}_2\end{pmatrix}, \text{ where } \boldsymbol{\theta}_1 = \begin{pmatrix}\boldsymbol{\gamma}_\lambda\\ \boldsymbol{\phi}\end{pmatrix} \text{ and } \boldsymbol{\theta}_2 = \begin{pmatrix}\boldsymbol{\lambda}\\ \boldsymbol{\sigma}_d\end{pmatrix}.$$

The dimensions associated with $\boldsymbol{\theta}$, $\boldsymbol{\theta}_1$, and $\boldsymbol{\theta}_2$ can be arranged as follows:

$$\dot{\mathbf{k}} = \begin{pmatrix} mp\\ \frac{1}{2}m(m-1)\\ p\\ p\end{pmatrix}, \quad k = mp + \frac{1}{2}m(m-1) + 2p, \quad \dot{\mathbf{k}}_1 = \begin{pmatrix} mp\\ \frac{1}{2}m(m-1)\end{pmatrix},$$

$$k_1 = mp + \frac{1}{2}m(m-1), \quad \dot{\mathbf{k}}_2 = \begin{pmatrix} p\\ p\end{pmatrix}, \text{ and } k_2 = 2p.$$

The constraint function in (28) depends solely on $\boldsymbol{\theta}_1$ and is

$$\mathbf{g}(\boldsymbol{\theta}_1) = \mathbf{0}, \text{ where}$$

$$\mathbf{g}(\boldsymbol{\theta}_1) = \begin{pmatrix}\mathbf{g}_1(\boldsymbol{\theta}_1)\\ \mathbf{g}_2(\boldsymbol{\theta}_1)\end{pmatrix},$$

$$\mathbf{g}_1(\boldsymbol{\theta}_1) = \mathbf{A}\left[\mathbf{I}_{m^2} - (\mathbf{I}_m \otimes \boldsymbol{\Phi})\,\mathbf{L}_{m,22}\right]\left(\boldsymbol{\Gamma}_\lambda' \otimes \mathbf{I}_m\right)\mathbf{I}_{(m,p)}\mathbf{D}^{(1)}_{Q;\boldsymbol{\gamma}_\lambda},$$

$$\mathbf{g}_2(\boldsymbol{\theta}_1) = \mathbf{L}_{p,21}'\operatorname{vec}\left(\boldsymbol{\Gamma}_\lambda\boldsymbol{\Phi}\boldsymbol{\Gamma}_\lambda'\right) - \mathbf{1}_p,$$

$\mathbf{A}$ is defined in (27), and $\mathbf{L}_{p,21}$ is defined in (3). The dimension of $\mathbf{g}$ is $q \times 1$, where $q = m(m-1) + p$. The dimension of the i^{th} component of $\mathbf{g}$ is $q_i \times 1$, where

$$\dot{\mathbf{q}} \overset{\text{def}}{=} \begin{pmatrix} q_1\\ q_2\end{pmatrix} = \begin{pmatrix} m(m-1)\\ p\end{pmatrix}.$$

Denote the identified parameters as $\boldsymbol{\tau}$, where

$$\boldsymbol{\tau} = \begin{pmatrix}\boldsymbol{\tau}_1\\ \boldsymbol{\tau}_2\end{pmatrix}, \quad \boldsymbol{\tau}_1 = \mathbf{G}'\boldsymbol{\theta}_1, \quad \nu_1 = k_1 - q, \quad \boldsymbol{\tau}_2 = \boldsymbol{\theta}_2, \quad \nu_2 = k_2, \quad \dot{\boldsymbol{\nu}}_2 = \dot{\mathbf{k}}_2,$$

and $\mathbf{G}$ is any $k \times \nu_1$ semiorthogonal matrix that satisfies $\mathcal{R}(\mathbf{G}) = \mathcal{N}\left(\mathbf{D}^{(1)}_{\mathbf{g};\boldsymbol{\theta}_1'}\right)$. The derivatives of $\mathbf{g}$ with respect to $\boldsymbol{\theta}_1$ can be written as

$$\mathbf{D}^{(1)}_{\mathbf{g};\boldsymbol{\theta}_1'} = \sum_{i=1}^{2}\sum_{j_1=1}^{2}\mathbf{E}_{i,\dot{\mathbf{q}}}\mathbf{D}^{(1)}_{\mathbf{g}_i;\boldsymbol{\theta}_{1,j_1}'}\mathbf{E}_{j_1,\dot{\mathbf{k}}_1}',$$

$$\mathbf{D}^{(2)}_{\mathbf{g};\boldsymbol{\theta}_1',\boldsymbol{\theta}_1'} = \sum_{i=1}^{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\mathbf{E}_{i,\dot{\mathbf{q}}}\mathbf{D}^{(2)}_{\mathbf{g}_i;\boldsymbol{\theta}_{1,j_1}',\boldsymbol{\theta}_{1,j_2}'}\left(\mathbf{E}_{j_1,\dot{\mathbf{k}}_1} \otimes \mathbf{E}_{j_2,\dot{\mathbf{k}}_1}\right)',$$

$$\mathbf{D}^{(3)}_{\mathbf{g};\boldsymbol{\theta}_1',\boldsymbol{\theta}_1',\boldsymbol{\theta}_1'} = \sum_{i=1}^{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\mathbf{E}_{i,\dot{\mathbf{q}}}\mathbf{D}^{(3)}_{\mathbf{g}_i;\boldsymbol{\theta}_{1,j_1}',\boldsymbol{\theta}_{1,j_2}',\boldsymbol{\theta}_{1,j_3}'}\left(\mathbf{E}_{j_1,\dot{\mathbf{k}}_1} \otimes \mathbf{E}_{j_2,\dot{\mathbf{k}}_1} \otimes \mathbf{E}_{j_3,\dot{\mathbf{k}}_1}\right)', \text{ and}$$

$$\begin{aligned}\mathbf{D}^{(4)}_{\mathbf{g};\boldsymbol{\theta}'_1,\boldsymbol{\theta}'_1,\boldsymbol{\theta}'_1,\boldsymbol{\theta}'_1} = \sum_{i=1}^{2}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{j_4=1}^{2}\mathbf{E}_{i,\dot{\mathbf{q}}}\mathbf{D}^{(4)}_{\mathbf{g}_i;\boldsymbol{\theta}'_{1,j_1},\boldsymbol{\theta}'_{1,j_2},\boldsymbol{\theta}'_{1,j_3},\boldsymbol{\theta}'_{1,j_4}} \\ \times\left(\mathbf{E}_{j_1,\dot{\mathbf{k}}_1}\otimes\mathbf{E}_{j_2,\dot{\mathbf{k}}_1}\otimes\mathbf{E}_{j_3,\dot{\mathbf{k}}_1}\otimes\mathbf{E}_{j_4,\dot{\mathbf{k}}_1}\right)'.\end{aligned}$$

The nonzero components of the above derivatives are the following:

$$\begin{aligned}
\mathbf{D}^{(1)}_{\mathbf{g}_1;\boldsymbol{\theta}'_{1,1}} &= \mathbf{A}\left[\mathbf{I}_{m^2}-\left(\mathbf{I}_m\otimes\boldsymbol{\Phi}\right)\mathbf{L}_{m,22}\right]\mathbf{I}'_{mp,3}\left(\mathbf{I}_{mp}\otimes\mathbf{I}_{(m,p)}\mathbf{D}^{(1)}_{Q;\boldsymbol{\gamma}_\lambda}\right) \\
&\quad +\mathbf{A}\left[\mathbf{I}_{m^2}-\left(\mathbf{I}_m\otimes\boldsymbol{\Phi}\right)\mathbf{L}_{m,22}\right]\mathbf{I}_{(m,m)}\left(\mathbf{I}_m\otimes\boldsymbol{\Gamma}'_\lambda\right)\mathbf{D}^{(2)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda}, \\
\mathbf{I}_{mp,3} &= \left(\mathbf{I}_m\otimes\operatorname{vec}\mathbf{I}_p\otimes\mathbf{I}_m\right), \\
\mathbf{D}^{(1)}_{\mathbf{g}_1;\boldsymbol{\theta}'_{1,2}} &= -\mathbf{A}\mathbf{I}_{(m,m)}\mathbf{I}'_{mm,3}\left[\mathbf{L}_m\otimes\mathbf{L}_{m,22}\left(\mathbf{I}_m\otimes\boldsymbol{\Gamma}'_\lambda\right)\mathbf{D}^{(1)}_{Q;\boldsymbol{\gamma}_\lambda}\right], \\
\mathbf{I}_{mm,3} &= \left(\mathbf{I}_m\otimes\operatorname{vec}\mathbf{I}_m\otimes\mathbf{I}_m\right), \\
\mathbf{D}^{(1)}_{\mathbf{g}_2;\boldsymbol{\theta}'_{1,1}} &= 2\mathbf{L}'_{p,21}\left(\boldsymbol{\Gamma}_\lambda\boldsymbol{\Phi}\otimes\mathbf{I}_p\right), \\
\mathbf{D}^{(1)}_{\mathbf{g}_2;\boldsymbol{\theta}'_{1,2}} &= \mathbf{L}'_{p,21}\left(\boldsymbol{\Gamma}_\lambda\otimes\boldsymbol{\Gamma}_\lambda\right)\mathbf{L}_m, \\
\mathbf{D}^{(2)}_{\mathbf{g}_1;\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,1}} &= \mathbf{A}\left[\mathbf{I}_{m^2}-\left(\mathbf{I}_m\otimes\boldsymbol{\Phi}\right)\mathbf{L}_{m,22}\right]\left\{\mathbf{I}'_{mp,3}\left(\mathbf{I}_{mp}\otimes\mathbf{I}_{(m,p)}\mathbf{D}^{(2)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda}\right)2\mathbf{N}_{mp}\right. \\
&\quad \left.+\mathbf{I}_{(m,m)}\left(\mathbf{I}_m\otimes\boldsymbol{\Gamma}'_\lambda\right)\mathbf{D}^{(3)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda,\boldsymbol{\gamma}'_\lambda}\right\} \\
\mathbf{D}^{(2)}_{\mathbf{g}_1;\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,2}} &= -\mathbf{A}\mathbf{I}'_{mm,3}\left\{\left[\mathbf{L}_{m,22}\left(\mathbf{I}_m\otimes\boldsymbol{\Gamma}'_\lambda\right)\mathbf{D}^{(2)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda}\otimes\mathbf{L}_m\right]\right. \\
&\quad \left.+\left[\mathbf{L}_{m,22}\mathbf{I}'_{mp,3}\left(\mathbf{I}_{mp}\otimes\mathbf{I}_{(m,p)}\mathbf{D}^{(1)}_{Q;\boldsymbol{\gamma}_\lambda}\right)\otimes\mathbf{L}_m\right]\right\} \\
\mathbf{D}^{(2)}_{\mathbf{g}_2;\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,1}} &= 2\mathbf{L}'_{p,21}\left(\mathbf{I}_p\otimes\operatorname{vec}\boldsymbol{\Phi}\otimes\mathbf{I}_p\right)'\left(\mathbf{I}_{(m,p)}\otimes\mathbf{I}_{mp}\right), \\
\mathbf{D}^{(2)}_{\mathbf{g}_2;\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,2}} &= 2\mathbf{L}'_{p,21}\left(\mathbf{I}_p\otimes\operatorname{vec}\mathbf{I}_m\otimes\boldsymbol{\Gamma}'_\lambda\right)'\left(\mathbf{I}_{(m,p)}\otimes\mathbf{L}_m\right), \\
\mathbf{D}^{(3)}_{\mathbf{g}_1;\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,1}} &= \mathbf{A}\left[\mathbf{I}_{m^2}-\left(\mathbf{I}_m\otimes\boldsymbol{\Phi}\right)\mathbf{L}_{m,22}\right]\left\{\mathbf{I}'_{mp,3}\left(\mathbf{I}_{mp}\otimes\mathbf{I}_{(m,p)}\mathbf{D}^{(3)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda,\boldsymbol{\gamma}'_\lambda}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&\times \left[(2\mathbf{N}_{mp} \otimes \mathbf{I}_{mp}) + \mathbf{I}_{(m^2p^2,mp)}\right] + \mathbf{I}_{(m,m)} \left(\mathbf{I}_m \otimes \boldsymbol{\Gamma}'_\lambda\right) \mathbf{D}^{(4)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda,\boldsymbol{\gamma}'_\lambda,\boldsymbol{\gamma}'_\lambda} \Bigg\},\\[1ex]
\mathbf{D}^{(3)}_{\mathbf{g}_1;\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,2}} &= -\mathbf{A}\mathbf{I}'_{mm,3}\Bigg\{\Big[\mathbf{L}_{m,22}\mathbf{I}'_{mp,3}\left(\mathbf{I}_{mp} \otimes \mathbf{I}_{(m,p)}\mathbf{D}^{(2)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda}\right) 2\mathbf{N}_{mp} \otimes \mathbf{L}_m\Big]\\
&\quad + \Big[\mathbf{L}_{m,22}\left(\mathbf{I}_m \otimes \boldsymbol{\Gamma}'_\lambda\right) \mathbf{D}^{(3)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda,\boldsymbol{\gamma}'_\lambda} \otimes \mathbf{L}_m\Big]\Bigg\},\\[1ex]
\mathbf{D}^{(3)}_{\mathbf{g}_2;\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,2},\boldsymbol{\theta}'_{1,1}} &= 2\mathbf{L}'_{p,21}\left(\mathbf{I}_p \otimes \operatorname{vec}\mathbf{I}_m \otimes \operatorname{vec}\mathbf{I}_m \otimes \mathbf{I}_p\right)'\left(\mathbf{I}_{(m,p)} \otimes \mathbf{L}_m \otimes \mathbf{I}_{mp}\right),\\[1ex]
\mathbf{D}^{(4)}_{\mathbf{g}_1;\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,1}} &= \mathbf{A}\left[\mathbf{I}_{m^2} - \left(\mathbf{I}_m \otimes \boldsymbol{\Phi}\right)\mathbf{L}_{m,22}\right]\mathbf{I}'_{mp,3}\left(\mathbf{I}_{mp} \otimes \mathbf{I}_{(m,p)}\mathbf{D}^{(4)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda,\boldsymbol{\gamma}'_\lambda,\boldsymbol{\gamma}'_\lambda}\right)\Big\{\mathbf{I}_{m^4p^4}\\
&\quad + \mathbf{I}_{(m^3p^3,mp)}\left[\mathbf{I}_{m^4p^4} + \left(\mathbf{I}_{mp} \otimes \mathbf{I}_{(mp,m^2p^2)}\right)\left(\mathbf{I}_{mp} \otimes 2\mathbf{N}_{mp} \otimes \mathbf{I}_{mp}\right)\right]\Big\}, \text{ and}\\[1ex]
\mathbf{D}^{(4)}_{\mathbf{g}_1;\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,1},\boldsymbol{\theta}'_{1,2}} &= -\mathbf{A}\mathbf{I}'_{mm,3}\Bigg\{\Big[\mathbf{L}_{m,22}\mathbf{I}'_{mp,3}\left(\mathbf{I}_{mp} \otimes \mathbf{I}_{(m,p)}\mathbf{D}^{(3)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda,\boldsymbol{\gamma}'_\lambda}\right)\\
&\quad \times \left(\mathbf{I}_{(m^2p^2,mp)}\left(\mathbf{I}_{mp} \otimes 2\mathbf{N}_{mp}\right) + \mathbf{I}_{m^3p^3}\right) \otimes \mathbf{L}_m\Big]\\
&\quad + \Big[\mathbf{L}_{m,22}\left(\mathbf{I}_m \otimes \boldsymbol{\Gamma}'_\lambda\right) \mathbf{D}^{(4)}_{Q;\boldsymbol{\gamma}_\lambda,\boldsymbol{\gamma}'_\lambda,\boldsymbol{\gamma}'_\lambda,\boldsymbol{\gamma}'_\lambda} \otimes \mathbf{L}_m\Big]\Bigg\}.
\end{aligned}
$$

