SPECTRAL MODELS FOR COVARIANCE MATRICES: SUPPLEMENT

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1 Errata

1. The published article contains a typographical error in the definition of $\mathbf{N}_{p^2}$. The error is corrected on page 13
2. A typographical error in the definition of $\mathbf{K}_4^*$ on page 16 was corrected May, 2002.
3. A typographical error in the definition of $\widetilde{\mathbf{U}}_i^{(2)}$ on page 16 was corrected May, 2003.
4. Two typographical errors in the multiplicative invariance properties of $\mathbf{F}_i^{(111)}$ on page 4 were corrected March, 2005.

2 Higher-Order Derivatives

For clarity, matrices and vectors are displayed in boldface font.

2.1 Arrangement of Derivatives

Let $\mathbf{A} = \mathbf{A}(\mathbf{B})$ be a $p \times q$ matrix whose elements are functions of the $r \times s$ matrix $\mathbf{B}$. Then the derivative of $\mathbf{A}$ with respect to $\mathbf{B}$ is a $pr \times qs$ matrix whose elements are arranged as follows:

$$\frac{\partial \mathbf{A}}{\partial \mathbf{B}} \overset{\text{def}}{=} \frac{\partial}{\partial \mathbf{B}} \otimes \mathbf{A}.$$

Partition $\boldsymbol{\theta}$ as $\boldsymbol{\theta} = (\,\boldsymbol{\theta}_1' \quad \cdots \quad \boldsymbol{\theta}_k'\,)'$. In particular, if $\boldsymbol{\theta}$ has the structure in (10) then $k = 3$ and $\boldsymbol{\theta} = (\,\boldsymbol{\mu}' \quad \boldsymbol{\tau}' \quad \boldsymbol{\varphi}'\,)'$; and if $\boldsymbol{\theta}$ has the structure in (27) then $k = 4$ and $\boldsymbol{\theta} = (\,\boldsymbol{\mu}' \quad \boldsymbol{\tau}' \quad \boldsymbol{\varphi}' \quad \boldsymbol{\beta}'\,)'$. Derivatives of $\operatorname{vec}\boldsymbol{\Sigma}_i$ with respect to $\boldsymbol{\theta}$ are arranged as follows:

$$\mathbf{F}_i^{(1)} \overset{\text{def}}{=} \left.\frac{\partial \operatorname{vec}(\boldsymbol{\Sigma}_i)}{\partial \boldsymbol{\theta}'}\right|_{\boldsymbol{\mu}=\mathbf{0},\boldsymbol{\tau}=\mathbf{0}} = \left.\sum_{s=1}^{k} \frac{\partial \operatorname{vec}(\boldsymbol{\Sigma}_i)}{\partial \boldsymbol{\theta}_s'} \mathbf{E}_{s,\boldsymbol{\nu}}\right|_{\boldsymbol{\mu}=\mathbf{0},\boldsymbol{\tau}=\mathbf{0}};$$

$$\mathbf{F}_i^{(2)} \overset{\text{def}}{=} \left.\frac{\partial^2 \operatorname{vec}(\boldsymbol{\Sigma}_i)}{\partial \boldsymbol{\theta}' \otimes \partial \boldsymbol{\theta}'}\right|_{\boldsymbol{\mu}=\mathbf{0},\boldsymbol{\tau}=\mathbf{0}} = \left.\sum_{s=1}^{k}\sum_{t=1}^{k} \frac{\partial^2 \operatorname{vec}(\boldsymbol{\Sigma}_i)}{\partial \boldsymbol{\theta}_s' \otimes \partial \boldsymbol{\theta}_t'} \left(\mathbf{E}_{s,\boldsymbol{\nu}} \otimes \mathbf{E}_{t,\boldsymbol{\nu}}\right)\right|_{\boldsymbol{\mu}=\mathbf{0},\boldsymbol{\tau}=\mathbf{0}};$$

$$\mathbf{F}_i^{(11)} \overset{\text{def}}{=} \left.\frac{\partial^2 \operatorname{vec}(\boldsymbol{\Sigma}_i)}{\partial \boldsymbol{\theta} \otimes \partial \boldsymbol{\theta}'}\right|_{\boldsymbol{\mu}=\mathbf{0},\boldsymbol{\tau}=\mathbf{0}} = \operatorname{dvec}\left(\mathbf{F}_i^{(2)}, p^2\, d, d\right);$$

$$\mathbf{F}_i^{(3)} \overset{\text{def}}{=} \left.\frac{\partial^3 \operatorname{vec}(\boldsymbol{\Sigma}_i)}{\partial \boldsymbol{\theta}' \otimes \partial \boldsymbol{\theta}' \otimes \partial \boldsymbol{\theta}'}\right|_{\boldsymbol{\mu}=\mathbf{0},\boldsymbol{\tau}=\mathbf{0}}$$

$$= \sum_{s=1}^{k}\sum_{t=1}^{k}\sum_{u=1}^{k} \frac{\partial^3 \operatorname{vec}(\boldsymbol{\Sigma}_i)}{\partial\boldsymbol{\theta}'_s \otimes \partial\boldsymbol{\theta}'_t \otimes \partial\boldsymbol{\theta}'_u} \left(\mathbf{E}_{s,\boldsymbol{\nu}} \otimes \mathbf{E}_{t,\boldsymbol{\nu}} \otimes \mathbf{E}_{u,\boldsymbol{\nu}}\right)\Bigg|_{\boldsymbol{\mu}=\mathbf{0},\boldsymbol{\tau}=\mathbf{0}}; \text{ and}$$

$$\mathbf{F}_i^{(111)} \overset{\text{def}}{=} \frac{\partial^3 \operatorname{vec}(\boldsymbol{\Sigma}_i)}{\partial\boldsymbol{\theta} \otimes \partial\boldsymbol{\theta} \otimes \partial\boldsymbol{\theta}'}\Bigg|_{\boldsymbol{\mu}=\mathbf{0},\boldsymbol{\tau}=\mathbf{0}} = \operatorname{dvec}\left(\mathbf{F}_i^{(3)}, p^2d^2, d\right);$$

where $\mathbf{E}_{s,\boldsymbol{\nu}}$ is defined in (12);

$$\boldsymbol{\nu} = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_k \end{pmatrix} = \begin{pmatrix} \dim(\boldsymbol{\theta}_1) \\ \vdots \\ \dim(\boldsymbol{\theta}_k) \end{pmatrix};$$

and $\operatorname{dvec}(\mathbf{A}, a, b)$ is the $a \times b$ matrix that satisfies $\operatorname{vec}(\mathbf{A}) = \operatorname{vec}\{\operatorname{dvec}(\mathbf{A}, a, b)\}$. The matrix $\mathbf{A}$ must have exactly ab elements. Higher-order derivatives can be rearranged by post-multiplying by commutation matrices. For example,

$$\begin{aligned} \frac{\partial^2 \operatorname{vec}(\boldsymbol{\Sigma}_i)}{\partial\boldsymbol{\theta}'_t \otimes \partial\boldsymbol{\theta}'_s} &= \frac{\partial^2 \operatorname{vec}(\boldsymbol{\Sigma}_i)}{\partial\boldsymbol{\theta}'_s \otimes \partial\boldsymbol{\theta}'_t}\mathbf{I}_{(\nu_t,\nu_s)} \text{ and} \\ \frac{\partial^3 \operatorname{vec}(\boldsymbol{\Sigma}_i)}{\partial\boldsymbol{\theta}'_t \otimes \partial\boldsymbol{\theta}'_s \otimes \partial\boldsymbol{\theta}'_u} &= \frac{\partial^3 \operatorname{vec}(\boldsymbol{\Sigma}_i)}{\partial\boldsymbol{\theta}'_s \otimes \partial\boldsymbol{\theta}'_t \otimes \partial\boldsymbol{\theta}'_u}\left(\mathbf{I}_{(\nu_t,\nu_s)} \otimes \mathbf{I}_{\nu_u}\right) \\ &= \frac{\partial^3 \operatorname{vec}(\boldsymbol{\Sigma}_i)}{\partial\boldsymbol{\theta}'_t \otimes \partial\boldsymbol{\theta}'_u \otimes \partial\boldsymbol{\theta}'_s}\left(\mathbf{I}_{\nu_t} \otimes \mathbf{I}_{(\nu_s,\nu_u)}\right). \end{aligned}$$

The derivatives of $\operatorname{vec}\boldsymbol{\Sigma}_i$ with respect to $\boldsymbol{\theta}$ satisfy the following multiplicative invariance properties:

$$\begin{aligned} \mathbf{F}_i^{(1)} &= \mathbf{I}_{(p,p)}\,\mathbf{F}_i^{(1)}; \\ \mathbf{F}_i^{(2)} &= \mathbf{I}_{(p,p)}\,\mathbf{F}_i^{(2)} = \mathbf{F}_i^{(2)}\,\mathbf{I}_{(d,d)}; \\ \mathbf{F}_i^{(11)} &= \left(\mathbf{I}_d \otimes \mathbf{I}_{(p,p)}\right)\mathbf{F}_i^{(11)}; \\ \mathbf{F}_i^{(3)} &= \mathbf{I}_{(p,p)}\,\mathbf{F}_i^{(3)} = \mathbf{F}_i^{(3)}\,\mathbf{I}_{(d,d^2)} = \mathbf{F}_i^{(3)}\,\mathbf{I}_{(d^2,d)}; \\ &= \mathbf{F}_i^{(3)}(\mathbf{I}_d \otimes \mathbf{I}_{(d,d)}) = \mathbf{F}_i^{(3)}(\mathbf{I}_{(d,d)} \otimes \mathbf{I}_d); \text{ and} \\ \mathbf{F}_i^{(111)} &= \left(\mathbf{I}_{d^2} \otimes \mathbf{I}_{(p,p)}\right)\mathbf{F}_i^{(111)} = \left(\mathbf{I}_{(d,d)} \otimes \mathbf{I}_{p^2}\right)\mathbf{F}_i^{(111)}. \end{aligned}$$

These properties are useful when simplifying expressions.

2.2 Expressions for Derivatives

2.2.1 Derivatives of $\boldsymbol{\lambda}_i$ With Respect to $\boldsymbol{\varphi}$

Analogous to $\mathbf{D}^{(1)}_{\lambda_i}$ in (13), define $\mathbf{D}^{(2)}_{\lambda_i}$ and $\mathbf{D}^{(3)}_{\lambda_i}$ as

$$\mathbf{D}^{(2)}_{\lambda_i} \overset{\text{def}}{=} \frac{\partial^2 \boldsymbol{\lambda}_i}{\partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\varphi}'} \text{ and } \mathbf{D}^{(3)}_{\lambda_i} \overset{\text{def}}{=} \frac{\partial^3 \boldsymbol{\lambda}_i}{\partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\varphi}'}.$$