4 Derivatives of $\boldsymbol{\Sigma}$ in Factor Analysis

Recall that $\boldsymbol{\Sigma}$ is parameterized as $\boldsymbol{\Sigma}(\boldsymbol{\theta})$, where

$$
\boldsymbol{\theta} = \begin{pmatrix} \boldsymbol{\theta}_{1*} \\ \boldsymbol{\theta}_{2*} \\ \boldsymbol{\theta}_{3*} \\ \boldsymbol{\theta}_{4*} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\gamma}_\lambda \\ \boldsymbol{\phi} \\ \boldsymbol{\lambda} \\ \boldsymbol{\sigma}_d \end{pmatrix} \text{ with dimensions } \dot{\mathbf{k}} = \begin{pmatrix} mp \\ \frac{1}{2}m(m-1) \\ p \\ p \end{pmatrix},
$$

where the $*$ subscript is employed to distinguish between

$$
\boldsymbol{\theta}_1 = \begin{pmatrix} \boldsymbol{\gamma}_\lambda \\ \boldsymbol{\phi} \end{pmatrix} \text{ and } \boldsymbol{\theta}_{1*} = \boldsymbol{\gamma}_\lambda \text{ and between } \boldsymbol{\theta}_2 = \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\sigma}_d \end{pmatrix} \text{ and } \boldsymbol{\theta}_{2*} = \boldsymbol{\phi}.
$$

The first three derivatives of $\operatorname{vec}\boldsymbol{\Sigma}$ can be written as

$$
\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'} = \sum_{s=1}^{4} \mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{s*}} \mathbf{E}'_{s,\dot{\mathbf{k}}},
$$

$$
\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}',\boldsymbol{\theta}'} = \sum_{s=1}^{4}\sum_{t=1}^{4} \mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{s*},\boldsymbol{\theta}'_{t*}} \left(\mathbf{E}_{s,\dot{\mathbf{k}}} \otimes \mathbf{E}_{t,\dot{\mathbf{k}}}\right)', \text{ and}
$$

$$\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}',\boldsymbol{\theta}',\boldsymbol{\theta}'} = \sum_{s=1}^{4}\sum_{t=1}^{4}\sum_{u=1}^{4}\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{s*},\boldsymbol{\theta}'_{t*},\boldsymbol{\theta}'_{u*}}\left(\mathbf{E}_{s,\dot{\mathbf{k}}}\otimes\mathbf{E}_{t,\dot{\mathbf{k}}}\otimes\mathbf{E}_{u,\dot{\mathbf{k}}}\right)'.$$

The nonzero components of the derivatives are as follows:

$$\begin{aligned}
\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*}} &= 2\mathbf{N}_p\left((\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda\boldsymbol{\Phi}\otimes(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\right),\\
\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{2*}} &= \left((\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda\otimes(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda\right)\mathbf{L}_m,\\
\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{3*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\left(\boldsymbol{\Gamma}_\lambda\boldsymbol{\Phi}\boldsymbol{\Gamma}'_\lambda-\mathbf{I}_p\right)\otimes(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\right]\mathbf{L}_{p,21},\\
\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{4*}} &= 2\mathbf{N}_p\left((\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Delta}\otimes\mathbf{I}_p\right)\mathbf{L}_{p,21},\\
\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{1*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\otimes\operatorname{vec}'\boldsymbol{\Phi}\otimes(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\right]\left(\mathbf{I}_{(m,p)}\otimes\mathbf{I}_{mp}\right),\\
\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{2*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\otimes\operatorname{vec}'\mathbf{I}_m\otimes(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda\right]\left(\mathbf{I}_{(m,p)}\otimes\mathbf{L}_m\right),\\
\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{3*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\otimes\operatorname{vec}'\mathbf{I}_p\otimes(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\right]\left[2\mathbf{N}_p\left(\boldsymbol{\Gamma}_\lambda\boldsymbol{\Phi}\otimes\mathbf{I}_p\right)\otimes\mathbf{L}_{p,21}\right],\\
\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{4*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\otimes\operatorname{vec}'\mathbf{I}_p\otimes\mathbf{I}_p\right]\left[2\mathbf{N}_p\left(\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda\boldsymbol{\Phi}\otimes\boldsymbol{\Lambda}\right)\otimes\mathbf{L}_{p,21}\right],\\
\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{2*},\boldsymbol{\theta}'_{3*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda\otimes\operatorname{vec}'\boldsymbol{\Gamma}_\lambda\otimes\mathbf{I}_p\right]\left(\mathbf{L}_m\otimes\mathbf{L}_{p,21}\right),\\
\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{2*},\boldsymbol{\theta}'_{4*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda\otimes\operatorname{vec}'(\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda)\otimes\mathbf{I}_p\right]\left(\mathbf{L}_m\otimes\mathbf{L}_{p,21}\right),\\
\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{3*},\boldsymbol{\theta}'_{3*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\otimes\operatorname{vec}'\left(\boldsymbol{\Gamma}_\lambda\boldsymbol{\Phi}\boldsymbol{\Gamma}'_\lambda-\mathbf{I}_p\right)\otimes(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\right]\left(\mathbf{L}_{p,21}\otimes\mathbf{L}_{p,21}\right),\\
\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{3*},\boldsymbol{\theta}'_{4*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\otimes\operatorname{vec}'\mathbf{I}_p\otimes\mathbf{I}_p\right]\left\{2\mathbf{N}_p\left[\boldsymbol{\Lambda}\left(\boldsymbol{\Gamma}_\lambda\boldsymbol{\Phi}\boldsymbol{\Gamma}'_\lambda-\mathbf{I}_p\right)\otimes\mathbf{I}_p\right]\mathbf{L}_{p,21}\otimes\mathbf{L}_{p,21}\right\},\\
\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{4*},\boldsymbol{\theta}'_{4*}} &= 2\mathbf{N}_p\left(\mathbf{I}_p\otimes\operatorname{vec}'\boldsymbol{\Delta}\otimes\mathbf{I}_p\right)\left(\mathbf{L}_{p,21}\otimes\mathbf{L}_{p,21}\right),\\
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{2*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\otimes\operatorname{vec}'\mathbf{I}_m\otimes(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\otimes\operatorname{vec}'\mathbf{I}_m\right]\left[\mathbf{I}_{(m,p)}\otimes\mathbf{I}_{(m^2p,m)}\left(\mathbf{I}_{(m,p)}\otimes\mathbf{L}_m\right)\right],\\
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{3*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\otimes\operatorname{vec}'\mathbf{I}_p\otimes(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\right]\\
&\quad\times\left[2\mathbf{N}_p\left(\mathbf{I}_p\otimes\operatorname{vec}'\boldsymbol{\Phi}\otimes\mathbf{I}_p\right)\left(\mathbf{I}_{(m,p)}\otimes\mathbf{I}_{mp}\right)\otimes\mathbf{L}_{p,21}\right],\\
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{4*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\otimes\operatorname{vec}'\mathbf{I}_p\otimes\mathbf{I}_p\right]\left[2\mathbf{N}_p\left(\boldsymbol{\Lambda}\otimes\operatorname{vec}'\boldsymbol{\Phi}\otimes\boldsymbol{\Lambda}\right)\left(\mathbf{I}_{(m,p)}\otimes\mathbf{I}_{mp}\right)\otimes\mathbf{L}_{p,21}\right],
\end{aligned}$$