Then, $\mathbf{D}^{(2)}_{\lambda_i}$ and $\mathbf{D}^{(3)}_{\lambda_i}$ can be computed as follows:

$$\mathbf{D}^{(2)}_{\lambda_i} = \begin{cases} \mathbf{0} & \text{if (2) is employed,} \\ \mathbf{H}^{(2)}_i & \text{if (3) is employed,} \\ (\boldsymbol{\lambda}'_i \otimes \mathbf{I}_p)\mathbf{W}^{(2)}_p(\mathbf{H}^{(1)}_i \otimes \mathbf{H}^{(1)}_i) + \boldsymbol{\Lambda}_i \mathbf{H}^{(2)}_i & \text{if (4) is employed;} \end{cases} \quad \text{and} \qquad (30)$$

$$\mathbf{D}^{(3)}_{\lambda_i} = \begin{cases} \mathbf{0} & \text{if (2) is employed,} \\ \mathbf{H}^{(3)}_i & \text{if (3) is employed,} \\ \begin{array}{l} (\boldsymbol{\lambda}'_i \otimes \mathbf{I}_p)\mathbf{W}^{(3)}_p(\mathbf{H}^{(1)}_i \otimes \mathbf{H}^{(1)}_i \otimes \mathbf{H}^{(1)}_i) + \boldsymbol{\Lambda}_i \mathbf{H}^{(3)}_i \\ +2(\boldsymbol{\lambda}'_i \otimes \mathbf{I}_p)\mathbf{W}^{(2)}_p(\mathbf{H}^{(2)}_i \otimes \mathbf{H}^{(1)}_i)\,(\mathbf{I}_{\nu_3} \otimes \mathbf{N}_{\nu_3}) \end{array} & \text{if (4) is employed;} \end{cases}$$

where

$$\mathbf{W}^{(2)}_h = \sum_{j=1}^{h} (\mathbf{e}^h_j \mathbf{e}^{h\prime}_j \otimes \mathbf{e}^h_j \mathbf{e}^{h\prime}_j); \quad \mathbf{W}^{(3)}_h = \sum_{j=1}^{h} (\mathbf{e}^h_j \mathbf{e}^{h\prime}_j \otimes \mathbf{e}^h_j \mathbf{e}^{h\prime}_j \otimes \mathbf{e}^{h\prime}_j);$$

$$\mathbf{H}^{(2)}_i = \{\exp(\odot \mathbf{T}_{2i}\boldsymbol{\varphi}) \otimes \mathbf{T}'_{1i}\}'\,\mathbf{W}^{(2)}_{c_i}(\mathbf{T}_{2i} \otimes \mathbf{T}_{2i});$$

$$c_i = \#\text{ columns of } \mathbf{T}_{1i} = \#\text{ rows of } \mathbf{T}_{2i};$$

$$\mathbf{H}^{(3)}_i = (\exp\{\odot \mathbf{T}_{2i}\boldsymbol{\varphi}\} \otimes \mathbf{T}'_{1i})'\,\mathbf{W}^{(3)}_{c_i}(\mathbf{T}_{2i} \otimes \mathbf{T}_{2i} \otimes \mathbf{T}_{2i});$$

$\mathbf{N}_{\nu_j} = \frac{1}{2}\left(\mathbf{I}_{\nu_j^2} + \mathbf{I}_{(\nu_j,\nu_j)}\right)$; and $\mathbf{H}^{(1)}_i$ is defined in (13).

2.2.2 Derivatives of $\operatorname{vec}\mathbf{J}_{i\cdot}$ With Respect to $\boldsymbol{\tau}$

Analogous to $\mathbf{D}^{(1)}_{J_{i\cdot}}$ in (13), define $\mathbf{D}^{(2)}_{J_{i\cdot}}$ and $\mathbf{D}^{(3)}_{J_{i\cdot}}$ as

$$\mathbf{D}^{(2)}_{J_{i\cdot}} \overset{\text{def}}{=} \left.\frac{\partial^2 \operatorname{vec}\mathbf{J}_{i\cdot}}{\partial\, \boldsymbol{\tau}' \otimes \boldsymbol{\tau}'}\right|_{\boldsymbol{\tau}=\mathbf{0}} \text{ and } \mathbf{D}^{(3)}_{J_{i\cdot}} \overset{\text{def}}{=} \left.\frac{\partial^3 \operatorname{vec}\mathbf{J}_{i\cdot}}{\partial\, \boldsymbol{\tau}' \otimes \boldsymbol{\tau}' \otimes \boldsymbol{\tau}'}\right|_{\boldsymbol{\tau}=\mathbf{0}}.$$

Then,

$$\mathbf{D}^{(2)}_{J_{i\cdot}} = \sum_{j=1}^{d_i} (\mathbf{E}_{j,\mathbf{r}_i} \otimes \mathbf{E}_{j,\mathbf{r}_i})' \mathbf{D}^{(2)}_{J_{t_{ij}}} \left(\mathbf{E}_{t_{ij},\mathbf{u}} \otimes \mathbf{E}_{t_{ij},\mathbf{u}}\right) \text{ and} \qquad (31)$$

$$\mathbf{D}^{(3)}_{J_{i\cdot}} = \sum_{j=1}^{d_i} (\mathbf{E}_{j,\mathbf{r}_i} \otimes \mathbf{E}_{j,\mathbf{r}_i})' \mathbf{D}^{(3)}_{J_{t_{ij}}} \left(\mathbf{E}_{t_{ij},\mathbf{u}} \otimes \mathbf{E}_{t_{ij},\mathbf{u}} \otimes \mathbf{E}_{t_{ij},\mathbf{u}} \right);$$

where $\mathbf{u}$ is defined in (13); $\mathbf{E}_{j,\mathbf{r}_i}$ is defined in (12); $\mathbf{r}_i$ is given in (1); d_i is the order of $\mathbf{r}_i$; and $\mathbf{D}^{(2)}_{J_{t_{ij}}}$ and $\mathbf{D}^{(3)}_{J_{t_{ij}}}$ are computed by partitioning $\mathbf{J}_{t_{ij}}$ according to $\mathbf{m}_t$ in (8) and then using Theorem A1.

2.2.3 First-Order Derivatives of $\operatorname{vec} \boldsymbol{\Sigma}_i$ With Respect to $\boldsymbol{\theta}$

Assume that the model is parameterized as (27). First-order derivatives of $\operatorname{vec} \boldsymbol{\Sigma}_i$ with respect to the components of $\boldsymbol{\theta}$, evaluated at $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\tau} = \mathbf{0}$ are

$$\frac{\partial \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\mu}'} = 2\,\mathbf{N}_p (\boldsymbol{\Sigma}_i \boldsymbol{\Gamma}_0 \otimes \boldsymbol{\Gamma}_0) \mathbf{D}^{(1)}_G; \tag{32}$$

$$\frac{\partial \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\tau}'} = 2\,\mathbf{N}_p (\boldsymbol{\Sigma}_i \boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_i) \mathbf{D}^{(1)}_{J_{i\cdot}};$$

$$\frac{\partial \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\varphi}'} = (\boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_i)(\mathbf{L}\,\mathbf{D}^{(1)}_{\lambda_i} + \boldsymbol{\xi}_i \mathbf{w}_i'); \text{ and}$$

$$\frac{\partial \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\beta}'} = (\boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_i) \mathbf{T}_{3i} \pi_i;$$

where $\pi_i = \exp(\mathbf{w}_i' \boldsymbol{\varphi})$; $\boldsymbol{\xi}_i = \operatorname{vec} \boldsymbol{\Xi}_i$; $\mathbf{D}^{(1)}_G$ is given in Theorem A1; and $\mathbf{D}^{(1)}_{\lambda_i}$ and $\mathbf{D}^{(1)}_{J_{i\cdot}}$ are given in (13).

2.2.4 Second-Order Derivatives of $\operatorname{vec} \boldsymbol{\Sigma}_i$ With Respect to $\boldsymbol{\theta}$

To obtain expressions for second- and third-order derivatives, the identities

$$(\mathbf{ABC} \otimes \mathbf{D})\mathbf{E} = [\mathbf{A} \otimes \{\operatorname{vec}(\mathbf{C}')\}' \otimes \mathbf{D}] \{\operatorname{vec}(\mathbf{B}') \otimes \mathbf{E}\} \text{ and}$$

$$(\mathbf{A} \otimes \mathbf{BCD})\mathbf{E} = \{\mathbf{A} \otimes (\operatorname{vec} \mathbf{D})' \otimes \mathbf{B}\} (\mathbf{E} \otimes \operatorname{vec} \mathbf{C}),$$

were repeatedly used, where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ are any matrices that are conformable for multiplication.

Second-order derivatives of $\operatorname{vec} \boldsymbol{\Sigma}_i$ with respect to the components of $\boldsymbol{\theta}$, evaluated at $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\tau} = \mathbf{0}$ are

$$\begin{aligned} \frac{\partial^2 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\mu}' \otimes \partial \boldsymbol{\mu}'} &= 2\,\mathbf{N}_p \Big[(\boldsymbol{\Sigma}_i \boldsymbol{\Gamma}_0 \otimes \boldsymbol{\Gamma}_0)\, \mathbf{D}^{(2)}_G \\ &\quad + \{\boldsymbol{\Gamma}_0 \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot} \boldsymbol{\Lambda}_i^* \boldsymbol{\Psi}_{i\cdot}')' \otimes \boldsymbol{\Gamma}_0\} \left(\mathbf{I}_{(p,p)} \mathbf{D}^{(1)}_G \otimes \mathbf{D}^{(1)}_G \right) \Big]; \end{aligned} \tag{33}$$

$$\begin{aligned}
\frac{\partial^2 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\tau}' \otimes \partial \boldsymbol{\mu}'} &= 2\,\mathbf{N}_p \Big[\{\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot}\boldsymbol{\Lambda}_i^*)' \otimes \boldsymbol{\Gamma}_0\} \left(\mathbf{I}_{(p,p)}\mathbf{D}_{J_i}^{(1)} \otimes \mathbf{D}_G^{(1)}\right) \\
&\quad + \{\boldsymbol{\Sigma}_i\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot})' \otimes \boldsymbol{\Gamma}_0\} \left(\mathbf{D}_{J_i}^{(1)} \otimes \mathbf{D}_G^{(1)}\right) \Big];
\end{aligned}$$

$$\frac{\partial^2 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\mu}'} = 2\,\mathbf{N}_p \{\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot})' \otimes \boldsymbol{\Gamma}_0\} \left\{(\mathbf{L}\,\mathbf{D}_{\lambda_i}^{(1)} + \boldsymbol{\xi}_i\mathbf{w}_i') \otimes \mathbf{D}_G^{(1)}\right\};$$

$$\frac{\partial^2 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\beta}' \otimes \partial \boldsymbol{\mu}'} = 2\,\mathbf{N}_p \{\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot})' \otimes \boldsymbol{\Gamma}_0\} (\pi_i\mathbf{T}_{3i} \otimes \mathbf{D}_G^{(1)});$$

$$\begin{aligned}
\frac{\partial^2 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\tau}' \otimes \partial \boldsymbol{\tau}'} &= 2\,\mathbf{N}_p \Big[(\boldsymbol{\Sigma}_i\boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_i)\,\mathbf{D}_{J_i}^{(2)} \\
&\quad + \left\{\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \boldsymbol{\Lambda}_i^*)' \otimes \boldsymbol{\Gamma}_i\right\} \left(\mathbf{I}_{(p,p)}\mathbf{D}_{J_i}^{(1)} \otimes \mathbf{D}_{J_i}^{(1)}\right) \Big];
\end{aligned}$$

$$\frac{\partial^2 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\tau}'} = 2\,\mathbf{N}_p \{\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \mathbf{I}_p)' \otimes \boldsymbol{\Gamma}_i\} \left\{(\mathbf{L}\,\mathbf{D}_{\lambda_i}^{(1)} + \boldsymbol{\xi}_i\mathbf{w}_i') \otimes \mathbf{D}_{J_i}^{(1)}\right\};$$

$$\frac{\partial^2 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\beta}' \otimes \partial \boldsymbol{\tau}'} = 2\,\mathbf{N}_p \{\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \mathbf{I}_p)' \otimes \boldsymbol{\Gamma}_i\} (\pi_i\mathbf{T}_{3i} \otimes \mathbf{D}_{J_i}^{(1)});$$

$$\frac{\partial^2 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\varphi}'} = (\boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_i) \left\{\mathbf{L}\,\mathbf{D}_{\lambda_i}^{(2)} + \boldsymbol{\xi}_i(\mathbf{w}_i \otimes \mathbf{w}_i)'\right\};$$

$$\frac{\partial^2 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\beta}'} = (\boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_i)(\pi_i\mathbf{w}_i' \otimes \mathbf{T}_{3i}); \text{ and}$$

$$\frac{\partial^2 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\beta}' \otimes \partial \boldsymbol{\beta}'} = \mathbf{0};$$

where $\boldsymbol{\Lambda}_i^* = \boldsymbol{\Lambda}_i + \boldsymbol{\Xi}_i$; π_i and $\boldsymbol{\xi}_i$ are defined in (32); $\mathbf{D}_G^{(s)}$ for $s = 1, 2$ is given in Theorem A1; $\mathbf{D}_{\lambda_i}^{(1)}$ and $\mathbf{D}_{J_{i\cdot}}^{(1)}$ are given in (13); $\mathbf{D}_{\lambda_i}^{(2)}$ is given in (30); and $\mathbf{D}_{J_{i\cdot}}^{(2)}$ is given in (31).