$$
\begin{aligned}
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{2*},\boldsymbol{\theta}'_{3*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\boldsymbol{\Lambda}\otimes\operatorname{vec}'\mathbf{I}_p\otimes(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\right]\\
&\quad\times\left[2\mathbf{N}_p\left(\mathbf{I}_p\otimes\operatorname{vec}'\mathbf{I}_m\otimes\boldsymbol{\Gamma}_\lambda\right)\left(\mathbf{I}_{(m,p)}\otimes\mathbf{L}_m\right)\otimes\mathbf{L}_{p,21}\right],\\
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{2*},\boldsymbol{\theta}'_{4*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\otimes\operatorname{vec}'\mathbf{I}_p\otimes\mathbf{I}_p\right]\left[2\mathbf{N}_p\left(\boldsymbol{\Lambda}\otimes\operatorname{vec}'\mathbf{I}_m\otimes\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda\right)\left(\mathbf{I}_{(m,p)}\otimes\mathbf{L}_m\right)\otimes\mathbf{L}_{p,21}\right],\\
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{3*},\boldsymbol{\theta}'_{3*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\otimes\operatorname{vec}'\mathbf{I}_p\otimes\operatorname{vec}'\mathbf{I}_p,(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\right]\left(\mathbf{I}_{(p^2,p^2)}\otimes\mathbf{I}_{p^2}\right)\\
&\quad\times\left[2\mathbf{N}_p\left(\boldsymbol{\Gamma}_\lambda\boldsymbol{\Phi}\otimes\mathbf{I}_p\right)\otimes\mathbf{L}_{p,21}\otimes\mathbf{L}_{p,21}\right],\\
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{3*},\boldsymbol{\theta}'_{4*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\otimes\operatorname{vec}'\mathbf{I}_p\otimes\mathbf{I}_p\right]\Big\{2\mathbf{N}_p\left(\boldsymbol{\Lambda}\otimes\operatorname{vec}'\mathbf{I}_p\otimes\mathbf{I}_p\right)\\
&\quad\times\left[2\mathbf{N}_p\left(\boldsymbol{\Gamma}_\lambda\boldsymbol{\Phi}\otimes\mathbf{I}_p\right)\otimes\mathbf{L}_{p,21}\right]\otimes\mathbf{L}_{p,21}\Big\},\\
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{4*},\boldsymbol{\theta}'_{4*}} &= 2\mathbf{N}_p\left(\mathbf{I}_p\otimes\operatorname{vec}'\mathbf{I}_p\otimes\operatorname{vec}'\mathbf{I}_p\otimes\mathbf{I}_p\right)\left(\mathbf{I}_{(p^2,p^2)}\otimes\mathbf{I}_{p^2}\right)\\
&\quad\times\left[2\mathbf{N}_p\left(\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda\boldsymbol{\Phi}\otimes\boldsymbol{\Lambda}\right)\otimes\mathbf{L}_{p,21}\otimes\mathbf{L}_{p,21}\right],\\
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{2*},\boldsymbol{\theta}'_{3*},\boldsymbol{\theta}'_{3*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\otimes\operatorname{vec}'\mathbf{I}_p\otimes\operatorname{vec}'\mathbf{I}_p\otimes(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\right]\left(\mathbf{I}_{(p^2,p^2)}\otimes\mathbf{I}_{p^2}\right)\\
&\quad\times\left[\left(\boldsymbol{\Gamma}_\lambda\otimes\boldsymbol{\Gamma}_\lambda\right)\mathbf{L}_m\otimes\mathbf{L}_{p,21}\otimes\mathbf{L}_{p,21}\right],\\
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{2*},\boldsymbol{\theta}'_{3*},\boldsymbol{\theta}'_{4*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\otimes\operatorname{vec}'\mathbf{I}_p\otimes\mathbf{I}_p\right]\left[2\mathbf{N}_p\left(\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda\otimes\operatorname{vec}'\boldsymbol{\Gamma}_\lambda\otimes\mathbf{I}_p\right)\left(\mathbf{L}_m\otimes\mathbf{L}_{p,21}\right)\otimes\mathbf{L}_{p,21}\right],\\
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{2*},\boldsymbol{\theta}'_{4*},\boldsymbol{\theta}'_{4*}} &= 2\mathbf{N}_p\left(\mathbf{I}_p\otimes\operatorname{vec}'\mathbf{I}_p\otimes\operatorname{vec}'\mathbf{I}_p\otimes\mathbf{I}_p\right)\\
&\quad\times\left(\mathbf{I}_{(p^2,p^2)}\otimes\mathbf{I}_{p^2}\right)\left[\left(\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda\otimes\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\lambda\right)\mathbf{L}_m\otimes\mathbf{L}_{p,21}\otimes\mathbf{L}_{p,21}\right],\\
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{3*},\boldsymbol{\theta}'_{3*},\boldsymbol{\theta}'_{4*}} &= 2\mathbf{N}_p\left[(\boldsymbol{\Sigma})_D^{\frac{1}{2}}\otimes\operatorname{vec}'\mathbf{I}_p\otimes\mathbf{I}_p\right]\\
&\quad\times\left\{2\mathbf{N}_p\left[\mathbf{I}_p\otimes\operatorname{vec}'\left(\boldsymbol{\Gamma}_\lambda\boldsymbol{\Phi}\boldsymbol{\Gamma}'_\lambda-\mathbf{I}_p\right)\otimes\mathbf{I}_p\right]\left(\mathbf{L}_{p,21}\otimes\mathbf{L}_{p,21}\right)\otimes\mathbf{L}_{p,21}\right\},\ \text{and}\\
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{3*},\boldsymbol{\theta}'_{4*},\boldsymbol{\theta}'_{4*}} &= 2\mathbf{N}_p\left(\mathbf{I}_p\otimes\operatorname{vec}'\mathbf{I}_p\otimes\operatorname{vec}'\mathbf{I}_p\otimes\mathbf{I}_p\right)\left(\mathbf{I}_{(p^2,p^2)}\otimes\mathbf{I}_{p^2}\right)\\
&\quad\times\left\{2\mathbf{N}_p\left[\boldsymbol{\Lambda}\left(\boldsymbol{\Gamma}_\lambda\boldsymbol{\Phi}\boldsymbol{\Gamma}'_\lambda-\mathbf{I}_p\right)\otimes\mathbf{I}_p\right]\mathbf{L}_{p,21}\otimes\mathbf{L}_{p,21}\otimes\mathbf{L}_{p,21}\right\},
\end{aligned}
$$

where $\mathbf{L}_{p,21}$ is defined in (3), and $\mathbf{L}_m$ is defined in §3 of this supplement. Permutations of the above derivatives also are required. For example,

$$
\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{2*},\boldsymbol{\theta}'_{1*}}=\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{2*}}\mathbf{I}_{(k_2,k_1)},\quad \mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{2*},\boldsymbol{\theta}'_{1*}}=\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{2*}}\left(\mathbf{I}_{\nu_1}\otimes\mathbf{I}_{(k_2,k_1)}\right),
$$

$$
\text{and }\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{2*},\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{1*}}=\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{1*},\boldsymbol{\theta}'_{2*}}\mathbf{I}_{(k_2,k_1^2)}.
$$

Derivatives of $\operatorname{vec}\boldsymbol{\Sigma}$ with respect to the identified parameter $\boldsymbol{\tau}$ can be computed as

$$\begin{aligned}
\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'} &= \mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'}\mathbf{D}^{(1)}_{\boldsymbol{\theta};\boldsymbol{\tau}'},\\
\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'} &= \mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}',\boldsymbol{\theta}'}\left(\mathbf{D}^{(1)}_{\boldsymbol{\theta};\boldsymbol{\tau}'}\otimes\mathbf{D}^{(1)}_{\boldsymbol{\theta};\boldsymbol{\tau}'}\right)+\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'}\mathbf{D}^{(2)}_{\boldsymbol{\theta};\boldsymbol{\tau}',\boldsymbol{\tau}'},\\
\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}',\boldsymbol{\tau}'} &= \mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\theta}',\boldsymbol{\theta}',\boldsymbol{\theta}'}\left(\mathbf{D}^{(1)}_{\boldsymbol{\theta};\boldsymbol{\tau}'}\otimes\mathbf{D}^{(1)}_{\boldsymbol{\theta};\boldsymbol{\tau}'}\otimes\mathbf{D}^{(1)}_{\boldsymbol{\theta};\boldsymbol{\tau}'}\right)\\
&\quad+\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\theta}',\boldsymbol{\theta}'}\left(\mathbf{D}^{(2)}_{\boldsymbol{\theta};\boldsymbol{\tau}',\boldsymbol{\tau}'}\otimes\mathbf{D}^{(1)}_{\boldsymbol{\theta};\boldsymbol{\tau}'}\right)\mathbf{J}_{21,\nu}+\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\theta}'}\mathbf{D}^{(3)}_{\boldsymbol{\theta};\boldsymbol{\tau}',\boldsymbol{\tau}',\boldsymbol{\tau}'},
\end{aligned}$$

where derivatives of $\boldsymbol{\theta}$ with respect to $\boldsymbol{\tau}$ are given in Theorem 2, derivatives of $\mathbf{g}$ with respect to $\boldsymbol{\theta}_1$ are given in §3 of this supplement, and $\mathbf{J}_{21,\nu}$ is defined in Corollary 1.1.

5 Derivatives of the Log Likelihood Function in Factor Analysis

If $\mathcal{L}$ is the Wishart log likelihood function, then expressions for the derivatives of $\mathcal{L}$ and for $\mathbf{Z}_i$ and $\mathbf{K}_i$ are the following:

$$\begin{aligned}
\frac{1}{\sqrt{n}}\mathbf{D}^{(1)}_{\mathcal{L};\boldsymbol{\tau}} &= \frac{1}{2}\ddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma}),\\
\mathbf{Z}_1 &= \frac{1}{2}\ddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma}),\\
\mathbf{K}_1 &= \mathbf{0},\\
\frac{1}{\sqrt{n}}\mathbf{D}^{(2)}_{\mathcal{L};\boldsymbol{\tau},\boldsymbol{\tau}'} &= -\frac{\sqrt{n}}{2}\ddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}+\frac{1}{2}\mathbf{D}^{(2)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau},\boldsymbol{\tau}'}\left[\mathbf{I}_\nu\otimes\left(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}\right)\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right]\\
&\quad-\dddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\left[\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right],\\
\mathbf{Z}_2 &= \frac{1}{2}\mathbf{D}^{(2)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau},\boldsymbol{\tau}'}\left[\mathbf{I}_\nu\otimes\left(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}\right)\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right]-\dddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\left[\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right],\\
\mathbf{K}_2 &= -\frac{1}{2}\ddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}=-\bar{\mathbf{I}}_{\boldsymbol{\tau}},\\
\frac{1}{\sqrt{n}}\mathbf{D}^{(3)}_{\mathcal{L};\boldsymbol{\tau},\boldsymbol{\tau}',\boldsymbol{\tau}'} &= \sqrt{n}\dddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\left(\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\right)2\mathbf{N}_\nu\\
&\quad-\sqrt{n}\mathbf{D}^{(2)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau},\boldsymbol{\tau}'}\left(\mathbf{I}_\nu\otimes\ddot{\mathbf{D}}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\right)\mathbf{N}_\nu-\frac{\sqrt{n}}{2}\ddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\\
&\quad+\left(\operatorname{vec}\ddot{\mathbf{D}}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes\ddot{\mathbf{D}}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\right)'\left(\mathbf{I}_{p\nu}\otimes\mathbf{I}_{(p,p)}\otimes\mathbf{I}_p\right)\left[\mathbf{I}_\nu\otimes\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right]
\end{aligned}$$

$$
\begin{aligned}
& +\dddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\left\{\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes 2\mathbf{N}_p\left(\mathbf{I}_p\otimes\operatorname{vec}\boldsymbol{\Sigma}^{-1}\otimes\mathbf{I}_p\right)'\left[\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right]\right\}\\
& +\frac{1}{2}\mathbf{D}^{(3)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau},\boldsymbol{\tau},\boldsymbol{\tau}'}\left[\mathbf{I}_{\nu^2}\otimes\left(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}\right)\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right]\\
& -\mathbf{D}^{(2)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau},\boldsymbol{\tau}'}\left(\mathbf{I}_\nu\otimes\boldsymbol{\Sigma}^{-1}\otimes\operatorname{vec}\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}\right)'\\
& \quad\times\left[\left(\mathbf{I}_\nu\otimes\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\right)2\mathbf{N}_\nu\otimes\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right]\\
& -\dddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\left[\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\otimes\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right],
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Z}_3 &= \left(\operatorname{vec}\ddot{\mathbf{D}}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes\ddot{\mathbf{D}}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\right)'\left(\mathbf{I}_{p\nu}\otimes\mathbf{I}_{(p,p)}\otimes\mathbf{I}_p\right)\left[\mathbf{I}_\nu\otimes\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right]\\
& +\dddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\left\{\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes 2\mathbf{N}_p\left(\mathbf{I}_p\otimes\operatorname{vec}\boldsymbol{\Sigma}^{-1}\otimes\mathbf{I}_p\right)'\left[\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right]\right\}\\
& +\frac{1}{2}\mathbf{D}^{(3)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau},\boldsymbol{\tau},\boldsymbol{\tau}'}\left[\mathbf{I}_{\nu^2}\otimes\left(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}\right)\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right]\\
& -\mathbf{D}^{(2)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau},\boldsymbol{\tau}'}\left(\mathbf{I}_\nu\otimes\boldsymbol{\Sigma}^{-1}\otimes\operatorname{vec}\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}\right)'\left[\left(\mathbf{I}_\nu\otimes\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\right)2\mathbf{N}_\nu\otimes\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right]\\
& -\dddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\left[\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\otimes\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right],\ \text{and}\\
\mathbf{K}_3 &= \dddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\left(\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\right)2\mathbf{N}_\nu-\mathbf{D}^{(2)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau},\boldsymbol{\tau}'}\left(\mathbf{I}_\nu\otimes\ddot{\mathbf{D}}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\right)\mathbf{N}_\nu\\
& -\frac{1}{2}\ddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'},\ \text{and}\\
\mathbf{K}_4 &= 2\mathbf{D}^{(2)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau},\boldsymbol{\tau}'}\mathbf{I}_{(p^2,\nu)}\left\{\left(\boldsymbol{\Sigma}^{-1}\otimes\operatorname{vec}\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}\right)'\left(\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\right)\otimes\mathbf{I}_\nu\right\}\mathbf{J}_{21,\nu}\\
&+\ 2\dddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\left(\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\otimes\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\right)\mathbf{J}_{21,\nu}-\frac{1}{2}\mathbf{D}^{(3)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau},\boldsymbol{\tau},\boldsymbol{\tau}'}\left(\mathbf{I}_{\nu^2}\otimes\ddot{\mathbf{D}}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\right)\mathbf{J}_{21,\nu}\\
&-\ \mathbf{D}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\left(\boldsymbol{\Sigma}^{-1}\otimes\operatorname{vec}\boldsymbol{\Sigma}^{-1}\otimes\operatorname{vec}\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}\right)'\left(\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\otimes\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\right)\\
&\times\ \left\{2\mathbf{I}_{(\nu,\nu^2)}+3\mathbf{I}_{(\nu^2,\nu)}+4\left(\mathbf{N}_\nu\otimes\mathbf{I}_\nu\right)\right\}\\
&-\ \frac{1}{2}\mathbf{D}^{(2)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau},\boldsymbol{\tau}'}\mathbf{I}_{(p^2,\nu)}\left(\ddot{\mathbf{D}}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\otimes\mathbf{I}_\nu\right)\mathbf{J}_{21,\nu}-\frac{1}{2}\ddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}',\boldsymbol{\tau}'},\ \text{where}\\
\dddot{\mathbf{D}}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'} &= \left(\boldsymbol{\Sigma}^{-1}\otimes\operatorname{vec}\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}\right)\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'},
\end{aligned}
$$

$$
\begin{aligned}
\ddot{\mathbf{D}}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'} &= \left(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}\right)\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'},\\
\ddot{\mathbf{D}}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'} &= \left(\boldsymbol{\Sigma}^{-1}\otimes\boldsymbol{\Sigma}^{-1}\right)\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'},
\end{aligned}
$$

$\mathbf{J}_{21,\nu}$ is defined in Corollary 1.1, $\mathbf{s}=\operatorname{vec}\mathbf{S}$, and $\mathbf{S}$ is defined in (29).