2.2.5 Third-Order Derivatives of $\operatorname{vec} \boldsymbol{\Sigma}_i$ With Respect to $\boldsymbol{\theta}$

Third-order derivatives of $\operatorname{vec} \boldsymbol{\Sigma}_i$ with respect to the components of $\boldsymbol{\theta}$, evaluated at $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\tau} = \mathbf{0}$ are

$$\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\mu}' \otimes \partial \boldsymbol{\mu}' \otimes \boldsymbol{\mu}'} = 2\,\mathbf{N}_p \Big[(\boldsymbol{\Sigma}_i\boldsymbol{\Gamma}_0 \otimes \boldsymbol{\Gamma}_0)\,\mathbf{D}_G^{(3)}$$

$$
\begin{aligned}
&+\{\boldsymbol{\Gamma}_0 \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot}\boldsymbol{\Lambda}_i^* \boldsymbol{\Psi}_{i\cdot}')' \otimes \boldsymbol{\Gamma}_0\} \\
&\times \Big\{(\mathbf{I}_{(p,p)}\mathbf{D}_G^{(1)} \otimes \mathbf{D}_G^{(2)}) + 2(\mathbf{I}_{(p,p)}\mathbf{D}_G^{(2)} \otimes \mathbf{D}_G^{(1)})(\mathbf{I}_{\nu_1} \otimes \mathbf{N}_{\nu_1})\Big\}\Big];
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\mu}' \otimes \partial \boldsymbol{\tau}' \otimes \boldsymbol{\mu}'} &= 2\,\mathbf{N}_p \Big[\{\boldsymbol{\Gamma}_0 \otimes (\operatorname{vec} \boldsymbol{\Lambda}_i^* \boldsymbol{\Psi}_{i\cdot}')' \otimes \boldsymbol{\Gamma}_i\}\, (\mathbf{I}_{(p,p)}\mathbf{D}_G^{(2)} \otimes \mathbf{D}_{J_i}^{(1)})(\mathbf{I}_{\nu_1} \otimes \mathbf{I}_{(\nu_2,\nu_1)}) \\
&+ \{\boldsymbol{\Gamma}_0 \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot}')' \otimes \boldsymbol{\Sigma}_i \boldsymbol{\Gamma}_i\}\, (\mathbf{I}_{(p,p)}\mathbf{D}_G^{(2)} \otimes \mathbf{I}_{(p,p)}\mathbf{D}_{J_i}^{(1)})(\mathbf{I}_{\nu_1} \otimes \mathbf{I}_{(\nu_2,\nu_1)}) \\
&+ \{\boldsymbol{\Gamma}_0 \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot}')' \otimes \boldsymbol{\Gamma}_0\} \\
&\times \Big[\mathbf{I}_{(p,p)}\mathbf{D}_G^{(1)} \otimes \{\mathbf{I}_p \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot}\boldsymbol{\Lambda}_i^*)' \otimes \mathbf{I}_p\}\, (\mathbf{I}_{(p,p)}\mathbf{D}_{J_i}^{(1)} \otimes \mathbf{D}_G^{(1)})\Big] \\
&+ \{\boldsymbol{\Gamma}_0 \otimes (\operatorname{vec} \mathbf{I}_p)' \otimes \boldsymbol{\Gamma}_0\} \\
&\times \Big[\mathbf{I}_{(p,p)}\mathbf{D}_G^{(1)} \otimes \{\boldsymbol{\Psi}_{i\cdot}\boldsymbol{\Lambda}_i^* \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot})' \otimes \mathbf{I}_p\}\, (\mathbf{D}_{J_i}^{(1)} \otimes \mathbf{D}_G^{(1)})\Big]\Big];
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\mu}' \otimes \boldsymbol{\mu}'} &= 2\,\mathbf{N}_p \Big[\{\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot})' \otimes \boldsymbol{\Gamma}_0\} \Big\{(\mathbf{L}\,\mathbf{D}_{\lambda_i}^{(1)} + \boldsymbol{\xi}_i \mathbf{w}_i') \otimes \mathbf{D}_G^{(2)}\Big\} \\
&+ \Big(\Big[\operatorname{vec}\Big\{(\boldsymbol{\Psi}_{i\cdot} \otimes \boldsymbol{\Psi}_{i\cdot})(\mathbf{L}\,\mathbf{D}_{\lambda_i}^{(1)} + \boldsymbol{\xi}_i \mathbf{w}_i')\Big\}\Big]' \otimes \boldsymbol{\Gamma}_0 \otimes \boldsymbol{\Gamma}_0\Big) \\
&\times \Big\{\mathbf{I}_{\nu_3} \otimes \Big(\mathbf{I}_p \otimes \mathbf{I}_{(p,p)} \otimes \mathbf{I}_p\Big)\Big(\mathbf{D}_G^{(1)} \otimes \mathbf{D}_G^{(1)}\Big)\Big\}\Big];
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\beta}' \otimes \partial \boldsymbol{\mu}' \otimes \boldsymbol{\mu}'} &= 2\,\mathbf{N}_p \Big[\{\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot})' \otimes \boldsymbol{\Gamma}_0\} \Big(\pi_i \mathbf{T}_{3i} \otimes \mathbf{D}_G^{(2)}\Big) \\
&+ \Big([\operatorname{vec}\{(\boldsymbol{\Psi}_{i\cdot} \otimes \boldsymbol{\Psi}_{i\cdot})\pi_i \mathbf{T}_{3i}\}]' \otimes \boldsymbol{\Gamma}_0 \otimes \boldsymbol{\Gamma}_0\Big) \\
&\times \Big\{\mathbf{I}_{\nu_4} \otimes \Big(\mathbf{I}_p \otimes \mathbf{I}_{(p,p)} \otimes \mathbf{I}_p\Big)\Big(\mathbf{D}_G^{(1)} \otimes \mathbf{D}_G^{(1)}\Big)\Big\}\Big];
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\tau}' \otimes \partial \boldsymbol{\tau}' \otimes \boldsymbol{\mu}'} &= 2\,\mathbf{N}_p \Big[\{\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot}\boldsymbol{\Lambda}_i^*)' \otimes \boldsymbol{\Gamma}_0\}\, (\mathbf{I}_{(p,p)}\mathbf{D}_{J_i}^{(2)} \otimes \mathbf{D}_G^{(1)}) \\
&+ \{\boldsymbol{\Sigma}_i \boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot})' \otimes \boldsymbol{\Gamma}_0\}\, (\mathbf{D}_{J_i}^{(2)} \otimes \mathbf{D}_G^{(1)}) \\
&+ \Big(\Big[\operatorname{vec}\Big\{(\boldsymbol{\Lambda}_i^* \otimes \boldsymbol{\Psi}_{i\cdot})\,\mathbf{D}_{J_i}^{(1)}\Big\}\Big]' \otimes \boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_0\Big)
\end{aligned}
$$

$$
\begin{aligned}
& \times \Big\{ \mathbf{I}_{\nu_2} \otimes \Big(\mathbf{I}_p \otimes \mathbf{I}_{(p,p)} \otimes \mathbf{I}_p \Big) \Big(\mathbf{D}^{(1)}_{J_i} \otimes \mathbf{D}^{(1)}_{G} \Big) \Big\} \\
& + \{ \boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \boldsymbol{\Lambda}^*_i)' \otimes \boldsymbol{\Gamma}_0 \} \\
& \times \Big[\mathbf{I}_{(p,p)} \mathbf{D}^{(1)}_{J_i} \otimes \{ \mathbf{I}_p \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot})' \otimes \mathbf{I}_p \} \, (\mathbf{D}^{(1)}_{J_i} \otimes \mathbf{D}^{(1)}_{G}) \Big] \Big];
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\tau}' \otimes \boldsymbol{\mu}'} &= 2\, \mathbf{N}_p \Big[\Big(\Big[\operatorname{vec} \Big\{ (\mathbf{I}_p \otimes \boldsymbol{\Psi}_{i\cdot}) (\mathbf{L}\, \mathbf{D}^{(1)}_{\lambda_i} + \boldsymbol{\xi}_i \mathbf{w}'_i) \Big\} \Big]' \otimes \boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_0 \Big) \\
& \times \Big\{ \mathbf{I}_{\nu_3} \otimes \Big(\mathbf{I}_p \otimes \mathbf{I}_{(p,p)} \otimes \mathbf{I}_p \Big) \Big(\mathbf{D}^{(1)}_{J_i} \otimes \mathbf{D}^{(1)}_{G} \Big) \Big\} \\
& + \{ \boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \mathbf{I}_p)' \otimes \boldsymbol{\Gamma}_0 \} \\
& \times \Big[(\mathbf{L}\, \mathbf{D}^{(1)}_{\lambda_i} + \boldsymbol{\xi}_i \mathbf{w}'_i) \otimes \{ \mathbf{I}_p \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot})' \otimes \mathbf{I}_p \} \, (\mathbf{D}^{(1)}_{J_i} \otimes \mathbf{D}^{(1)}_{G}) \Big] \Big];
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\beta}' \otimes \partial \boldsymbol{\tau}' \otimes \boldsymbol{\mu}'} &= 2\, \mathbf{N}_p \Big[\Big([\operatorname{vec} \{ (\mathbf{I}_p \otimes \boldsymbol{\Psi}_{i\cdot}) \pi_i \mathbf{T}_{3i} \}]' \otimes \boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_0 \Big) \\
& \times \Big\{ \mathbf{I}_{\nu_4} \otimes \Big(\mathbf{I}_p \otimes \mathbf{I}_{(p,p)} \otimes \mathbf{I}_p \Big) \Big(\mathbf{D}^{(1)}_{J_i} \otimes \mathbf{D}^{(1)}_{G} \Big) \Big\} \\
& + \{ \boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \mathbf{I}_p)' \otimes \boldsymbol{\Gamma}_0 \} \\
& \times \Big[\pi_i \mathbf{T}_{3i} \otimes \{ \mathbf{I}_p \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot})' \otimes \mathbf{I}_p \} \, (\mathbf{D}^{(1)}_{J_i} \otimes \mathbf{D}^{(1)}_{G}) \Big] \Big];
\end{aligned}
$$

$$
\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\varphi}' \otimes \boldsymbol{\mu}'} = 2\, \mathbf{N}_p \{ \boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot})' \otimes \boldsymbol{\Gamma}_0 \} \Big[\Big\{ \mathbf{L}\, \mathbf{D}^{(2)}_{\lambda_i} + (\boldsymbol{\xi}_i \mathbf{w}'_i \otimes \mathbf{w}'_i) \Big\} \otimes \mathbf{D}^{(1)}_{G} \Big];
$$

$$
\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\beta}' \otimes \partial \boldsymbol{\varphi}' \otimes \boldsymbol{\mu}'} = 2\, \mathbf{N}_p \{ \boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \boldsymbol{\Psi}_{i\cdot})' \otimes \boldsymbol{\Gamma}_0 \} \, (\pi_i \mathbf{T}_{3i} \otimes \mathbf{w}'_i \otimes \mathbf{D}^{(1)}_{G});
$$

$$
\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\mu}' \otimes \partial \boldsymbol{\beta}' \otimes \boldsymbol{\beta}'} = \mathbf{0};
$$

$$
\begin{aligned}
\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\tau}' \otimes \partial \boldsymbol{\tau}' \otimes \boldsymbol{\tau}'} &= 2\, \mathbf{N}_p \Big[(\boldsymbol{\Sigma}_i \boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_i) \, \mathbf{D}^{(3)}_{J_i} + \{ \boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \boldsymbol{\Lambda}^*_i)' \otimes \boldsymbol{\Gamma}_i \} \\
& \times \Big\{ (\mathbf{I}_{(p,p)} \mathbf{D}^{(1)}_{J_i} \otimes \mathbf{D}^{(2)}_{J_i}) + 2 (\mathbf{I}_{(p,p)} \mathbf{D}^{(2)}_{J_i} \otimes \mathbf{D}^{(1)}_{J_i}) (\mathbf{I}_{\nu_2} \otimes \mathbf{N}_{\nu_2}) \Big\} \Big];
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\tau}' \otimes \boldsymbol{\tau}'} &= 2\,\mathbf{N}_p \Big[\{\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \mathbf{I}_p)' \otimes \boldsymbol{\Gamma}_i\} \Big\{ (\mathbf{L}\,\mathbf{D}^{(1)}_{\lambda_i} + \boldsymbol{\xi}_i \mathbf{w}_i') \otimes \mathbf{D}^{(2)}_{J_i} \Big\} \\
&\quad + \Big[\Big\{ \operatorname{vec}(\mathbf{L}\,\mathbf{D}^{(1)}_{\lambda_i} + \boldsymbol{\xi}_i \mathbf{w}_i') \Big\}' \otimes \boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_i \Big] \\
&\quad \times \Big\{ \mathbf{I}_{\nu_3} \otimes \Big(\mathbf{I}_p \otimes \mathbf{I}_{(p,p)} \otimes \mathbf{I}_p \Big) \Big(\mathbf{D}^{(1)}_{J_i} \otimes \mathbf{D}^{(1)}_{J_i} \Big) \Big\} \Big];
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\beta}' \otimes \partial \boldsymbol{\tau}' \otimes \boldsymbol{\tau}'} &= 2\,\mathbf{N}_p \Big[\{\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \mathbf{I}_p)' \otimes \boldsymbol{\Gamma}_i\} \Big(\pi_i \mathbf{T}_{3i} \otimes \mathbf{D}^{(2)}_{J_i} \Big) \\
&\quad + \Big\{ (\operatorname{vec} \pi_i \mathbf{T}_{3i})' \otimes \boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_i \Big\} \\
&\quad \times \Big\{ \mathbf{I}_{\nu_4} \otimes \Big(\mathbf{I}_p \otimes \mathbf{I}_{(p,p)} \otimes \mathbf{I}_p \Big) \Big(\mathbf{D}^{(1)}_{J_i} \otimes \mathbf{D}^{(1)}_{J_i} \Big) \Big\} \Big];
\end{aligned}$$

$$\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\varphi}' \otimes \boldsymbol{\tau}'} = 2\,\mathbf{N}_p \{\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \mathbf{I}_p)' \otimes \boldsymbol{\Gamma}_i\} \Big[\Big\{ \mathbf{L}\,\mathbf{D}^{(2)}_{\lambda_i} + (\boldsymbol{\xi}_i \mathbf{w}_i' \otimes \mathbf{w}_i') \Big\} \otimes \mathbf{D}^{(1)}_{J_i} \Big];$$

$$\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\beta}' \otimes \partial \boldsymbol{\varphi}' \otimes \boldsymbol{\tau}'} = 2\,\mathbf{N}_p \{\boldsymbol{\Gamma}_i \otimes (\operatorname{vec} \mathbf{I}_p)' \otimes \boldsymbol{\Gamma}_i\} \Big(\pi_i \mathbf{T}_{3i} \otimes \mathbf{w}_i' \otimes \mathbf{D}^{(1)}_{J_i} \Big);$$

$$\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\tau}' \otimes \partial \boldsymbol{\beta}' \otimes \boldsymbol{\beta}'} = \mathbf{0};$$

$$\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\varphi}' \otimes \boldsymbol{\varphi}'} = (\boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_i) \Big\{ \mathbf{L}\,\mathbf{D}^{(3)}_{\lambda_i} + (\boldsymbol{\xi}_i \mathbf{w}_i' \otimes \mathbf{w}_i' \otimes \mathbf{w}_i') \Big\};$$

$$\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\beta}' \otimes \partial \boldsymbol{\varphi}' \otimes \boldsymbol{\varphi}'} = (\boldsymbol{\Gamma}_i \otimes \boldsymbol{\Gamma}_i)(\pi_i \mathbf{T}_{3i} \otimes \mathbf{w}_i' \otimes \mathbf{w}_i');$$

$$\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\varphi}' \otimes \partial \boldsymbol{\beta}' \otimes \boldsymbol{\beta}'} = \mathbf{0};\ \text{and}$$

$$\frac{\partial^3 \operatorname{vec} \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\beta}' \otimes \partial \boldsymbol{\beta}' \otimes \boldsymbol{\beta}'} = \mathbf{0};$$

where $\mathbf{N}_{\nu_j}$ is defined in (30); $\boldsymbol{\Lambda}_i^*$ is defined in (33); π_i and $\boldsymbol{\xi}_i$ are defined in (32); $\mathbf{D}^{(s)}_G$ for $s = 1, 2, 3$ is given in Theorem A1; $\mathbf{D}^{(1)}_{\lambda_i}$ and $\mathbf{D}^{(1)}_{J_i.}$ are given in (13); $\mathbf{D}^{(s)}_{\lambda_i}$ for $s = 2, 3$ is given in (30); and $\mathbf{D}^{(s)}_{J_i.}$ for $s = 2, 3$ is given in (31).

3 Third- and Fourth-Order Moments of the Sample Covariance Matrix: Proof of Theorem A2

Define $D(\widehat{\boldsymbol{\theta}};\boldsymbol{\theta})$ as $D(\widehat{\boldsymbol{\theta}};\boldsymbol{\theta}) \stackrel{\text{def}}{=} 2\{\ell(\widehat{\boldsymbol{\theta}}) - \ell(\boldsymbol{\theta})\}$, where $\ell(\boldsymbol{\theta})$ is the log likelihood function in (10). The statistic $D(\widehat{\boldsymbol{\theta}};\boldsymbol{\theta})$ is a multiplicative function of the random matrices $\mathbf{Z}_i$ for $i = 1, \ldots, 3$ (see equations 14 and 26). To compute the expectation of $D(\widehat{\boldsymbol{\theta}};\boldsymbol{\theta})$, third- and fourth-order moments of $\mathbf{s}_i - \boldsymbol{\sigma}_i$ must be obtained, at least to order $O(n_i^{-2})$. The expectation of the fourth moment of $\mathbf{s}_i - \boldsymbol{\sigma}_i$ consists of terms of order $O(n_i^{-2})$ and terms of order $O(n_i^{-3})$. The $O(n_i^{-2})$ term is easily obtained by using

$$\sqrt{n_i}\,\text{vec}(\mathbf{S}_i - \boldsymbol{\Sigma}_i) \sim \mathbf{u}_i + O_p\left(n_i^{-\frac{1}{2}}\right), \text{ where } \mathbf{u}_i \sim \text{N}\left\{\mathbf{0}, 2\mathbf{N}_p(\boldsymbol{\Sigma}_i \otimes \boldsymbol{\Sigma}_i)\right\}.$$

Nonetheless, an exact expression for the fourth moment of $\mathbf{s}_i - \boldsymbol{\sigma}_i$ can be obtained and it is given here for completeness. Magnus and Neudecker (1979, theorem 4.1) gave a rather complicated expression for sixth-order moments of normal random vectors. Their expression could be used to obtain the third-order moments of $\mathbf{s}_i - \boldsymbol{\sigma}_i$, but the algebra would be very messy. It is easier to get these moments directly from the cumulant generating function.

Define $\mathbf{S}$ as $\mathbf{S} \stackrel{\text{def}}{=} n^{-1}\sum_{i=1}^{n}\mathbf{u}_i\mathbf{u}_i'$, where $\mathbf{u}_i \sim \text{iid N}(\mathbf{0}, \boldsymbol{\Sigma})$; $\boldsymbol{\Sigma}$ is a $p \times p$ positive semi-definite matrix and $n \geq 1$. Without loss of generality, it can be assumed that $\boldsymbol{\Sigma}$ has full-rank. If $\text{rank}(\boldsymbol{\Sigma}) = r < p$, then replace $\mathbf{u}_i$ by $\mathbf{u}_i^* = \mathbf{V}'\mathbf{u}_i$ and replace $\mathbf{S}$ by $\mathbf{S}^*$, where $\boldsymbol{\Sigma} = \mathbf{V}\mathbf{D}\mathbf{V}'$; $\mathbf{V}$ is $p \times r$; $\mathbf{V}'\mathbf{V} = \mathbf{I}_r$; $\mathbf{u}_i^* \sim \text{iid N}(\mathbf{0}, \mathbf{D})$; and $\mathbf{S}^* \stackrel{\text{def}}{=} n^{-1}\sum_{i=1}^{b}\mathbf{u}_i^*\mathbf{u}_i^{*\prime}$. Moments of $\text{vec}\,\mathbf{S}$ can be obtained from moments of $\text{vec}\,\mathbf{S}^*$ because $\mathbf{S} = \mathbf{V}\mathbf{S}^*\mathbf{V}'$.

Let $\text{vech}(\,\cdot\,)$ be the vector-half operator that stacks the distinct elements of a symmetric matrix (Searle, 1982, p. 332). It is readily shown that the cumulant generating function of $\text{vech}(\mathbf{S})$ is

$$\begin{aligned}\mathbf{C}^{(0)}(\mathbf{t}) &= \frac{n}{2}\ln|\boldsymbol{\Sigma}^*| - \frac{n}{2}\ln|\boldsymbol{\Sigma}|; \text{ where} \\ \text{vech}\,\boldsymbol{\Sigma}^{*-1} &= \boldsymbol{\sigma}^* - \frac{2}{n}(\mathbf{D}_p'\mathbf{D}_p)^{-1}\mathbf{t}; \quad \boldsymbol{\sigma}^* = \text{vech}\,\boldsymbol{\Sigma}^{-1};\end{aligned}$$

and $\mathbf{D}_p$ is the duplication matrix (Magnus and Neudecker, 1999). The first four derivatives of the cumulant generating function are the following:

$$\begin{aligned}\mathbf{C}^{(1)}(\mathbf{t}) &= \frac{\partial\,\mathbf{C}^{(0)}(\mathbf{t})}{\partial\mathbf{t}} = \mathbf{H}_p\,\text{vec}\,\boldsymbol{\Sigma}^*; \\ \mathbf{C}^{(2)}(\mathbf{t}) &= \frac{\partial^2\,\mathbf{C}^{(0)}(\mathbf{t})}{\partial\mathbf{t}' \otimes \partial\mathbf{t}} = \frac{2}{n}\mathbf{H}_p(\boldsymbol{\Sigma}^* \otimes \boldsymbol{\Sigma}^*)\mathbf{H}_p';\end{aligned}$$