6 Expressions for Cumulant Functions in Factor Analysis Under Normality

The quantities ω_i for $i=1,2,3,4,6$ in Corollary 4.2 can be written as follows:

$$
\begin{aligned}
\omega_1 &= \frac{1}{2}\mathbf{a}_2'\operatorname{vec}\left(\bar{\mathbf{I}}_{\boldsymbol{\tau}}\right)-\frac{1}{4}\mathbf{v}'\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\operatorname{vec}\left(\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\right),\\
\omega_2 &= \omega_1^2-2\operatorname{tr}\left[\mathbf{D}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\left(\mathbf{V}\otimes\boldsymbol{\Sigma}^{-1}\right)\left(\mathbf{I}_{p^2}-\mathbf{P}_{\boldsymbol{\Sigma}}\right)\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}(\mathbf{a}\otimes\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1})\right]\\
&\quad+\mathbf{a}'\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{A}_3\operatorname{vec}\left(\bar{\mathbf{I}}_{\boldsymbol{\tau}}\right)+4\operatorname{tr}\left[\mathbf{P}_{\boldsymbol{\Sigma}}\left(\boldsymbol{\Sigma}\mathbf{V}\otimes\mathbf{I}_p\right)\left(\mathbf{I}_{p^2}-\mathbf{P}_{\boldsymbol{\Sigma}}\right)\mathbf{N}_p\left(\boldsymbol{\Sigma}\mathbf{V}\otimes\mathbf{I}_p\right)\right]\\
&\quad-\frac{1}{2}\mathbf{a}'\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{A}_2\mathbf{D}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\ddot{\mathbf{D}}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\operatorname{vec}\left(\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\right)\\
&\quad+\frac{1}{2}\operatorname{tr}\left[\ddot{\mathbf{D}}^{(2)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\left(\mathbf{I}_{p^2}-\mathbf{P}_{\boldsymbol{\Sigma}}\right)\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}(\mathbf{a}\mathbf{a}'\otimes\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1})\right]\\
&\quad+\operatorname{tr}\left[\mathbf{A}_2\mathbf{D}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\left(\mathbf{V}\otimes\boldsymbol{\Sigma}^{-1}\right)\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\right]-\frac{1}{2}\mathbf{v}'\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\mathbf{a}_2\\
&\quad-\frac{1}{2}\mathbf{v}'\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}',\boldsymbol{\tau}'}\left[\mathbf{a}\otimes\operatorname{vec}\left(\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\right)\right]+\frac{1}{8}\mathbf{v}'\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\left(\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\otimes\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\right)\mathbf{D}^{(2)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\mathbf{v}\\
&\quad-\frac{1}{2}\mathbf{v}'\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\operatorname{vec}\left\{\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\mathbf{D}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\left[(\mathbf{V}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}-\ddot{\mathbf{D}}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\left(\mathbf{a}\otimes\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\right)\right]\right\}\\
&\quad-\operatorname{tr}\left[\mathbf{D}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\ddot{\mathbf{D}}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\left(\mathbf{a}\otimes\mathbf{A}_2\right)\right]+\left[\operatorname{vec}(\mathbf{V}\boldsymbol{\Sigma}\mathbf{V})\right]'\left(\mathbf{I}_{p^2}-\mathbf{P}_{\boldsymbol{\Sigma}}\right)\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\operatorname{vec}\left(\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\right)\\
&\quad+\frac{1}{4}\mathbf{v}'\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\left[\mathbf{a}\otimes\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\mathbf{D}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\ddot{\mathbf{D}}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\operatorname{vec}\left(\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\right)\right]+\frac{1}{2}\operatorname{tr}\left[\mathbf{A}_2\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{A}_2\bar{\mathbf{I}}_{\boldsymbol{\tau}}\right]\\
&\quad-\frac{1}{2}\left[\operatorname{vec}\left(\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\right)\right]'\ddot{\mathbf{D}}^{(2)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\left(\mathbf{I}_{p^2}-\mathbf{P}_{\boldsymbol{\Sigma}}\right)\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}(\mathbf{a}\otimes\mathbf{a}),\\
\omega_3 &= -\frac{3}{2}\mathbf{v}'\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}(\mathbf{a}\otimes\mathbf{a})+3\mathbf{a}'\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{A}_2\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{a}+\operatorname{tr}(\boldsymbol{\Sigma}\mathbf{V})^3,\\
\omega_4 &= 4\omega_1\omega_3-6\mathbf{v}'\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\left[\mathbf{a}\otimes\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\mathbf{D}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\operatorname{vec}(\mathbf{V}\boldsymbol{\Sigma}\mathbf{V})\right]\\
&\quad+12\left[\operatorname{vec}(\mathbf{V}\boldsymbol{\Sigma}\mathbf{V})\right]'\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\mathbf{A}_2\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{a}+12\mathbf{a}'\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{A}_2\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{A}_2\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{a}
\end{aligned}
$$

$$\begin{aligned}
&-12\mathbf{v}'\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\left[\mathbf{a}\otimes\mathbf{A}_2\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{a}\right]+3\,\mathrm{tr}(\boldsymbol{\Sigma}\mathbf{V})^4-2\mathbf{v}'\mathbf{D}^{(3)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}',\boldsymbol{\tau}'}(\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a})\\
&-6\mathbf{a}'\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{A}_2\mathbf{D}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\ddot{\mathbf{D}}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}(\mathbf{a}\otimes\mathbf{a})+4\mathbf{a}'\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{A}_3\left[\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{a}\otimes\bar{\mathbf{I}}_{\boldsymbol{\tau}}\mathbf{a}\right]\\
&+3\mathbf{v}'\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\left[\mathbf{a}\otimes\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\mathbf{D}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\ddot{\mathbf{D}}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}(\mathbf{a}\otimes\mathbf{a})\right]\\
&+3\mathbf{v}'\mathbf{D}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\left(\mathbf{a}\mathbf{a}'\otimes\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\right)\mathbf{D}^{(2)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}\mathbf{v},
\end{aligned}$$

$$\begin{aligned}
\omega_6 &= 10\omega_3^2,\\
\mathbf{P}_{\boldsymbol{\Sigma}} &= \frac{1}{2}\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\ddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'},\quad \mathbf{a}=\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\mathbf{D}_{\beta;\boldsymbol{\tau}},\quad \mathbf{v}=\ddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}\mathbf{a},\\
\mathbf{V} &= \mathrm{dvec}(\mathbf{v},p,p),\quad \mathbf{a}_2=\left(\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\otimes\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\right)\mathbf{D}_{\beta;\boldsymbol{\tau},\boldsymbol{\tau}},\quad \mathbf{A}_2=\mathrm{dvec}(\mathbf{a}_2,\nu,\nu),\\
\mathbf{a}_3 &= \left(\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\otimes\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\otimes\bar{\mathbf{I}}_{\boldsymbol{\tau}}^{-1}\right)\mathbf{D}_{\beta;\boldsymbol{\tau},\boldsymbol{\tau},\boldsymbol{\tau}},\quad \mathbf{A}_3=\mathrm{dvec}(\mathbf{a}_3,\nu,\nu^2),
\end{aligned}$$

and $\ddot{\mathbf{D}}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'}$ and $\ddot{\mathbf{D}}^{(2)}_{\boldsymbol{\sigma};\boldsymbol{\tau}',\boldsymbol{\tau}'}$ are defined in §5 of this supplement. The quantity $\mathbf{P}_{\boldsymbol{\Sigma}}$ is the projection operator that projects onto $\mathcal{R}(\mathbf{D}^{(1)}_{\boldsymbol{\sigma};\boldsymbol{\tau}'})$ along $\mathcal{N}(\ddot{\mathbf{D}}^{(1)\prime}_{\boldsymbol{\sigma};\boldsymbol{\tau}'})$.

7 Expressions for Cumulant Functions in Factor Analysis Under Arbitrary Distributions with Finite 12^{th} Cumulant

Under the simulation conditions described in §6 of the article, the random covariance matrix can be written as

$$\mathbf{S}=n^{-1}\mathbf{Y}'\mathbf{M}\mathbf{Y},\ \text{where } n=N-1,\quad \mathbf{M}=\mathbf{I}_N-\left(\frac{1}{N}\right)\mathbf{1}_N\mathbf{1}_N',\tag{30}$$

$$\mathbf{Y}=\mathbf{1}_N\boldsymbol{\mu}'+\mathbf{X}_2\boldsymbol{\Phi}^{\frac{1}{2}}\boldsymbol{\Xi}'(\boldsymbol{\Sigma})_D^{\frac{1}{2}}+\mathbf{X}_3\boldsymbol{\Psi}^{\frac{1}{2}}(\boldsymbol{\Sigma})_D^{\frac{1}{2}}$$

$$=\mathbf{1}_N\mu'+\mathbf{X}\mathbf{C},\quad \mathbf{X}=\begin{pmatrix}\mathbf{X}_2 & \mathbf{X}_3\end{pmatrix},\quad \mathbf{C}=\begin{pmatrix}\boldsymbol{\Phi}^{\frac{1}{2}}\boldsymbol{\Xi}'\\ \boldsymbol{\Psi}^{\frac{1}{2}}\end{pmatrix}(\boldsymbol{\Sigma})_D^{\frac{1}{2}},$$

and the components of the $N\times(p+m)$ matrix $\mathbf{X}$ are iid random variables scaled to have mean zero and unit variance. The non-zero components of the diagonal matrix $(\boldsymbol{\Sigma})_D^{\frac{1}{2}}$ are arbitrary positive constants. Without loss of generality, $(\boldsymbol{\Sigma})_D^{\frac{1}{2}}$ can be equated to $\mathbf{I}_p$. The $m\times m$ matrix $\boldsymbol{\Phi}$ is the matrix of correlations among the rotated factors. The components of $\boldsymbol{\Xi}$ and $\boldsymbol{\Psi}$ are scaled such that the matrix of correlations among the columns of $\mathbf{Y}$ is $\boldsymbol{\Delta}=\boldsymbol{\Xi}\boldsymbol{\Phi}\boldsymbol{\Xi}'+\boldsymbol{\Psi}$.