$$
\begin{aligned}
\mathbf{C}^{(3)}(\mathbf{t}) &= \frac{\partial^3 \mathbf{C}^{(0)}(\mathbf{t})}{\partial \mathbf{t}' \otimes \partial \mathbf{t}' \otimes \partial \mathbf{t}} \qquad (34)\\
&= \frac{8}{n^2}\mathbf{H}_p \left\{\boldsymbol{\Sigma}^* \otimes (\operatorname{vec}\boldsymbol{\Sigma}^*)' \otimes \boldsymbol{\Sigma}^*\right\}(\mathbf{H}_p \otimes \mathbf{H}_p)'; \text{ and}\\
\mathbf{C}^{(4)}(\mathbf{t}) &= \frac{\partial^4 \mathbf{C}^{(0)}(\mathbf{t})}{\partial \mathbf{t} \otimes \partial \mathbf{t}' \otimes \partial \mathbf{t}' \otimes \partial \mathbf{t}} \qquad (35)\\
&= \frac{32}{n^3}(\mathbf{H}_p \otimes \mathbf{H}_p)\mathbf{N}_{p^2}\left\{\boldsymbol{\Sigma}^* \otimes (\operatorname{vec}\boldsymbol{\Sigma}^*)' \otimes \operatorname{vec}\boldsymbol{\Sigma}^* \otimes \boldsymbol{\Sigma}^*\right\}(\mathbf{H}_p \otimes \mathbf{H}_p)'\\
&\quad + \frac{16}{n^3}(\mathbf{H}_p \otimes \mathbf{H}_p)(\mathbf{I}_p \otimes \mathbf{I}_{(p,p)} \otimes \mathbf{I}_p)(\boldsymbol{\Sigma}^* \otimes \boldsymbol{\Sigma}^* \otimes \boldsymbol{\Sigma}^* \otimes \boldsymbol{\Sigma}^*)(\mathbf{H}_p \otimes \mathbf{H}_p)';
\end{aligned}
$$

where $\mathbf{H}_p = (\mathbf{D}_p'\mathbf{D}_p)^{-1}\mathbf{D}_p'$ and $\mathbf{N}_{p^2} = \frac{1}{2}(\mathbf{I}_{(p^2,p^2)} + \mathbf{I}_{p^4})$. Note that $\mathbf{D}_p$ and $\mathbf{H}_p$ satisfy $\mathbf{D}_p \operatorname{vech}\mathbf{A} = \operatorname{vec}\mathbf{A}$, $\mathbf{H}_p \operatorname{vec}\mathbf{A} = \operatorname{vech}\mathbf{A}$, and $\mathbf{D}_p\mathbf{H}_p = \mathbf{N}_p$ where $\mathbf{A}$ is any $p \times p$ symmetric matrix and $\mathbf{N}_p$ is defined in (13). The matrix $\boldsymbol{\Sigma}^*$, evaluated at $\mathbf{t} = \mathbf{0}$, simplifies to $\boldsymbol{\Sigma}$. Denote $\operatorname{vec}(\mathbf{S} - \boldsymbol{\Sigma})$ by $\mathbf{s} - \boldsymbol{\sigma}$. It can be shown by straightforward algebra that the third and fourth cumulants of $\operatorname{vech}\mathbf{S}$ are

$$
\mathbf{C}^{(3)}(\mathbf{0}) = \mathbf{H}_p\mathrm{E}\{(\mathbf{s}-\boldsymbol{\sigma})' \otimes (\mathbf{s}-\boldsymbol{\sigma})(\mathbf{s}-\boldsymbol{\sigma})'\}(\mathbf{H}_p \otimes \mathbf{H}_p)' \text{ and} \qquad (36)
$$

$$
\begin{aligned}
\mathbf{C}^{(4)}(\mathbf{0}) &= (\mathbf{H}_p \otimes \mathbf{H}_p)\mathrm{E}\{(\mathbf{s}-\boldsymbol{\sigma})(\mathbf{s}-\boldsymbol{\sigma})' \otimes (\mathbf{s}-\boldsymbol{\sigma})(\mathbf{s}-\boldsymbol{\sigma})'\}(\mathbf{H}_p \otimes \mathbf{H}_p)' \qquad (37)\\
&\quad - 2(\mathbf{H}_p \otimes \mathbf{H}_p)\mathbf{N}_{p^2}\left[\mathrm{E}\{(\mathbf{s}-\boldsymbol{\sigma})(\mathbf{s}-\boldsymbol{\sigma})'\} \otimes \mathrm{E}\{(\mathbf{s}-\boldsymbol{\sigma})(\mathbf{s}-\boldsymbol{\sigma})'\}\right](\mathbf{H}_p \otimes \mathbf{H}_p)'\\
&\quad - (\mathbf{H}_p \otimes \mathbf{H}_p)\mathrm{E}\left\{(\mathbf{s}-\boldsymbol{\sigma}) \otimes (\mathbf{s}-\boldsymbol{\sigma})\right\}\mathrm{E}\left\{(\mathbf{s}-\boldsymbol{\sigma})' \otimes (\mathbf{s}-\boldsymbol{\sigma})'\right\}(\mathbf{H}_p \otimes \mathbf{H}_p)'.
\end{aligned}
$$

Pre-multiplying $\mathbf{C}^{(3)}(\mathbf{t})$ in (34) by $\mathbf{D}_p$; post-multiplying by $(\mathbf{D}_p \otimes \mathbf{D}_p)'$; and then using equation (36) reveals that

$$
\mathrm{E}\{(\mathbf{s}-\boldsymbol{\sigma})' \otimes (\mathbf{s}-\boldsymbol{\sigma})(\mathbf{s}-\boldsymbol{\sigma})'\} = \frac{8}{n^2}\mathbf{N}_p\left\{\boldsymbol{\Sigma} \otimes (\operatorname{vec}\boldsymbol{\Sigma})' \otimes \boldsymbol{\Sigma}\right\}(\mathbf{N}_p \otimes \mathbf{N}_p).
$$

Similarly, pre-multiplying $\mathbf{C}^{(4)}(\mathbf{t})$ in (35) by $(\mathbf{D}_p \otimes \mathbf{D}_p)$; post-multiplying by $(\mathbf{D}_p \otimes \mathbf{D}_p)'$; and then using equation (37) reveals that

$$
\begin{aligned}
&\mathrm{E}\{(\mathbf{s}-\boldsymbol{\sigma})(\mathbf{s}-\boldsymbol{\sigma})' \otimes (\mathbf{s}-\boldsymbol{\sigma})(\mathbf{s}-\boldsymbol{\sigma})'\} =\\
&\quad \frac{32}{n^3}\mathbf{N}_{p^2}(\mathbf{N}_p \otimes \mathbf{N}_p)\left\{\boldsymbol{\Sigma} \otimes (\operatorname{vec}\boldsymbol{\Sigma})' \otimes \operatorname{vec}\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}\right\}(\mathbf{N}_p \otimes \mathbf{N}_p)\\
&\quad + \frac{16}{n^3}(\mathbf{N}_p \otimes \mathbf{N}_p)(\mathbf{I}_p \otimes \mathbf{I}_{(p,p)} \otimes \mathbf{I}_p)(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})(\mathbf{N}_p \otimes \mathbf{N}_p)
\end{aligned}
$$

$$+\ \frac{8}{n^2}\mathbf{N}_{p^2}(\mathbf{N}_p\otimes\mathbf{N}_p)\left(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}\right)(\mathbf{N}_p\otimes\mathbf{N}_p)$$

$$+\ \frac{4}{n^2}(\mathbf{N}_p\otimes\mathbf{N}_p)\,\mathrm{vec}(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})\left\{\mathrm{vec}(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})\right\}'(\mathbf{N}_p\otimes\mathbf{N}_p),$$

where $\mathbf{N}_{p^2}=\frac{1}{2}\left(\mathbf{I}_{p^4}+\mathbf{I}_{(p^2,p^2)}\right)$.

4 Matrix Expressions for $\mathbf{Z}_j$ and $\mathbf{K}_j$

The terms $\mathbf{Z}_j$ for $j=1,\ldots$ in (14) are linear functions of $\mathbf{s}_i-\boldsymbol{\sigma}_i=\mathrm{vec}(\mathbf{S}_i-\boldsymbol{\Sigma}_i)$ whose coefficients depend on the derivatives of $\boldsymbol{\Sigma}_i$ with respect to $\boldsymbol{\theta}$. For convenience, define $\ddot{\mathbf{F}}_i^{(1)}$, $\ddot{\mathbf{F}}_i^{(2)}$, $\dddot{\mathbf{F}}_i^{(1)}$, and $\dddot{\mathbf{F}}_i^{(2)}$ as

$$\begin{aligned}
\ddot{\mathbf{F}}_i^{(1)} &= \left(\boldsymbol{\Sigma}_i^{-1}\otimes\boldsymbol{\Sigma}_i^{-1}\right)\mathbf{F}_i^{(1)};\quad \ddot{\mathbf{F}}_i^{(2)}=\left(\boldsymbol{\Sigma}_i^{-1}\otimes\boldsymbol{\Sigma}_i^{-1}\right)\mathbf{F}_i^{(2)};\\
\dddot{\mathbf{F}}_i^{(1)} &= \left(\boldsymbol{\Sigma}_i^{-1}\otimes\mathrm{vec}\,\boldsymbol{\Sigma}_i^{-1}\otimes\boldsymbol{\Sigma}_i^{-1}\right)\mathbf{F}_i^{(1)}\ \text{ and }\ \dddot{\mathbf{F}}_i^{(2)}=\left(\boldsymbol{\Sigma}_i^{-1}\otimes\mathrm{vec}\,\boldsymbol{\Sigma}_i^{-1}\otimes\boldsymbol{\Sigma}_i^{-1}\right)\mathbf{F}_i^{(2)}.
\end{aligned}$$

Expressions for $\mathbf{Z}_j$ and $\mathbf{K}_j$ are as follows:

$$\begin{aligned}
\mathbf{Z}_1 &= \sum_{i=1}^{g}\frac{n_i}{2\bar{n}^{\frac{1}{2}}}\,\ddot{\mathbf{F}}_i^{(1)\prime}\left(\mathbf{s}_i-\boldsymbol{\sigma}_i\right);\\
\mathbf{Z}_2 &= \sum_{i=1}^{g}\frac{n_i}{2\bar{n}^{\frac{1}{2}}}\,\mathbf{F}_i^{(11)\prime}\left\{\mathbf{I}_d\otimes\left(\boldsymbol{\Sigma}_i^{-1}\otimes\boldsymbol{\Sigma}_i^{-1}\right)(\mathbf{s}_i-\boldsymbol{\sigma}_i)\right\}\\
&- \sum_{i=1}^{g}\frac{n_i}{\bar{n}^{\frac{1}{2}}}\,\dddot{\mathbf{F}}_i^{(1)\prime}\left\{\mathbf{F}_i^{(1)}\otimes(\mathbf{s}_i-\boldsymbol{\sigma}_i)\right\};\\
\mathbf{Z}_3 &= \sum_{i=1}^{g}\frac{n_i}{\bar{n}^{\frac{1}{2}}}\left(\mathrm{vec}\,\ddot{\mathbf{F}}_i^{(1)}\otimes\ddot{\mathbf{F}}_i^{(1)}\right)'\left(\mathbf{I}_{pd}\otimes\mathbf{I}_{(p,p)}\otimes\mathbf{I}_p\right)\left\{\mathbf{I}_d\otimes\mathbf{F}_i^{(1)}\otimes(\mathbf{s}_i-\boldsymbol{\sigma}_i)\right\}\\
&+ 2\sum_{i=1}^{g}\frac{n_i}{\bar{n}^{\frac{1}{2}}}\,\dddot{\mathbf{F}}_i^{(1)\prime}\left[\mathbf{F}_i^{(1)}\otimes\mathbf{N}_p\left(\mathbf{I}_p\otimes\mathrm{vec}\,\boldsymbol{\Sigma}_i^{-1}\otimes\mathbf{I}_p\right)'\left\{\mathbf{F}_i^{(1)}\otimes(\mathbf{s}_i-\boldsymbol{\sigma}_i)\right\}\right]\\
&+ \sum_{i=1}^{g}\frac{n_i}{2\bar{n}^{\frac{1}{2}}}\,\mathbf{F}_i^{(111)\prime}\left\{\mathbf{I}_{d^2}\otimes\left(\boldsymbol{\Sigma}_i^{-1}\otimes\boldsymbol{\Sigma}_i^{-1}\right)(\mathbf{s}_i-\boldsymbol{\sigma}_i)\right\}\\
&- 2\sum_{i=1}^{g}\frac{n_i}{\bar{n}^{\frac{1}{2}}}\,\mathbf{F}_i^{(11)\prime}\left(\mathbf{I}_d\otimes\boldsymbol{\Sigma}_i^{-1}\otimes\mathrm{vec}\,\boldsymbol{\Sigma}_i^{-1}\otimes\boldsymbol{\Sigma}_i^{-1}\right)'\left\{\left(\mathbf{I}_d\otimes\mathbf{F}_i^{(1)}\right)\mathbf{N}_d\otimes(\mathbf{s}_i-\boldsymbol{\sigma}_i)\right\}
\end{aligned}$$

$$
\begin{aligned}
& - && \sum_{i=1}^{g} \frac{n_i}{\bar{n}^{\frac{1}{2}}} \dddot{\mathbf{F}}_i^{(1)\prime} \left\{ \mathbf{F}_i^{(2)} \otimes (\mathbf{s}_i - \boldsymbol{\sigma}_i) \right\}; \\
\mathbf{K}_2 & = && -\sum_{i=1}^{g} \frac{n_i}{2\bar{n}} \ddot{\mathbf{F}}_i^{(1)\prime} \mathbf{F}_i^{(1)} = -\bar{\mathbf{I}}_{\boldsymbol{\theta}}; \\
\mathbf{K}_3 & = && 2\sum_{i=1}^{g} \frac{n_i}{\bar{n}} \dddot{\mathbf{F}}_i^{(1)\prime} \left(\mathbf{F}_i^{(1)} \otimes \mathbf{F}_i^{(1)} \right) \mathbf{N}_d - \sum_{i=1}^{g} \frac{n_i}{\bar{n}} \mathbf{F}_i^{(11)\prime} \left(\mathbf{I}_d \otimes \ddot{\mathbf{F}}_i^{(1)} \right) \mathbf{N}_d \\
& - && \sum_{i=1}^{g} \frac{n_i}{2\bar{n}} \ddot{\mathbf{F}}_i^{(1)\prime} \mathbf{F}_i^{(2)}; \text{ and} \\
\mathbf{K}_4 & = && \sum_{i=1}^{g} \frac{n_i}{\bar{n}} \mathbf{F}_i^{(11)\prime} \left\{ \mathbf{I}_d \otimes \left(\boldsymbol{\Sigma}_i^{-1} \otimes \operatorname{vec} \boldsymbol{\Sigma}_i^{-1} \otimes \boldsymbol{\Sigma}_i^{-1} \right)' \left(\mathbf{F}_i^{(1)} \otimes \mathbf{F}_i^{(1)} \right) \right\} \\
& \times && \left\{ 4 \left(\mathbf{N}_d \otimes \mathbf{I}_d \right) + 2\mathbf{I}_{(d^2,d)} \right\} \\
& + && 4\sum_{i=1}^{g} \frac{n_i}{\bar{n}} \dddot{\mathbf{F}}_i^{(1)\prime} \left(\mathbf{F}_i^{(2)} \otimes \mathbf{F}_i^{(1)} \right) \left\{ \left(\mathbf{I}_d \otimes \mathbf{N}_d \right) + \frac{1}{2} \mathbf{I}_{(d,d^2)} \right\} \\
& - && \sum_{i=1}^{g} \frac{n_i}{\bar{n}} \mathbf{F}_i^{(1)\prime} (\boldsymbol{\Sigma}_i^{-1} \otimes \operatorname{vec} \boldsymbol{\Sigma}_i^{-1} \otimes \operatorname{vec} \boldsymbol{\Sigma}_i^{-1} \otimes \boldsymbol{\Sigma}_i^{-1})' \left(\mathbf{F}_i^{(1)} \otimes \mathbf{F}_i^{(1)} \otimes \mathbf{F}_i^{(1)} \right) \\
& \times && \left\{ 2\mathbf{I}_{(d,d^2)} + 3\mathbf{I}_{(d^2,d)} + 4 \left(\mathbf{N}_d \otimes \mathbf{I}_d \right) \right\} \\
& - && \sum_{i=1}^{g} \frac{n_i}{\bar{n}} \mathbf{F}_i^{(111)\prime} \left(\mathbf{I}_{d^2} \otimes \ddot{\mathbf{F}}_i^{(1)} \right) \left\{ \left(\mathbf{I}_d \otimes \mathbf{N}_d \right) + \frac{1}{2} \mathbf{I}_{(d,d^2)} \right\} \\
& - && \sum_{i=1}^{g} \frac{n_i}{\bar{n}} \mathbf{F}_i^{(11)\prime} \left(\mathbf{I}_d \otimes \ddot{\mathbf{F}}_i^{(2)} \right) \left\{ \left(\mathbf{N}_d \otimes \mathbf{I}_d \right) + \frac{1}{2} \mathbf{I}_{(d^2,d)} \right\} - \sum_{i=1}^{g} \frac{n_i}{2\bar{n}} \ddot{\mathbf{F}}_i^{(1)\prime} \mathbf{F}_i^{(3)};
\end{aligned}
$$

where $\mathbf{N}_d = \frac{1}{2}\left(\mathbf{I}_d + \mathbf{I}_{(d,d)}\right)$. In the current application, the expressions for $\mathbf{K}_3$ and $\mathbf{K}_4$ can be simplified because the Bartlett correction depends on these terms only through the expectations of functions such as $\mathbf{Z}_1^{*\prime}\mathbf{K}_3(\mathbf{Z}_1^* \otimes \mathbf{Z}_1^*)$ and $\mathbf{Z}_1^{*\prime}\mathbf{K}_4(\mathbf{Z}_1^* \otimes \mathbf{Z}_1^* \otimes \mathbf{Z}_1^*)$; see equation 26 for details. The simplified quantities are denoted by $\mathbf{K}_3^*$ and $\mathbf{K}_4^*$ and are defined in the next section.

5 Expectation of Likelihood Ratio Statistic

Evaluation of the expectation of the right-hand-side of equation 26 reveals that the expectation of the likelihood ratio statistic is the following:

$$\mathrm{E}\left\{D\left(\widehat{\boldsymbol{\theta}};\boldsymbol{\theta}\right)\right\} = \dim(\boldsymbol{\theta}) + \frac{1}{\bar{n}}\sum_{i=1}^{10}\zeta_i + O\left(\bar{n}^{-2}\right),$$

where

$$\begin{aligned}
\zeta_1 &= \sum_{i=1}^{g}\frac{n_i}{\bar{n}}\,\mathrm{tr}\left\{\left(\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\,\ddot{\mathbf{F}}_i^{(1)\prime}\otimes\mathbf{N}_p\right)\left(\mathbf{I}_p\otimes\mathrm{vec}\,\boldsymbol{\Sigma}_i\otimes\mathbf{I}_p\right)\mathbf{F}_i^{(1)}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\,\mathbf{U}_i^{(1)}\right\};\\
\mathbf{U}_i^{(1)} &= \mathbf{F}_i^{(11)\prime}\left(\mathbf{I}_d\otimes\boldsymbol{\Sigma}_i^{-1}\otimes\boldsymbol{\Sigma}_i^{-1}\right) - 2\,\dddot{\mathbf{F}}_i^{(1)\prime}\left(\mathbf{F}_i^{(1)}\otimes\mathbf{I}_{p^2}\right);\\
\zeta_2 &= \sum_{i=1}^{g}\frac{n_i}{3\bar{n}}\,\mathrm{tr}\left\{\left(\mathbf{F}_i^{(1)}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\otimes\mathbf{F}_i^{(1)}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\right)'\,\dddot{\mathbf{F}}_i^{(1)}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\,\mathbf{K}_3^*\right\};\\
\mathbf{K}_3^* &= \sum_{i=1}^{g}\frac{n_i}{\bar{n}}\left\{2\,\dddot{\mathbf{F}}_i^{(1)\prime}\left(\mathbf{F}_i^{(1)}\otimes\mathbf{F}_i^{(1)}\right) - \mathrm{dvec}\left(\ddot{\mathbf{F}}_i^{(1)\prime}\,\mathbf{F}_i^{(2)},d^2,d\right)' - \frac{1}{2}\,\ddot{\mathbf{F}}_i^{(1)\prime}\,\mathbf{F}_i^{(2)}\right\};\\
\zeta_3 &= \sum_{i=1}^{g}\frac{n_i}{2\bar{n}}\,\mathrm{tr}\left[\left\{\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\otimes\mathbf{N}_p\left(\boldsymbol{\Sigma}_i\otimes\boldsymbol{\Sigma}_i\right)\right\}\mathbf{U}_i^{(1)\prime}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\,\mathbf{U}_i^{(1)}\right];\\
\zeta_4 &= \mathbf{L}_1'\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\,\mathbf{L}_1 + \mathbf{L}_2'\left(\mathbf{I}_{(d,d)}\otimes\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\right)\mathbf{L}_2;\\
\mathbf{L}_1 &= \sum_{i=1}^{g}\frac{n_i}{2\bar{n}}\,\mathbf{U}_i^{(1)}\,\mathrm{vec}\left(\mathbf{F}_i^{(1)}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\right);\quad \mathbf{L}_2 = \sum_{i=1}^{g}\frac{n_i}{2\bar{n}}\,\mathrm{vec}\left\{\mathbf{U}_i^{(1)}\left(\mathbf{I}_d\otimes\mathbf{F}_i^{(1)}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\right)\right\};\\
\zeta_5 &= \frac{1}{2}\,\mathrm{tr}\left\{\mathbf{N}_d\left(\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\otimes\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\right)\mathbf{K}_3^{*\prime}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\,\mathbf{K}_3^*\right\} + \frac{1}{4}\left(\mathrm{vec}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\right)'\,\mathbf{K}_3^{*\prime}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\,\mathbf{K}_3^*\,\mathrm{vec}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1};\\
\zeta_6 &= \sum_{i=1}^{g}\frac{n_i}{\bar{n}}\,\mathrm{tr}\left\{\mathbf{N}_d\left(\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\otimes\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\,\mathbf{F}_i^{(1)\prime}\right)\mathbf{U}_i^{(1)\prime}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\,\mathbf{K}_3^*\right\};\\
\zeta_7 &= \sum_{i=1}^{g}\frac{n_i}{2\bar{n}}\left(\mathrm{vec}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\right)'\,\mathbf{K}_3^{*\prime}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\,\mathbf{U}_i^{(1)}\,\mathrm{vec}\left(\mathbf{F}_i^{(1)}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\right);\\
\zeta_8 &= \sum_{i=1}^{g}\frac{n_i}{3\bar{n}}\,\mathrm{tr}\left\{\mathbf{N}_d\left(\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\,\mathbf{F}_i^{(1)\prime}\otimes\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\right)\widetilde{\mathbf{U}}_i^{(2)}\right\};
\end{aligned}$$