If $\beta=\beta(\boldsymbol{\theta})$ is a scalar, then it follows from §5 in this supplement and Theorem 4 in the article that

$$Q_1=\boldsymbol{\ell}_1^{*\prime}\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma}),\quad Q_2=\boldsymbol{\ell}_2^{*\prime}\left[\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\otimes\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right],$$

$$\text{and } Q_3 = \boldsymbol{\ell}_3^{*\prime}\left[\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\otimes\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\otimes\sqrt{n}\,(\mathbf{s}-\boldsymbol{\sigma})\right],$$

where $n = N-1$ and $\boldsymbol{\ell}_i^*$ is a p^{2i}-vector of constants. It follows from (30) that the random Q quantities also can be written as

$$Q_1 = \boldsymbol{\ell}_1'\sqrt{n}\,(\mathbf{s}^*-\boldsymbol{\sigma}^*),\quad Q_2 = \boldsymbol{\ell}_2'\left[\sqrt{n}\,(\mathbf{s}^*-\boldsymbol{\sigma}^*)\otimes\sqrt{n}\,(\mathbf{s}^*-\boldsymbol{\sigma}^*)\right], \tag{31}$$

$$\text{and } Q_3 = \boldsymbol{\ell}_3'\left[\sqrt{n}\,(\mathbf{s}^*-\boldsymbol{\sigma}^*)\otimes\sqrt{n}\,(\mathbf{s}^*-\boldsymbol{\sigma}^*)\otimes\sqrt{n}\,(\mathbf{s}^*-\boldsymbol{\sigma}^*)\right],\ \text{where}$$

$$\boldsymbol{\ell}_1 = \mathbf{C}\boldsymbol{\ell}_1^*,\quad \boldsymbol{\ell}_2 = [\mathbf{C}\otimes\mathbf{C}]\,\boldsymbol{\ell}_2^*,\quad \boldsymbol{\ell}_3 = [\mathbf{C}\otimes\mathbf{C}\otimes\mathbf{C}]\,\boldsymbol{\ell}_3^*,$$

$\boldsymbol{\sigma}^* = \text{vec}(\mathbf{I}_k)$, $k = m+p$, $\mathbf{s}^* = \text{vec}(\mathbf{S}^*)$, and $\mathbf{S}^* = n^{-1}\mathbf{X}'\mathbf{M}\mathbf{X}$. Furthermore, $\boldsymbol{\ell}_2$ can be replace by $2\mathbf{N}_{k^2}\,(2\mathbf{N}\otimes 2\mathbf{N}_k)\,\boldsymbol{\ell}_2/8$ so that $\boldsymbol{\ell}_2$ satisfies

$$\boldsymbol{\ell}_2 = \mathbf{N}_{k^2}\boldsymbol{\ell}_2 = (\mathbf{I}_{k^2}\otimes\mathbf{N}_k)\,\boldsymbol{\ell}_2 = (\mathbf{N}_k\otimes\mathbf{I}_{k^2})\,\boldsymbol{\ell}_2. \tag{32}$$

For later use, the following variants of $\boldsymbol{\ell}_1$, $\boldsymbol{\ell}_2$, and $\boldsymbol{\ell}_3$ are defined:

$$\mathbf{L}_1 = \text{dvec}(\boldsymbol{\ell}_1, k, k),\quad \mathbf{L}_2 = \text{dvec}(\boldsymbol{\ell}_2, k^2, k^2), \tag{33}$$

$$\mathbf{L}_3 = \text{dvec}(\boldsymbol{\ell}_3, k^4, k^2),\ \text{and } \mathbf{L}_3^* = \text{dvec}(\boldsymbol{\ell}_3, k^2, k^4),$$

where $\text{dvec}(\mathbf{A}, b, c)$ is the $a\times b$ matrix that satisfies $\text{vec}(\mathbf{A}) = \text{vec}\left[\text{dvec}(\mathbf{A}, b, c)\right]$, provided that $\text{vec}(\mathbf{A})$ is an $ab\times 1$ vector.

7.1 Covariance Matrix for $\sqrt{n}\,(\mathbf{s}^*-\boldsymbol{\sigma}^*)$

Denote the ij^{th} component of $\mathbf{X}$ by X_{ij} and denote the r^{th} cumulant of X_{ij} by κ_r. That is,

$$\kappa_1 = 0,\quad \kappa_2 = 1,\quad \kappa_3 = \text{E}(X_{ij}^3),\ \text{and } \kappa_4 = \text{E}(X_{ij}^4) - 3,$$

etc. Also, define $\mathbf{W}_{qr}$ as

$$\mathbf{W}_{qr} = \sum_{i=1}^{k}\mathbf{e}_i^{\otimes q}\mathbf{e}_i^{\otimes r\prime}, \tag{34}$$

where $\mathbf{e}_i$ is the i^{th} column of $\mathbf{I}_k$ and $\mathbf{e}_i^{\otimes q}$ is the q^{th}-order Kronecker product of $\mathbf{e}_i$ with itself. For example, $\mathbf{e}_i^{\otimes 3} = \mathbf{e}_i\otimes\mathbf{e}_i\otimes\mathbf{e}_i$.

Define $\boldsymbol{\Omega}_{22}$ as

$$\boldsymbol{\Omega}_{22} \overset{\text{def}}{=} \text{Var}\left[\sqrt{n}\,(\mathbf{s}^*-\boldsymbol{\sigma}^*)\right] = n\text{E}(\mathbf{s}^*\mathbf{s}^{*\prime}) - n\boldsymbol{\sigma}^*\boldsymbol{\sigma}^{*\prime}. \tag{35}$$

To compute the expectation, it is convenient to write $\mathbf{S}^*$ as

$$\mathbf{S}^* = \frac{1}{n}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{X}_i m_{ij}\mathbf{X}_j' = \frac{1}{N}\sum_{i=1}^{N}\mathbf{X}_i\mathbf{X}_i' - \frac{1}{nN}\sum_{i\neq j}^{N}\mathbf{X}_i\mathbf{X}_j', \tag{36}$$

where $\mathbf{X}_i'$ is the i^{th} row of $\mathbf{X}$, and m_{ij} is the ij^{th} component of $\mathbf{M}$; i.e., $m_{ii} = n/N$ and $m_{ij} = -1/N$ if $i\neq j$. Then,

$$n\text{E}(\mathbf{s}^*\mathbf{s}^{*\prime}) = \frac{1}{n}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{i_3=1}^{N}\sum_{i_4=1}^{N} m_{i_1i_2}m_{i_3i_4}\text{E}\left(\mathbf{X}_{i_1}\mathbf{X}_{i_3}'\otimes\mathbf{X}_{i_2}\mathbf{X}_{i_4}'\right)$$

$$= \frac{n}{N}\text{E}\left(\mathbf{X}_{i_1}\mathbf{X}_{i_1}'\otimes\mathbf{X}_{i_1}\mathbf{X}_{i_1}'\right) + \frac{1}{N}\text{E}\left(\mathbf{X}_1\mathbf{X}_1'\otimes\mathbf{X}_2\mathbf{X}_2'\right) + \frac{n^2}{N}\text{E}\left(\mathbf{X}_1\mathbf{X}_2'\otimes\mathbf{X}_1\mathbf{X}_2'\right) + \frac{1}{N}\text{E}\left(\mathbf{X}_1\mathbf{X}_2'\otimes\mathbf{X}_2\mathbf{X}_1'\right)$$

$$= \frac{n}{N}\left[\mathbf{W}_{22}\kappa_4 + 2\mathbf{N}_k + \boldsymbol{\sigma}^*\boldsymbol{\sigma}^{*\prime}\right] + \frac{1}{N}2\mathbf{N}_k + \frac{n^2}{N}\boldsymbol{\sigma}^*\boldsymbol{\sigma}^{*\prime}$$

$$= \frac{n}{N}\mathbf{W}_{22}\kappa_4 + 2\mathbf{N}_k + n\boldsymbol{\sigma}^*\boldsymbol{\sigma}^{*\prime},$$

where $\mathbf{W}_{22}$ is defined in (34). The matrix $\mathbf{W}_{22}$ also can be computed as $\mathbf{W}_{22} = \text{Diag}[\text{vec}(\mathbf{I}_k)]$, where $\text{Diag}(\mathbf{a})$ is a diagonal matrix whose diagonal components are a_1, a_2, …. It follows from the above analysis that

$$\boldsymbol{\Omega}_{22} = \frac{n\kappa_4}{N}\mathbf{W}_{22} + 2\mathbf{N}_k. \tag{37}$$

7.2 Functions of Even Powers of $\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)$

7.2.1 $\text{E}(Q_1^2)$

The expectation of Q_1^2 can be computed as

$$\text{E}(Q_1^2) = \boldsymbol{\ell}_1'\text{E}\left[\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)'\right]\boldsymbol{\ell}_1 = \boldsymbol{\ell}_1'\boldsymbol{\Omega}_{22}\boldsymbol{\ell}_1.$$

7.2.2 $\text{E}(Q_2)$

The expectation of Q_2 can be computed as

$$\text{E}(Q_2) = \boldsymbol{\ell}_2'\text{E}\left[\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*) \otimes \sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)\right] = \boldsymbol{\ell}_2'\boldsymbol{\omega}_{22} = \text{trace}\left(\mathbf{L}_2\boldsymbol{\Omega}_{22}\right), \tag{38}$$

$$\text{where } \boldsymbol{\omega}_{22} = \text{vec}(\boldsymbol{\Omega}_{22}),$$

and $\mathbf{L}_2$ is defined in (33).

7.2.3 $\text{E}(Q_2^2)$

The expectation of Q_2^2 requires fourth order moments of $\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)$. It follows from the central limit theorem that

$$\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*) \otimes \sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*) \sim \mathbf{Z} + O_p\left(n^{-\frac{1}{2}}\right), \text{ where} \tag{39}$$

$$\mathbf{Z} \sim \text{N}\left[\boldsymbol{\omega}_{22}, 2\mathbf{N}_{k^2}\left(\boldsymbol{\Omega}_{22} \otimes \boldsymbol{\Omega}_{22}\right)\right].$$

Accordingly, the expectation of Q_2^2 can be computed as

$$\text{E}(Q_2^2) = \boldsymbol{\ell}_2'\text{E}\left[\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)' \otimes \sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)'\right]\boldsymbol{\ell}_2$$

$$= \boldsymbol{\ell}_2'\left[\boldsymbol{\omega}_{22}\boldsymbol{\omega}_{22}' + 2\mathbf{N}_{k^2}\left(\boldsymbol{\Omega}_{22} \otimes \boldsymbol{\Omega}_{22}\right)\right]\boldsymbol{\ell}_2 + O\left(n^{-1}\right)$$

$$= \left[\text{trace}(\mathbf{L}_2\boldsymbol{\Omega}_{22})\right]^2 + 2\,\text{trace}\left(\mathbf{L}_2\boldsymbol{\Omega}_{22}\mathbf{L}_2\boldsymbol{\Omega}_{22}\right) + O\left(n^{-1}\right).$$

The $O(n^{-1})$ term is a function of κ_8 and lower-order cumulants. The κ_8 term is

$$\frac{n^2\kappa_8}{N^3}\sum_{i=1}^{k}\left[(\mathbf{e}_i \otimes \mathbf{e}_i)'\mathbf{L}_2(\mathbf{e}_i \otimes \mathbf{e}_i)\right]^2,$$

where $\mathbf{e}_i$ is the i^{th} column of $\mathbf{I}_k$. Inclusion of this term does not change the order of accuracy of the cumulant functions, nor does it affect the accuracy of the Edgeworth approximation.

7.2.4 E(Q_1Q_3)

Employing (39), the expectation of Q_1Q_3 can be computed as

$$\mathrm{E}(Q_1Q_3) = (\boldsymbol{\ell}_1 \otimes \boldsymbol{\ell}_3)' \operatorname{vec}\left\{\mathrm{E}\left[\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)' \otimes \sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)'\right]\right\}$$

$$= (\boldsymbol{\ell}_1 \otimes \boldsymbol{\ell}_3)' \operatorname{vec}\left\{2\mathbf{N}_{k^2}(\boldsymbol{\Omega}_{22} \otimes \boldsymbol{\Omega}_{22}) + \boldsymbol{\omega}_{22}\boldsymbol{\omega}_{22}'\right\} + O\left(n^{-1}\right)$$

$$= \operatorname{trace}\left[\left(\boldsymbol{\ell}_1'\boldsymbol{\Omega}_{22} \otimes \boldsymbol{\Omega}_{22}\right) 2\mathbf{N}_{k^2}\mathbf{L}_3\right] + \boldsymbol{\omega}_{22}'\mathbf{L}_3\boldsymbol{\Omega}_{22}\boldsymbol{\ell}_1 + O\left(n^{-1}\right),$$

where $\mathbf{L}_3$ is defined in (33). The $O(n^{-1})$ term is a function of κ_8 and lower-order cumulants. The κ_8 term is

$$\frac{n^2\kappa_8}{N^3}\sum_{i=1}^{k}(\mathbf{e}_i'\mathbf{L}_1\mathbf{e}_i)(\mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_i)'\mathbf{L}_3(\mathbf{e}_i \otimes \mathbf{e}_i),$$

where $\mathbf{L}_1$ is defined in (33). Inclusion of this term does not change the order of accuracy of the cumulant functions, nor does it affect the accuracy of the Edgeworth approximation.

7.2.5 E($Q_1^2Q_2$)

Employing (39), the expectation of $Q_1^2Q_2$ can be computed as

$$\mathrm{E}(Q_1^2Q2) = (\boldsymbol{\ell}_1 \otimes \boldsymbol{\ell}_1)'\mathrm{E}\left[\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)' \otimes \sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)\sqrt{n}\,(\mathbf{s}^* - \boldsymbol{\sigma}^*)'\right]\boldsymbol{\ell}_2$$

$$= (\boldsymbol{\ell}_1 \otimes \boldsymbol{\ell}_1)'\left[\boldsymbol{\omega}_{22}\boldsymbol{\omega}_{22}' + 2\mathbf{N}_{k^2}\left(\boldsymbol{\Omega}_{22} \otimes \boldsymbol{\Omega}_{22}\right)\right]\boldsymbol{\ell}_2 + O\left(n^{-1}\right)$$

$$= \operatorname{trace}(\mathbf{L}_2\boldsymbol{\Omega}_{22})\boldsymbol{\ell}_1'\boldsymbol{\Omega}_{22}\boldsymbol{\ell}_1 + 2\boldsymbol{\ell}_1'\boldsymbol{\Omega}_{22}\mathbf{L}_2\boldsymbol{\Omega}_{22}\boldsymbol{\ell}_1 + O\left(n^{-1}\right).$$

The $O(n^{-1})$ term is a function of κ_8 and lower-order cumulants. The κ_8 term is

$$\frac{n^2\kappa_8}{N^3}\sum_{i=1}^{k}(\mathbf{e}_i'\mathbf{L}_1\mathbf{e}_i)^2(\mathbf{e}_i \otimes \mathbf{e}_i)'\mathbf{L}_2(\mathbf{e}_i \otimes \mathbf{e}_i).$$

Inclusion of this term does not change the order of accuracy of the cumulant functions, nor does it affect the accuracy of the Edgeworth approximation.

7.2.6 E($Q_1^3Q_3$)

Let $M_{\mathbf{Z}}(\mathbf{t})$ be the moment generating function of the random vector $\mathbf{Z} \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Omega}_{22})$. Then,

$$\mathrm{E}\left(\mathbf{ZZ}' \otimes \mathbf{ZZ}' \otimes \mathbf{ZZ}'\right) = \left.\frac{\partial^6 M_{\mathbf{Z}}(\mathbf{t})}{\partial\,\mathbf{t} \otimes \partial\,\mathbf{t} \otimes \partial\,\mathbf{t} \otimes \partial\,\mathbf{t}' \otimes \partial\,\mathbf{t}' \otimes \partial\,\mathbf{t}'}\right|_{\mathbf{t}=\mathbf{0}}$$

$$= \left[\mathbf{I}_{(k^4,k^2)} + (\mathbf{I}_{k^2} \otimes 2\mathbf{N}_{k^2})\right]\left(\boldsymbol{\omega}_{22}\boldsymbol{\omega}_{22}' \otimes \boldsymbol{\Omega}_{22}\right)\left[(\mathbf{I}_{k^2} \otimes 2\mathbf{N}_{k^2}) + \mathbf{I}_{(k^2,k^4)}\right]$$

$$+ \left[\mathbf{I}_{(k^4,k^2)} + (\mathbf{I}_{k^2} \otimes 2\mathbf{N}_{k^2})\right]\left(2\mathbf{N}_{k^2} \otimes \mathbf{I}_{k^2}\right)\left(\boldsymbol{\Omega}_{22} \otimes \boldsymbol{\Omega}_{22} \otimes \boldsymbol{\Omega}_{22}\right).$$

Employing (39), the expectation of $Q_1^3Q_3$ can be computed as

$$\mathrm{E}(Q_1^3Q_3) = (\boldsymbol{\ell}_1 \otimes \boldsymbol{\ell}_1 \otimes \boldsymbol{\ell}_1)'\,\mathrm{E}\left(\mathbf{ZZ}' \otimes \mathbf{ZZ}' \otimes \mathbf{ZZ}'\right)\boldsymbol{\ell}_3 + O\left(n^{-1}\right)$$

$$= 3\boldsymbol{\ell}_1'\boldsymbol{\Omega}_{22}\boldsymbol{\ell}_1\left\{\boldsymbol{\ell}_1'\boldsymbol{\Omega}_{22}\mathbf{L}_3^*\boldsymbol{\omega}_{22} + \operatorname{trace}\left[\left(\boldsymbol{\ell}_1'\boldsymbol{\Omega}_{22} \otimes \mathbf{I}_{k^2}\right)\mathbf{L}_3\boldsymbol{\Omega}_{22}\right] + \boldsymbol{\omega}_{22}'\mathbf{L}_3\boldsymbol{\Omega}_{22}\boldsymbol{\ell}_1\right\}$$

$$+6\left(\boldsymbol{\ell}_1'\boldsymbol{\Omega}_{22}\otimes\boldsymbol{\ell}_1'\boldsymbol{\Omega}_{22}\right)\mathbf{L}_3\boldsymbol{\Omega}_{22}\boldsymbol{\ell}_1+O\left(n^{-1}\right),$$

where $\mathbf{L}_3$ and $\mathbf{L}_3^*$ are defined in (33).

7.2.7 $\mathrm{E}(Q_1^2Q_2^2)$

Let $M_{\mathbf{Z}}(\mathbf{t})$ be the moment generating function of the random vector $\mathbf{Z}\sim\mathrm{N}(\mathbf{0},\boldsymbol{\Omega}_{22})$. Then,

$$\mathrm{E}\left(\mathbf{ZZ}'\otimes\mathbf{ZZ}'\otimes\mathbf{Z}\otimes\mathbf{Z}\right)=\left.\frac{\partial^6 M_{\mathbf{Z}}(\mathbf{t})}{\partial\,\mathbf{t}\otimes\partial\,\mathbf{t}\otimes\partial\,\mathbf{t}\otimes\partial\,\mathbf{t}\otimes\partial\,\mathbf{t}'\otimes\partial\,\mathbf{t}'}\right|_{\mathbf{t}=\mathbf{0}}$$

$$=\left[\left(\mathbf{I}_{k^2}\otimes\mathbf{I}_{(k^2,k^4)}\right)+\left(\mathbf{I}_{k^2}\otimes 2\mathbf{N}_{k^2}\otimes\mathbf{I}_{k^2}\right)\right]\left(\boldsymbol{\omega}_{22}\otimes\boldsymbol{\omega}_{22}\right)\boldsymbol{\omega}_{22}'$$

$$+\left[\left(\mathbf{I}_{k^2}\otimes\mathbf{I}_{(k^2,k^4)}\right)+\left(\mathbf{I}_{k^2}\otimes 2\mathbf{N}_{k}\otimes\mathbf{I}_{k^2}\right)\right]2\mathbf{N}_{k^4}\left(\boldsymbol{\Omega}_{22}\otimes\boldsymbol{\Omega}_{22}\otimes\boldsymbol{\omega}_{22}\right)2\mathbf{N}_{k^2}.$$

Employing (39), the expectation of $Q_1^2Q_2^2$ can be computed as

$$\mathrm{E}(Q_1^2Q_2^2)=\left(\boldsymbol{\ell}_2\otimes\boldsymbol{\ell}_2\right)'\mathrm{E}\left(\mathbf{ZZ}'\otimes\mathbf{ZZ}'\otimes\mathbf{Z}\otimes\mathbf{Z}\right)\left(\boldsymbol{\ell}_1\otimes\boldsymbol{\ell}_1\right)+O\left(n^{-1}\right)$$

$$=8\boldsymbol{\ell}_1'\boldsymbol{\Omega}_{22}\mathbf{L}_2\boldsymbol{\Omega}_{22}\mathbf{L}_2\boldsymbol{\Omega}_{22}\boldsymbol{\ell}_1+4\,\mathrm{trace}(\mathbf{L}_2\boldsymbol{\Omega}_{22})(\boldsymbol{\ell}_1'\boldsymbol{\Omega}_{22}\mathbf{L}_2\boldsymbol{\Omega}_{22}\boldsymbol{\ell}_1)$$

$$+2(\boldsymbol{\ell}_1'\boldsymbol{\Omega}_{22}\boldsymbol{\ell}_1)\,\mathrm{trace}(\boldsymbol{\Omega}_{22}\mathbf{L}_2\boldsymbol{\Omega}_{22}\mathbf{L}_2)+(\boldsymbol{\ell}_1'\boldsymbol{\Omega}_{22}\boldsymbol{\ell}_1)\left[\mathrm{trace}(\mathbf{L}_2\boldsymbol{\Omega}_{22})\right]^2+O\left(n^{-1}\right),$$

where $\mathbf{L}_2$ is defined in (33).

7.2.8 $n\left[\mathrm{E}(Q_1^4)-3\mathrm{E}(Q_1^2)^2\right]$

Because of the multiplier n, it is not sufficient to use the asymptotic normality of $\sqrt{n}\,(\mathbf{s}^*-\boldsymbol{\sigma}^*)\otimes\sqrt{n}\,(\mathbf{s}^*-\boldsymbol{\sigma}^*)$ in (39) to compute $n\left[\mathrm{E}(Q_1^4)-3\mathrm{E}(Q_1^2)^2\right]$. To proceed, note that

$$Q_1=\boldsymbol{\ell}_1'\sqrt{n}\,(\mathbf{s}^*-\boldsymbol{\sigma}^*)=\sqrt{n}\left[\frac{1}{N}\sum_{i=1}^{N}V_{ii}-\frac{1}{nN}\sum_{i\neq j}^{N}V_{ij}\right],\ \text{where}\tag{40}$$

$$V_{ij}=\begin{cases}\displaystyle\sum_{q=1}^{k}\sum_{r=1}^{k}L_{1,qr}\left(X_{iq}X_{ir}-\delta_{qr}\right) & \text{if } i=j,\\ \displaystyle\sum_{q=1}^{k}\sum_{r=1}^{k}L_{1,qr}X_{iq}X_{jr} & \text{if } i\neq j,\end{cases}$$

$$\delta_{qr}=\begin{cases}1 & \text{if } q=r,\\ 0 & \text{if } q\neq r,\end{cases}$$

and $L_{1,ij}$ is the ij^{th} component of $\mathbf{L}_1$ in (33). Note that V_{ii} are iid random variables for $i=1,\ldots,N$. It follows that

$$n\mathrm{E}(Q_1^4)=\mathrm{E}\left(T_{40}-4T_{31}+6T_{22}-4T_{13}+T_{04}\right),\ \text{where}\tag{41}$$

$$T_{fg}=\frac{n^3}{n^gN^4}\left(\sum_{i=1}^{N}V_{ii}\right)^f\left(\sum_{i\neq j}^{N}V_{ij}\right)^g.$$

The expectations of T_{ij} and like terms are functions of $\boldsymbol{\ell}_1$ as well as the cumulants of X_{ij}. Specifically, the following functions of $\boldsymbol{\ell}_1$ play important roles:

$$\begin{aligned}
c_1 &= \mathbf{1}_k'\mathbf{L}_1^{\odot 2}\mathbf{1}_k, \quad c_2 = \mathbf{1}_k'\mathbf{L}_1^{\odot 2}\mathbf{W}_{21}'\boldsymbol{\ell}_1^{\odot 2}, \quad c_3 = \operatorname{tr}\left(\mathbf{L}^{\odot 4}\right), \quad c_4 = \operatorname{tr}\left(\mathbf{L}_1^{\odot 2}\right), \qquad (42)\\
c_5 &= \mathbf{1}_k'\mathbf{L}^{\odot 4}\mathbf{1}_k, \quad c_6 = \mathbf{1}_k'\mathbf{L}_1^{\odot 2}\mathbf{L}_1^{\odot 2}\mathbf{1}_k, \quad c_7 = \operatorname{tr}\left(\mathbf{L}_1\mathbf{L}_1\mathbf{L}_1\mathbf{L}_1\right), \quad c_8 = \operatorname{tr}\left(\mathbf{D}_1\mathbf{L}_1\mathbf{L}_1\mathbf{L}_1\right),\\
c_9 &= \boldsymbol{\ell}_1'\mathbf{W}_{21}\mathbf{L}_1^{\odot 2}\mathbf{W}_{21}'\boldsymbol{\ell}_1, \quad c_{10} = \boldsymbol{\ell}_1'\mathbf{W}_{21}\mathbf{L}_1\mathbf{W}_{21}'\boldsymbol{\ell}_1^{\odot 2}, \quad c_{11} = \mathbf{1}_k'\mathbf{L}_1^{\odot 2}\mathbf{L}_1\mathbf{W}_{21}'\boldsymbol{\ell}_1,\\
c_{12} &= \mathbf{1}_k'\mathbf{L}^{\odot 3}\mathbf{W}_{21}'\boldsymbol{\ell}_1, \quad c_{13} = \operatorname{tr}\left(\mathbf{L}_1\mathbf{L}_1^{\odot 2}\mathbf{L}_1\right), \quad c_{14} = \boldsymbol{\ell}_1'\mathbf{W}_{21}\mathbf{L}_1\mathbf{L}_1\mathbf{W}_{21}'\boldsymbol{\ell}_1,\\
c_{15} &= \operatorname{tr}\left(\mathbf{L}_1^{\odot 3}\right) \quad c_{16} = \mathbf{1}_k'\mathbf{L}_1^{\odot 3}\mathbf{1}_k, \quad c_{17} = \operatorname{tr}\left(\mathbf{L}_1\mathbf{L}_1\mathbf{L}_1\right),\\
c_{18} &= \boldsymbol{\ell}_1'\mathbf{W}_{21}\mathbf{L}_1\mathbf{W}_{21}'\boldsymbol{\ell}_1, \quad c_{19} = \mathbf{1}_k'\mathbf{L}_1^{\odot 2}\mathbf{W}_{21}'\boldsymbol{\ell}_1,
\end{aligned}$$