$$
\begin{aligned}
\widetilde{\mathbf{U}}_i^{(2)} &= -6\,\mathrm{dvec}\left(\dddot{\mathbf{F}}_i^{(1)\prime}, p^2\,d, p^2\right)\mathbf{F}_i^{(2)} \\
&+ \mathrm{dvec}\left[\mathrm{dvec}\left\{(\boldsymbol{\Sigma}_i^{-1}\otimes\boldsymbol{\Sigma}_i^{-1})\,\mathbf{F}_i^{(3)}, p^2 d^2, d\right\}', p^2\,d, d^2\right] \\
&+ 6(\mathbf{I}_{p^2}\otimes\mathbf{F}_i^{(1)})'(\boldsymbol{\Sigma}_i^{-1}\otimes\mathrm{vec}\,\boldsymbol{\Sigma}_i^{-1}\otimes\mathbf{I}_p)\mathbf{I}_{(p,p)}\mathbf{L}_{3i}; \\
\mathbf{L}_{3i} &= \left(\boldsymbol{\Sigma}_i^{-1}\otimes\mathrm{vec}\,\boldsymbol{\Sigma}_i^{-1}\otimes\mathbf{I}_p\right)'\left(\mathbf{F}_i^{(1)}\otimes\mathbf{F}_i^{(1)}\right); \\
\zeta_9 &= \sum_{i=1}^{g}\frac{n_i}{6\bar{n}}\left\{\mathrm{vec}\left(\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\,\mathbf{F}_i^{(1)\prime}\right)\right\}'\widetilde{\mathbf{U}}_i^{(2)}\left(\mathrm{vec}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\right); \\
\zeta_{10} &= \frac{1}{6}\,\mathrm{tr}\left\{\mathbf{N}_d\left(\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\otimes\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\right)\mathbf{K}_4^*\right\} + \frac{1}{12}\left(\mathrm{vec}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1}\right)'\mathbf{K}_4^*\,\mathrm{vec}\,\bar{\mathbf{I}}_{\boldsymbol{\theta}}^{-1};\ \text{and} \\
\mathbf{K}_4^* &= \sum_{i=1}^{g}\frac{n_i}{\bar{n}}\left\{12\left(\mathbf{F}_i^{(1)}\otimes\mathbf{F}_i^{(1)}\right)'\dddot{\mathbf{F}}_i^{(2)} - 9\mathbf{L}_{3i}'\mathbf{I}_{(p,p)}\mathbf{L}_{3i}\right. \\
&\left. - 2\ \mathrm{dvec}\left(\ddot{\mathbf{F}}_i^{(1)\prime}\,\mathbf{F}_i^{(3)}, d^2, d^2\right) - \frac{3}{2}\,\mathbf{F}_i^{(2)\prime}\,\ddot{\mathbf{F}}_i^{(2)}\right\}.
\end{aligned}
$$

6 Examples from Table 1

6.1 Null is Model 10

Suppose that Model 10 in Table 1 is the null model. Use of Corollary 2 to generate alternative models and use of the extended model in (27) to reparameterize the alternative models is illustrated below. The reparameterizations also can be used when the null model is 9, 8, 7, 6, or 4. The case in which the null model is 5 is discussed in §6.2 of this supplement. Eigen-values are parameterized using (3).

6.1.1 Null is Model 10, Alternative is Model 9

To generate model 9 from model 10 use operation (i) to expand $\mathbf{r}_1$ and m_{11} from

$$
\mathbf{r}_1 = \begin{pmatrix}5\\2\\3\end{pmatrix}\ \text{into}\ \mathbf{r}_1 = \begin{pmatrix}3\\2\\2\\3\end{pmatrix}\ \text{and from}\ m_{11} = 5\ \text{into}\ \begin{pmatrix}m_{11}\\m_{12}\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}.
$$

Model 9 need not be reparameterized to compute the Bartlett correction. Nonetheless, use caution. The first five components are spherical under H_0, so the first two distinct eigen-values in group 1 cannot be ordered when computing the Bartlett correction. Suitable matrices for $\mathbf{T}_{1i}$ and $\mathbf{T}_{2i}$ are

$$\mathbf{T}_{11} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$\mathbf{T}_{21} = \begin{pmatrix} \mathbf{I}_4 & \mathbf{0}_{4\times 3} \end{pmatrix}; \text{ and } \mathbf{T}_{22} = \begin{pmatrix} \mathbf{0}_{3\times 4} & \mathbf{I}_3 \end{pmatrix}.$$

6.1.2 Null is Model 10, Alternative is Model 8

To generate model 8 from model 10 use operation (i) to expand $\mathbf{r}_1$ and m_{11} from

$$\mathbf{r}_1 = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \text{ into } \mathbf{r}_1 = \begin{pmatrix} 3 \\ 2 \\ 2 \\ 3 \end{pmatrix} \text{ and from } m_{11} = 5 \text{ into } \begin{pmatrix} m_{11} \\ m_{12} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Use operation (ii) to expand the multiplicity corresponding to the new r_{11} from

$$m_{11} = 3 \text{ into } \mathbf{m}_{11} = \mathbf{1}_3.$$

To satisfy Theorem 1, model 8 can be reparameterized as

$$\mathbf{r}_1 = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix};$$

$$\mathbf{T}_{11} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$\mathbf{T}_{21} = (\,\mathbf{I}_3 \quad \mathbf{0}_{3\times 3}\,)\,; \quad \mathbf{T}_{22} = (\,\mathbf{0}_{3\times 3} \quad \mathbf{I}_3\,)\,;$$

$$\boldsymbol{\Xi}_1 = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_2 & \beta_4 & \beta_5 \\ \beta_3 & \beta_5 & \beta_6 \end{pmatrix} \oplus \mathbf{0}_{7\times 7} \text{ and } \boldsymbol{\Xi}_2 = \mathbf{0}.$$

The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 37$, $\dim(\boldsymbol{\tau}) = 0$, $\dim(\boldsymbol{\varphi}) = 6$, $\dim(\boldsymbol{\beta}) = 6$, and $d = 49$.

6.1.3 Null is Model 10, Alternative is Model 7

To generate model 7 from model 10 use operation (i) to expand $\mathbf{r}_1$ and m_{11} from

$$\mathbf{r}_1 = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \text{ into } \mathbf{r}_1 = \begin{pmatrix} 3 \\ 2 \\ 2 \\ 3 \end{pmatrix} \text{ and from } m_{11} = 5 \text{ into } \begin{pmatrix} m_{11} \\ m_{12} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Use operation (ii) to expand the multiplicities corresponding to the new r_{11} and r_{12} from

$$m_{11} = 3 \text{ into } \mathbf{m}_{11} = \mathbf{1}_3 \text{ and from } m_{12} = 2 \text{ into } \mathbf{m}_{12} = \mathbf{1}_2.$$

To satisfy Theorem 1, model 7 can be reparameterized as

$$\mathbf{r}_1 = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}; \quad \mathbf{r}_2 = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix};$$

$$\mathbf{T}_{11} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$\mathbf{T}_{21} = (\,\mathbf{I}_3 \quad \mathbf{0}_{3\times 3}\,)\,; \quad \mathbf{T}_{22} = (\,\mathbf{0}_{3\times 3} \quad \mathbf{I}_3\,)\,;$$

$$\boldsymbol{\Xi}_1 = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_2 & \beta_4 & \beta_5 \\ \beta_3 & \beta_5 & \beta_6 \end{pmatrix} \oplus \begin{pmatrix} \beta_7 & \beta_8 \\ \beta_8 & -\beta^* \end{pmatrix} \oplus \mathbf{0}_{5\times 5};$$

$$\begin{aligned}
\beta^* &= \beta_1 + \beta_4 + \beta_6 + \beta_7; \text{ and} \\
\boldsymbol{\Xi}_2 &= \mathbf{0}.
\end{aligned}$$

The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 37$, $\dim(\boldsymbol{\tau}) = 0$, $\dim(\boldsymbol{\varphi}) = 6$, $\dim(\boldsymbol{\beta}) = 8$, and $d = 51$.

6.1.4 Null is Model 10, Alternative is Model 6

To generate model 6 from model 10 use operation (*ii*) to expand m_{11} from

$$m_{11} = 5 \text{ into } \mathbf{m}_{11} = \mathbf{1}_5.$$

To satisfy Theorem 1, model 6 can be reparameterized as

$$\begin{aligned}
\mathbf{r}_1 &= \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}; \quad \mathbf{r}_2 = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}; \\
\mathbf{T}_{11} &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \\
\mathbf{T}_{21} &= \left(\mathbf{I}_3 \quad \mathbf{0}_{3\times 3} \right); \quad \mathbf{T}_{22} = \left(\mathbf{0}_{3\times 3} \quad \mathbf{I}_3 \right); \\
\boldsymbol{\Xi}_1 &= \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \beta_2 & \beta_6 & \beta_7 & \beta_8 & \beta_9 \\ \beta_3 & \beta_7 & \beta_{10} & \beta_{11} & \beta_{12} \\ \beta_4 & \beta_8 & \beta_{11} & \beta_{13} & \beta_{14} \\ \beta_5 & \beta_9 & \beta_{12} & \beta_{14} & -\beta^* \end{pmatrix} \oplus \mathbf{0}_{5\times 5}; \\
\beta^* &= \beta_1 + \beta_6 + \beta_{10} + \beta_{13}; \text{ and} \\
\boldsymbol{\Xi}_2 &= \mathbf{0}.
\end{aligned}$$

The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 37$, $\dim(\boldsymbol{\tau}) = 0$, $\dim(\boldsymbol{\varphi}) = 6$, $\dim(\boldsymbol{\beta}) = 14$, and $d = 57$.

6.1.5 Null is Model 10, Alternative is Model 4

To generate model 4 from model 10 use operation (*iii*) to change $\mathbf{r}_1$ from

$$\mathbf{r}_1 = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \text{ to } \mathbf{r}_1 = \begin{pmatrix} 7 \\ 3 \end{pmatrix}.$$

The new multiplicity vector induced by operation (*iii*) is

$$\mathbf{m}_{11} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

Then use operation (*ii*) to expand the new $\mathbf{m}_{11}$ from

$$\mathbf{m}_{11} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \text{ into } \mathbf{m}_{11} = \mathbf{1}_7.$$

To satisfy Theorem 1, model 4 can be reparameterized as

$$\begin{aligned}
\mathbf{r}_1 &= \begin{pmatrix} 7 \\ 3 \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}, \text{ and} \\
\mathbf{T}_{11} &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}; \quad \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \\
\mathbf{T}_{21} &= \left(\mathbf{I}_2 \quad \mathbf{0}_{2\times 3} \right); \quad \mathbf{T}_{22} = \left(\mathbf{0}_{3\times 2} \quad \mathbf{I}_3 \right); \\
\boldsymbol{\Xi}_1 &= \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 \\ \beta_2 & \beta_8 & \beta_9 & \beta_{10} & \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_3 & \beta_9 & \beta_{14} & \beta_{15} & \beta_{16} & \beta_{17} & \beta_{18} \\ \beta_4 & \beta_{10} & \beta_{15} & \beta_{19} & \beta_{20} & \beta_{21} & \beta_{22} \\ \beta_5 & \beta_{11} & \beta_{16} & \beta_{20} & \beta_{23} & \beta_{24} & \beta_{25} \\ \beta_6 & \beta_{12} & \beta_{17} & \beta_{21} & \beta_{24} & \beta_{26} & \beta_{27} \\ \beta_7 & \beta_{13} & \beta_{18} & \beta_{22} & \beta_{25} & \beta_{27} & -\beta^* \end{pmatrix} \oplus \mathbf{0}_{3\times 3}; \\
\beta^* &= \beta_1 + \beta_8 + \beta_{14} + \beta_{19} + \beta_{23} + \beta_{26}; \text{ and}
\end{aligned}$$

$$\boldsymbol{\Xi}_2 = \mathbf{0}.$$

The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 33$, $\dim(\boldsymbol{\tau}) = 0$, $\dim(\boldsymbol{\varphi}) = 5$, $\dim(\boldsymbol{\beta}) = 27$, and $d = 65$.