$\mathbf{A}^{\odot r}$ is the r^{th} component-wise power (i.e., r^{th}-order Hadamard product) of the matrix $\mathbf{A}$, $\mathbf{L}_1$ is defined in (33), $\mathbf{W}_{21}$ is defined in (34), and $\mathbf{D}_1$ is a diagonal matrix that has the same diagonal components as $\mathbf{L}_1$. That is,

$$\mathbf{D}_1 = \sum_{i=1}^{k}\mathbf{e}_i L_{1,ii}\mathbf{e}_i'.$$

Expectations of T_{13} and T_{04} have magnitude $O(n^{-1})$ or smaller and, therefore, these terms can be dropped. The first term can be expanded as

$$\begin{aligned}
\mathrm{E}(T_{40}) = \frac{n^3}{N^4}\mathrm{E}\Big(&NV_{ii}^4 + 4NnV_{ii}^3V_{jj} + 3NnV_{ii}^2V_{jj}^2\\
&+6Nn(N-2)V_{ii}^2V_{jj}V_{rr} + Nn(N-2)(N-3)V_{ii}V_{jj}V_{rr}V_{ss}\Big)\\
= {}&\frac{n^3}{N^3}\mathrm{E}(V_{ii}^4) + \frac{3n^4}{N^3}\left[\mathrm{E}(V_{ii}^2)\right]^2,
\end{aligned}$$

where i, j, r, and s each have different values. Straightforward, but tedious, algebra reveals that

$$\begin{aligned}
\mathrm{E}(V_{ii}^2) &= c_4\kappa_4 + 2c_1, \text{ and} \qquad (43)\\
\mathrm{E}(V_{ii}^4) &= -56m_3c_3m_5 - 36c_1c_4 + c_3m_8 - 28c_3m_6 + 420c_3m_4 - 18c_4^2m_4 - 35c_3m_4^2\\
&\quad -240c_{10}m_3^2 + 560c_3m_3^2 - 360c_2m_4 + 48m_3^2c_{14} - 240m_3^2c_2 + 24c_9m_4^2 - 144c_9m_4\\
&\quad +24c_2m_6 + 96m_3^2c_{11} - 320m_3^2c_{12} + 96c_8m_4 + 48m_4c_6 - 48m_4c_5 + 96m_3^2c_{13}\\
&\quad +3c_4^2m_4^2 + 8m_4^2c_5 + 720c_2 - 630c_3 + 72c_5 - 144c_6 + 24m_3c_{10}m_5 + 12c_1c_4m_4\\
&\quad +32m_3c_{12}m_5 + 48c_7 + 216c_9 - 288c_8 + 12c_1^2 + 27c_4^2, \text{ where}\\
m_r &= \mathrm{E}(X_{ij}^r),
\end{aligned}$$

and c_r is defined in (42).

The expectation of the second term in (41) can be expanded as

$$\mathrm{E}(T_{31}) = \frac{n^2}{N^4}\mathrm{E}\left[\left(\sum_{i=1}^{N} V_{ii}\right)^3 \left(\sum_{i\neq j}^{N} V_{ij}\right)\right]$$

$$= \frac{n^2}{N^4}\mathrm{E}\Bigg[\left(\sum_{i=1}^{N} V_{ii}^3\right)\left(\sum_{i\neq j}^{N} V_{ij}\right) + 3\left(\sum_{i\neq j}^{N} V_{ii}^2 V_{jj}\right)\left(\sum_{i\neq j}^{N} V_{ij}\right) + \left(\sum_{i\neq j\neq g}^{N} V_{ii}V_{jj}V_{gg}\right)\left(\sum_{i\neq j}^{N} V_{ij}\right)\Bigg]$$

$$= \frac{6n^3}{N^3}\mathrm{E}\left(V_{ii}^2 V_{jj} V_{ij}\right), \text{ because}$$

$$\mathrm{E}\left(V_{ii}^3 V_{jk}\right) = 0 \;\forall\; j \neq k \text{ and } \mathrm{E}\left(V_{ii}V_{jj}V_{gg}V_{rs}\right) = 0 \;\forall\; i \neq j \neq g, \quad r \neq s.$$

Straightforward, but tedious, algebra reveals that

$$\mathrm{E}\left(V_{ii}^2 V_{jj} V_{ij}\right) = c_{10} m_3(m_5 - 2m_3) + 4(c_{14} - c_{10})m_3^2 + 4(c_{11} - c_{10})m_3^2, \tag{44}$$

where c_s is defined in (42) and m_t is defined in (43).

The expectation of the third term in (41) can be expanded as

$$\mathrm{E}(T_{22}) = \frac{n}{N^4}\mathrm{E}\left[\left(\sum_{i=1}^{N} V_{ii}\right)^2 \left(\sum_{i\neq j}^{N} V_{ij}\right)^2\right]$$

$$= \frac{n}{N^4}\mathrm{E}\Bigg[\left(\sum_{i=1}^{N} V_{ii}^2\right)\left(\sum_{i\neq j}^{N}\sum_{r\neq s}^{N} V_{ij}V_{rs}\right) + \left(\sum_{i\neq j}^{N} V_{ii}^2 V_{jj}\right)\left(\sum_{i\neq j}^{N}\sum_{r\neq s}^{N} V_{ij}V_{rs}\right)\Bigg]$$

$$= \frac{n}{N^4}\mathrm{E}\left[4NnV_{ii}^2V_{ij}^2 + 2Nn(N-2)V_{ii}^2V_{rs}^2 + 4NnV_{ii}V_{jj}V_{ij}^2 + 8Nn(N-2)V_{ii}V_{jj}V_{ik}V_{jk}\right]$$

$$= \frac{2n^2(N-2)}{N^3}\mathrm{E}\left(V_{ii}^2V_{rs}^2 + 4V_{ii}V_{jj}V_{ik}V_{jk}\right) + O\left(n^{-1}\right).$$

Straightforward, but tedious, algebra reveals that

$$\begin{aligned}\mathrm{E}\left(V_{ij}^2\right) &= c_1, \text{ and} \\ \mathrm{E}\left(V_{ii}V_{jj}V_{ij}V_{jk}\right) &= c_{14}m_3^2,\end{aligned}$$

where c_s is defined in (42) and m_t is defined in (43).

Further algebra, together with (43), (44), and (45), reveals that

$$\begin{aligned} n\left[\mathrm{E}(Q_1^4) - 3\mathrm{E}(Q_1^2)^2\right] &= c_3\kappa_8 + 24c_2\kappa_6 + 32\kappa_3\kappa_5c_{12} + 8\kappa_4^2(c_5 + 3c_9) \\ &\quad +48\kappa_4(c_6 + 2c_8) + 96\kappa_3^2c_{13} + 48c_7 + O\left(n^{-1}\right), \end{aligned} \tag{45}$$

where c_s is defined in (42) and m_t is defined in (43).

7.3 Functions of Odd Powers of $\sqrt{n}\left(\mathbf{s}^* - \boldsymbol{\sigma}^*\right)$

Expectations of functions of odd powers of $\sqrt{n}\left(\mathbf{s}^* - \boldsymbol{\sigma}^*\right)$ depend on $\boldsymbol{\ell}_1$, $\boldsymbol{\ell}_2$, and $\boldsymbol{\ell}_3$ through the following functions:

$$\begin{aligned}
d_1 &= \boldsymbol{\ell}_1'\mathbf{W}_{22}\mathbf{L}_2\mathbf{W}_{22}\boldsymbol{\ell}_1^{\odot 2}, \quad d_1^* = \boldsymbol{\ell}_1'\left(\mathbf{I}_{k^2} - \mathbf{W}_{22}\right)\mathbf{L}_2\mathbf{W}_{22}\boldsymbol{\ell}_1^{\odot 2}, \qquad (46)\\
d_2 &= \boldsymbol{\ell}_1'\mathbf{W}_{22}\mathbf{L}_2\mathbf{W}_{21}\mathbf{L}_1\mathbf{W}_{21}'\boldsymbol{\ell}_1, \quad d_2^* = \boldsymbol{\ell}_1'\left(\mathbf{I}_{k^2} - \mathbf{W}_{22}\right)\mathbf{L}_2\mathbf{W}_{21}\mathbf{L}_1\mathbf{W}_{21}'\boldsymbol{\ell}_1,\\
d_3 &= \boldsymbol{\ell}_1'\mathbf{W}_{22}\mathbf{L}_2\left(\mathbf{I}_k \otimes \mathbf{L}_1\right)\mathbf{W}_{22}\boldsymbol{\ell}_1, \quad d_3^* = \boldsymbol{\ell}_1'\left(\mathbf{I}_{k^2} - \mathbf{W}_{22}\right)\mathbf{L}_2\left(\mathbf{I}_k \otimes \mathbf{L}_1\right)\mathbf{W}_{22}\boldsymbol{\ell}_1,\\
d_4 &= \boldsymbol{\ell}_1'\mathbf{W}_{22}\mathbf{L}_2\left(\mathbf{I}_{k^2} - \mathbf{W}_{22}\right)\boldsymbol{\ell}_1^{\odot 2}, \quad d_4^* = \boldsymbol{\ell}_1'\left(\mathbf{I}_{k^2} - \mathbf{W}_{22}\right)\mathbf{L}_2\left(\mathbf{I}_{k^2} - \mathbf{W}_{22}\right)\boldsymbol{\ell}_1^{\odot 2},\\
d_5 &= \boldsymbol{\ell}_1'\mathbf{W}_{22}\mathbf{L}_2\operatorname{vec}\left(\mathbf{L}_1\mathbf{L}_1\right), \quad d_5^* = \boldsymbol{\ell}_1'\left(\mathbf{I}_{k^2} - \mathbf{W}_{22}\right)\mathbf{L}_2\operatorname{vec}\left(\mathbf{L}_1\mathbf{L}_1\right),\\
d_6 &= \boldsymbol{\ell}_1'\mathbf{W}_{22}\mathbf{L}_2\mathbf{W}_{21}\mathbf{L}_1^{\odot 2}\mathbf{1}_k, \quad d_6^* = \boldsymbol{\ell}_1'\left(\mathbf{I}_{k^2} - \mathbf{W}_{22}\right)\mathbf{L}_2\mathbf{W}_{21}\mathbf{L}_1^{\odot 2}\mathbf{1}_k,\\
d_7 &= \boldsymbol{\ell}_1'\mathbf{W}_{22}\mathbf{L}_2\left(\mathbf{W}_{21}'\boldsymbol{\ell}_1 \otimes \mathbf{W}_{21}'\boldsymbol{\ell}_1\right), \quad d_7^* = \boldsymbol{\ell}_1'\left(\mathbf{I}_{k^2} - \mathbf{W}_{22}\right)\mathbf{L}_2\left(\mathbf{W}_{21}'\boldsymbol{\ell}_1 \otimes \mathbf{W}_{21}'\boldsymbol{\ell}_1\right),\\
d_8 &= \boldsymbol{\ell}_2'\mathbf{W}_{42}\boldsymbol{\ell}_1, \quad d_9 = \operatorname{tr}\left[\mathbf{W}_{21}'\mathbf{L}_2\left(\mathbf{W}_{21}'\boldsymbol{\ell}_1 \otimes \mathbf{I}_k\right)\right], \quad d_{10} = \operatorname{tr}\left[\left(\mathbf{D}_1 \otimes \mathbf{I}_k\right)\mathbf{L}_2\right],\\
d_{11} &= \operatorname{tr}\left\{\left[\mathbf{W}_{21}'\left(\mathbf{I}_k \otimes \mathbf{L}_1\right) \otimes \mathbf{I}_k\right]\left(\mathbf{I}_k \otimes \mathbf{W}_{21}\right)\mathbf{L}_2\right\}, \quad d_{12} = \operatorname{tr}\left(\mathbf{W}_{21}'\mathbf{L}_2\mathbf{W}_{21}\mathbf{L}_1\right),\\
d_{13} &= \operatorname{tr}\left[\mathbf{W}_{22}\mathbf{L}_2\left(\mathbf{I}_k \otimes \mathbf{L}_1\right)\right], \quad d_{14} = \operatorname{tr}\left[\left(\mathbf{I}_k \otimes \mathbf{L}_1\right)\mathbf{L}_2\right],\\
d_{15} &= \operatorname{trace}\left(\mathbf{W}_{22}\mathbf{L}_2\right), \quad d_{16} = \operatorname{trace}\left(\mathbf{L}_2\right),
\end{aligned}$$

$\mathbf{D}_1$ is defined in (42), $\mathbf{W}_{21}$ and $\mathbf{W}_{22}$ are defined in (34), and $\mathbf{L}_1$ and $\mathbf{L}_2$ are defined in (33).

7.3.1 E(Q_1Q_2)

Using (32), it can be shown that Q_2 can be written as

$$Q_2 = \boldsymbol{\ell}_2'\left[\sqrt{n}\left(\mathbf{s}^* - \boldsymbol{\sigma}^*\right) \otimes \sqrt{n}\left(\mathbf{s}^* - \boldsymbol{\sigma}^*\right)\right] \qquad (47)$$

$$= \frac{n}{N^2}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} V_{iijj} - \frac{2}{n}\sum_{i=1}^{N}\sum_{f \neq g}^{N} V_{iifg} + \frac{1}{n^2}\sum_{i \neq j}^{N}\sum_{f \neq g}^{N} V_{ijfg}\right),$$

$$\text{where } V_{ijfg} = \sum_{q=1}^{k}\sum_{r=1}^{k}\sum_{s=1}^{k}\sum_{t=1}^{k}\left(\mathbf{e}_q \otimes \mathbf{e}_r\right)'\mathbf{L}_2\left(\mathbf{e}_s \otimes \mathbf{e}_t\right)R_{ijfg},$$

$$R_{ijfg} = \begin{cases} \left(X_{iq}X_{ir} - \delta_{qr}\right)\left(X_{fs}X_{ft} - \delta_{st}\right) & \text{if } i = j \text{ and } f = g,\\ \left(X_{iq}X_{ir} - \delta_{qr}\right)X_{fs}X_{gt} & \text{if } i = j \text{ and } f \neq g,\\ X_{iq}X_{jr}\left(X_{fs}X_{ft} - \delta_{st}\right) & \text{if } i \neq j \text{ and } f = g,\\ X_{iq}X_{jr}X_{fs}X_{gt} & \text{if } i \neq j \text{ and } f \neq g, \end{cases}$$

δ_{qr} is defined in (40), $\mathbf{e}_q$ is the q^{th} column of $\mathbf{I}_q$ and $\mathbf{L}_2$ is defined in (33). It is readily shown that

$$\mathrm{E}(V_{ijfg}) = \mathrm{E}(V_{fgij}) = \mathrm{E}(V_{ijgf}) = \mathrm{E}(V_{jifg}) \ \forall\ i, j, f, g,$$

$$\begin{gathered}\mathrm{E}\left(V_{iijj}\right)=\mathrm{E}\left(V_{iifg}\right)=\mathrm{E}\left(V_{ijfg}\right)=\mathrm{E}\left(V_{ijif}\right)=0\ \forall\ i\neq j\neq f\neq g,\\ \mathrm{E}\left(V_{iiii}\right)=\operatorname{tr}\left(\mathbf{W}_{22}\mathbf{L}_2\right)\kappa_4+2\operatorname{tr}(\mathbf{L}_2),\ \text{and}\qquad(48)\\ \mathrm{E}\left(V_{ijij}\right)=\operatorname{tr}(\mathbf{L}_2).\end{gathered}$$

Note that

$$\mathrm{E}(Q_2)=\frac{n}{N^2}\mathrm{E}\left(NV_{iiii}+\frac{2nN}{N^2}V_{ijij}\right)=\frac{n}{N}\operatorname{tr}(\mathbf{W}_{22}\mathbf{L}_2)\kappa_4+2\operatorname{tr}(\mathbf{L}_2),$$

which matches the result in §7.2.1.

Using (47) and (40), the expectation of $\sqrt{n}Q_1Q_2$ can be written as

$$\begin{aligned}&\sqrt{n}\mathrm{E}\left(Q_1Q_2\right)\\ &=\frac{n^2}{N^3}\mathrm{E}\left[\left(\sum_{i=1}^{N}V_{ii}-\frac{1}{n}\sum_{i\neq j}^{N}V_{ij}\right)\left(\sum_{i=1}^{N}\sum_{j=1}^{N}V_{iijj}-\frac{2}{n}\sum_{i=1}^{n}\sum_{f\neq g}^{N}V_{iifg}+\frac{1}{n^2}\sum_{i\neq j}^{N}\sum_{f\neq g}^{N}W_{ijfg}\right)\right]\\ &=\frac{n^2}{N^3}\mathrm{E}\Bigg[NV_{ii}V_{iiii}-4NV_{ii}V_{jjij}+\frac{4N}{n}V_{ii}V_{ijij}-2NV_{ij}V_{iijj}+\frac{8N}{n}V_{ij}V_{iiij}\\ &\qquad-\frac{4N}{n^2}V_{ij}V_{ijij}-\frac{8N(N-2)}{n^2}V_{ij}V_{ikjk}\Bigg]\\ &=\frac{n^2}{N^2}\mathrm{E}\left[V_{ii}V_{iiii}-4V_{ii}V_{jjij}-2V_{ij}V_{iijj}\right]+O\left(n^{-1}\right).\end{aligned}$$

Straightforward, but tedious, algebra reveals that

$$\begin{aligned}\mathrm{E}\left(V_{ii}V_{iiii}\right)&=d_8(m_6-3m_4+2)+4(d_9-d_8)m_3^2+4(d_{10}-d_8)(m_4-1)\qquad(49)\\ &\quad+4(d_{11}-d_8)m_3^2+2(d_{12}-d_8)m_3^2+8(d_{13}-d_8)(m_4-1)\\ &\quad+8(d_{14}-d_{10}-2d_{13}+2d_8),\\ \mathrm{E}\left(V_{ii}V_{jjij}\right)&=d_9m_3^2,\\ \mathrm{E}\left(V_{ij}V_{iijj}\right)&=d_{17}m_3^2,\end{aligned}$$

where m_r is defined in (43) and d_s is defined in (46). Further algebra reveals that

$$\sqrt{n}\mathrm{E}\left(Q_1Q_2\right)=d_8\kappa_6+4\kappa_4(d_{10}+2d_{13})+4d_{11}\kappa_3^2+8d_{14}+O\left(n^{-1}\right).\qquad(50)$$

7.3.2 $\mathrm{E}(Q_1^3)$

It follows from (40) that

$$\sqrt{n}\mathrm{E}\left(Q_1^3\right)=\frac{n^2}{N^3}\mathrm{E}\left[\left(\sum_{i=1}^{N}V_{ii}-\frac{1}{n}\sum_{i\neq j}^{N}V_{ij}\right)^3\right]$$

$$= \frac{n^2}{N^3}\mathrm{E}\left[\sum_{i=1}^{N} V_{ii}^3 + 3\sum_{i\neq j}^{N} V_{ii}^2 V_{jj} + \sum_{i\neq j\neq r}^{N} V_{ii}V_{jj}V_{rr} - \frac{3}{n}\sum_{i=1}^{N}\sum_{f\neq g}^{N} V_{ii}^2 V_{fg}\right.$$

$$\left. -\frac{3}{n}\sum_{i\neq j}^{N}\sum_{f\neq g}^{N} V_{ii}V_{jj}V_{fg} + \frac{3}{n^2}\sum_{i=1}^{N}\sum_{f\neq g}^{N}\sum_{r\neq s}^{N} V_{ii}V_{fg}V_{rs} - \frac{1}{n^3}\sum_{i\neq j}^{N}\sum_{f\neq g}^{N}\sum_{r\neq s}^{N} V_{ij}V_{fg}V_{rs}\right]$$

$$= \frac{n^2}{N^2}\mathrm{E}\left(V_{ii}^3 - 6V_{ii}V_{jj}V_{ij}\right) + O\left(n^{-1}\right),$$

because the remaining terms have expectation zero or are $O(n^{-1})$. Straightforward, but tedious, algebra reveals that

$$\begin{aligned} \mathrm{E}\left(V_{ii}^3\right) &= c_{15}(m_6 - 3m_4 + 2) + 6(c_{18} - c_{15})m_3^2 + 12(c_{19} - c_{15})(m_4 - 1) \qquad (51) \\ &\quad +4(c_{16} - c_{15})m_3^2 + 8(c_{17} - 3c_{19} + 2c_{15}), \text{ and} \\ \mathrm{E}\left(V_{ii}V_{jj}V_{ij}\right) &= c_{18}m_3^2. \end{aligned}$$

Further algebra reveals that

$$\sqrt{n}\mathrm{E}\left(Q_1^3\right) = c_{15}\kappa_6 + 12c_{19}\kappa_4 + 4c_{16}\kappa_3^2 + 8c_{17} + O\left(n^{-1}\right). \qquad (52)$$

7.3.3 $\mathrm{E}(Q_1^3 Q_2)$

It follows from (40) that

$$\sqrt{n}\mathrm{E}\left(Q_1^3 Q_2\right)$$

$$= \frac{n^3}{N^5}\mathrm{E}\left[\left(\sum_{i=1}^{N} V_{ii} - \frac{1}{n}\sum_{i\neq j}^{N} V_{ij}\right)^3 \left(\sum_{i=1}^{N}\sum_{j=1}^{N} V_{iijj} - \frac{2}{n}\sum_{i=1}^{N}\sum_{f\neq g}^{N} V_{iifg} + \frac{1}{n^2}\sum_{i\neq j}^{N}\sum_{f\neq g}^{N} V_{ijfg}\right)\right].$$

Most of the terms in $\sqrt{n}\mathrm{E}\left(Q_1^3 Q_2\right)$ are smaller than $O(n^{-1})$ and can be dropped. The remaining terms are given below:

$$\sqrt{n}\mathrm{E}\left(Q_1^3 Q_2\right) = \frac{n^4}{N^4}\left[\mathrm{E}\left(V_{ii}^3\right)\mathrm{E}\left(V_{iiii}\right) + 6\mathrm{E}\left(V_{ii}^2 V_{jj} V_{iijj}\right) + 3\mathrm{E}\left(V_{ii}^2\right)\mathrm{E}\left(V_{ii}V_{iiii}\right)\right]$$

$$-\frac{12n^3(N-2)}{N^4}\left[\mathrm{E}\left(V_{ii}^2\right)\mathrm{E}\left(V_{ii}V_{jjij}\right) + \mathrm{E}\left(V_{ii}V_{jj}V_{ff}V_{iijf}\right)\right]$$

$$-\frac{6n^3(N-2)}{N^4}\left[\mathrm{E}\left(V_{ii}^2\right)\mathrm{E}\left(V_{ij}V_{iijj}\right) + 4\mathrm{E}\left(V_{ii}V_{jj}V_{jf}V_{iiff}\right) + \mathrm{E}\left(V_{ii}V_{jj}V_{ij}\right)\mathrm{E}\left(V_{iiii}\right)\right] + O\left(n^{-1}\right).$$

Straightforward, but tedious, algebra reveals that

$$\begin{aligned} \mathrm{E}\left(V_{ii}^2 V_{jj} V_{iijj}\right) &= \left[d_1\left(m_4 - 1\right) + 2d_1^*\right]\left(m_6 - 3m_4 + 2\right) \qquad (53) \\ &\quad +4\left[\left(d_2 - d_1\right)\left(m_4 - 1\right) + 2(d_2^* - d_1^*)\right] m_3^2 \\ &\quad +8\left[\left(d_3 - d_1\right)\left(m_4 - 1\right) + 2(d_3^* - d_1^*)\right]\left(m_4 - 1\right) \end{aligned}$$

$$
\begin{aligned}
&+4\left[d_4\left(m_4-1\right)+2d_4^*\right]m_3^2\\
&+8\left[\left(d_5-2d_3-d_6+2d_1\right)\left(m_4-1\right)+2(d_5^*-2d_3^*-d_6^*+2d_1^*)\right]\\
&+4\left[\left(d_6-d_1\right)\left(m_4-1\right)+2(d_6^*-d_1^*)\right]\left(m_4-1\right)\\
&+2\left[\left(d_7-d_1\right)\left(m_4-1\right)+2(d_7^*-d_1^*)\right]m_3^2,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left(V_{ii}V_{jjij}\right) &= d_9 m_3^2,\\
\mathrm{E}\left(V_{ii}V_{jj}V_{ff}V_{iijf}\right) &= \left[(d_7-d_1)(\kappa_4+2)+2(d_7^*-d_1^*)\right]m_3^2,\\
\mathrm{E}\left(V_{ij}V_{iijj}\right) &= d_{12}m_3^2,\\
\mathrm{E}\left(V_{ii}V_{jj}V_{if}V_{iiff}\right) &= \left[d_2(\kappa_4+2)+2d_2^*\right]m_3^2,
\end{aligned}
$$

where d_r and d_s^* are defined in (46). Further algebra reveals that

$$
\begin{gathered}
\sqrt{n}\mathrm{E}\left(Q_1^3Q_2\right)=(c_{15}d_{15}+3c_4d_8+6d_1)\kappa_6\kappa_4 \qquad (54)\\
2\left[6(d_1+d_1^*)+3c_1d_8+c_{15}d_{16}\right]\kappa_6\\
+12\left[4d_3+2(c_4d_{13}+d_6)+c_4d_{10}+c_{19}d_{15}\right]\kappa_4^2\\
+4\left[c_{16}d_{15}+6(d_4+d_1)+3c_4d_{11}\right]\kappa_4\kappa_3^2\\
+8\left[6(d_5+d_6+d_6^*+c_1d_{13})+12(d_3+d_3^*)+3(c_{19}d_{16}+c_4d_{14}+c_1d_{10})+c_{17}d_{15}\right]\kappa_4\\
+8\left[6(d_1+d_1^*+d_4+d_4^*)+3c_1d_{11}+c_{16}d_{16}\right]\kappa_3^2\\
+16\left[3c_1d_{14}+6(d_5+d_5^*)+c_{17}d_{16}\right]+O\left(n^{-1}\right),
\end{gathered}
$$

where c_r is defined in (42) and d_f and d_g^* are defined in (46).