6.1.6 Null is Model 10, Alternative is Model 2

Table 12 displays an alternate parameterization of model 2. To generate this alternate form from model 10 use operation (*iii*) to change $\mathbf{r}_1$ and $\mathbf{r}_2$ from

$$\mathbf{r}_1 = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \text{ to } \mathbf{r}_1 = \begin{pmatrix} 7 \\ 3 \end{pmatrix} \text{ and from } \mathbf{r}_2 = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} \text{ to } \mathbf{r}_2 = \begin{pmatrix} 7 \\ 3 \end{pmatrix}.$$

The new multiplicity vectors induced by operation (*iii*) are

$$\mathbf{m}_{11} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \text{ and } \mathbf{m}_{21} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Then use operation (*ii*) to expand the new $\mathbf{m}_{11}$ and $\mathbf{m}_{12}$ from

$$\mathbf{m}_{11} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \text{ into } \mathbf{m}_{11} = \mathbf{1}_7 \text{ and from } \mathbf{m}_{12} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \text{ into } \mathbf{m}_{11} = \mathbf{1}_7.$$

To satisfy Theorem 1, model 2 can be reparameterized as

$$\begin{aligned}
\mathbf{r}_1 &= \begin{pmatrix} 7 \\ 3 \end{pmatrix}; \quad \mathbf{r}_2 = \begin{pmatrix} 7 \\ 3 \end{pmatrix}; \\
\mathbf{T}_{11} &= \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}; \\
\mathbf{T}_{21} &= (\,\mathbf{I}_2 \quad \mathbf{0}_{2\times 2}\,); \quad \mathbf{T}_{22} = (\,\mathbf{0}_{2\times 2} \quad \mathbf{I}_2\,); \\
\boldsymbol{\Xi}_1 &= \begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 \\
\beta_2 & \beta_8 & \beta_9 & \beta_{10} & \beta_{11} & \beta_{12} & \beta_{13} \\
\beta_3 & \beta_9 & \beta_{14} & \beta_{15} & \beta_{16} & \beta_{17} & \beta_{18} \\
\beta_4 & \beta_{10} & \beta_{15} & \beta_{19} & \beta_{20} & \beta_{21} & \beta_{22} \\
\beta_5 & \beta_{11} & \beta_{16} & \beta_{20} & \beta_{23} & \beta_{24} & \beta_{25} \\
\beta_6 & \beta_{12} & \beta_{17} & \beta_{21} & \beta_{24} & \beta_{26} & \beta_{27} \\
\beta_7 & \beta_{13} & \beta_{18} & \beta_{22} & \beta_{25} & \beta_{27} & -\beta_1^*
\end{pmatrix} \oplus \mathbf{0}_{3\times 3};
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\Xi}_2 &= \begin{pmatrix} \beta_{28} & \beta_{29} & \beta_{30} & \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{29} & \beta_{35} & \beta_{36} & \beta_{37} & \beta_{38} & \beta_{39} & \beta_{40} \\ \beta_{30} & \beta_{36} & \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} & \beta_{45} \\ \beta_{31} & \beta_{37} & \beta_{42} & \beta_{46} & \beta_{47} & \beta_{48} & \beta_{49} \\ \beta_{32} & \beta_{38} & \beta_{43} & \beta_{47} & \beta_{50} & \beta_{51} & \beta_{52} \\ \beta_{33} & \beta_{39} & \beta_{44} & \beta_{48} & \beta_{51} & \beta_{53} & \beta_{54} \\ \beta_{34} & \beta_{40} & \beta_{45} & \beta_{49} & \beta_{52} & \beta_{54} & -\beta_2^* \end{pmatrix} \oplus \mathbf{0}_{3\times 3}; \\
\beta_1^* &= \beta_1 + \beta_8 + \beta_{14} + \beta_{19} + \beta_{23} + \beta_{26}; \text{ and} \\
\beta_2^* &= \beta_{28} + \beta_{35} + \beta_{41} + \beta_{46} + \beta_{50} + \beta_{53}.
\end{aligned}
$$

The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 21$, $\dim(\boldsymbol{\tau}) = 0$, $\dim(\boldsymbol{\varphi}) = 4$, $\dim(\boldsymbol{\beta}) = 54$, and $d = 79$.

6.1.7 Null is Model 10, Alternative is Model 1

Corollary 1 verifies that model 1 can be reparameterized in such a manner as to satisfy Theorem 1. A suitable reparameterization is described in Corollary 1.

6.2 Null is Model 5

Suppose that Model 5 in Table 1 is the null model. Corollary 2 can be used to generate the alternate form of model 2 (see Table 12) by using operation (*iii*) change $\mathbf{r}_2$ from

$$\mathbf{r}_2 = \begin{pmatrix} \mathbf{1}_7 \\ 3 \end{pmatrix} \text{ to } \mathbf{r}_2 = \begin{pmatrix} 7 \\ 3 \end{pmatrix}.$$

The new multiplicity vector induced by operation (*iii*) is

$$\mathbf{m}_{21} = \mathbf{1}_7.$$

Model 2 (H_a) and model 5 (H_0) satisfy the conditions of theorem 1 without reparameterizing model 2 because the eigen-value model is identical under H_0 and H_a.

7 Examples from Table 3

Suppose that Model 6 in Table 3 is the null model. Use of Corollary 2 to generate alternative models and use of the extended model in (27) to reparameterize the alternative models is illustrated below. The reparameterizations also can be used when the null model is 5 or 3. Eigen-values are parameterized using (3).

7.1 Null is Model 6, Alternative is Model 5

To generate model 5a from model 6 use operation (*iii*) to change $\mathbf{r}_i$ from

$$\mathbf{r}_i = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ to } \mathbf{r}_i = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \text{ for all } i.$$

The new multiplicity vectors induced by operation (*iii*) are

$$\mathbf{m}_{i1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ for all } i.$$

Use operation (*ii*) to expand the new m_{i1} from

$$m_{i1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ into } \mathbf{m}_{i1} = \mathbf{1}_5 \text{ for all } i.$$

To satisfy Theorem 1, model 5 can be reparameterized as

$$\begin{aligned}
\mathbf{r}_i &= \begin{pmatrix} 5 \\ 4 \end{pmatrix} \text{ for all } i; \\
\mathbf{T}_{1i} &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ for all } i; \\
\mathbf{T}_{21} &= \begin{pmatrix} \mathbf{I}_2 & \mathbf{0}_{2\times(g-1)} \end{pmatrix}; \quad \mathbf{T}_{2i} = \begin{pmatrix} \mathbf{I}_2 & \mathbf{1}_2\mathbf{e}_{i-1}^{(g-1)\prime} \end{pmatrix} \text{ for } i \ge 2; \\
\mathbf{w}_1 &= \mathbf{0}_{g+1\times 1}; \quad \mathbf{w}_i = \begin{pmatrix} \mathbf{0}_{2\times 1} \\ \mathbf{e}_{i-1}^{(g-1)} \end{pmatrix} \text{ for } i \ge 2; \\
\boldsymbol{\Xi}_1 &= \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \beta_2 & \beta_6 & \beta_7 & \beta_8 & \beta_9 \\ \beta_3 & \beta_7 & \beta_{10} & \beta_{11} & \beta_{12} \\ \beta_4 & \beta_8 & \beta_{11} & \beta_{13} & \beta_{14} \\ \beta_5 & \beta_9 & \beta_{12} & \beta_{14} & -\beta^* \end{pmatrix} \oplus \mathbf{0}_{4\times 4}; \\
\beta^* &= \beta_1 + \beta_6 + \beta_{10} + \beta_{13}; \text{ and} \\
\boldsymbol{\Xi}_i &= e^{\varphi_{i+1}}\boldsymbol{\Xi}_1 \text{ for } i \ge 2.
\end{aligned}$$

The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 20$, $\dim(\boldsymbol{\tau}) = 0$, $\dim(\boldsymbol{\varphi}) = g + 1$, $\dim(\boldsymbol{\beta}) = 14$, and $d = g + 35$.

7.2 Null is Model 6, Alternative is Model 4

An alternate parameterization of model 4 is

$$r_i = 9 \text{ for all } i; \text{ and } \mathbf{m}_{i1} = \mathbf{1}_9 \text{ for all } i.$$

The multiplicity vector for partitioning $\mathbf{G}$ is $\widetilde{m} = 9$ and the parameter dimensions are $\dim(\boldsymbol{\mu}) = 0$, $\dim(\boldsymbol{\tau}) = 36$, and $\dim(\boldsymbol{\varphi}) = 8 + g$. The design matrices for this alternate parameterization of model 4 are

$$\mathbf{T}_{1i} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ for all } i;$$

$$\mathbf{T}_{21} = \left\{ \mathbf{I}_9 \quad \mathbf{0}_{9\times(g-1)} \right\}; \quad \mathbf{T}_{2i} = \left\{ \mathbf{I}_9 \quad \mathbf{1}_9 \mathbf{e}_{i-1}^{(g-1)\prime} \right\} \text{ for } i \geq 2;$$

and $t_{i1} = 1$ for all i.

To generate the alternate form of model 4 from model 6 use operation (*iii*) to change $\mathbf{r}_i$ from

$$\mathbf{r}_i = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ to } \mathbf{r}_i = (\, 9 \,) \text{ for all } i.$$

The new multiplicity vectors induced by operation (*iii*) are

$$\mathbf{m}_{i1} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ for all } i.$$

Use operation (*ii*) to expand the new m_{i1} from

$$m_{i1} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ into } \mathbf{m}_{i1} = \mathbf{1}_9 \text{ for all } i.$$

Proportionality requires that $t_{i1} = 1$ for all i. To satisfy Theorem 1, model 4 can be reparameterized as

$$\begin{aligned}
\mathbf{r}_i &= 9 \text{ for all } i; \\
\mathbf{T}_{1i} &= \mathbf{1}_9 \text{ for all } i; \\
\mathbf{T}_{21} &= \begin{pmatrix} 1 & \mathbf{0}_{1\times(g-1)} \end{pmatrix}; \quad \mathbf{T}_{2i} = \begin{pmatrix} 1 & \mathbf{1}_1 \mathbf{e}_{i-1}^{(g-1)\prime} \end{pmatrix} \text{ for } i \geq 2; \\
\mathbf{w}_1 &= \mathbf{0}_{g\times 1}; \quad \mathbf{w}_i = \begin{pmatrix} 0 \\ \mathbf{e}_{i-1}^{(g-1)} \end{pmatrix} \text{ for } i \geq 2; \\
\boldsymbol{\Xi}_1 &= \begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_6 & \beta_8 & \beta_9 \\
\beta_2 & \beta_{10} & \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} & \beta_{15} & \beta_{16} & \beta_{17} \\
\beta_3 & \beta_{11} & \beta_{18} & \beta_{19} & \beta_{20} & \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\
\beta_4 & \beta_{12} & \beta_{19} & \beta_{25} & \beta_{26} & \beta_{27} & \beta_{28} & \beta_{29} & \beta_{30} \\
\beta_5 & \beta_{13} & \beta_{20} & \beta_{26} & \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} & \beta_{35} \\
\beta_6 & \beta_{14} & \beta_{21} & \beta_{27} & \beta_{32} & \beta_{36} & \beta_{37} & \beta_{38} & \beta_{39} \\
\beta_7 & \beta_{15} & \beta_{22} & \beta_{28} & \beta_{33} & \beta_{37} & \beta_{40} & \beta_{41} & \beta_{42} \\
\beta_8 & \beta_{16} & \beta_{23} & \beta_{29} & \beta_{34} & \beta_{38} & \beta_{41} & \beta_{43} & \beta_{44} \\
\beta_9 & \beta_{17} & \beta_{24} & \beta_{30} & \beta_{35} & \beta_{39} & \beta_{42} & \beta_{44} & -\beta^*
\end{pmatrix}; \\
\beta^* &= \beta_1 + \beta_{10} + \beta_{18} + \beta_{25} + \beta_{31} + \beta_{36} + \beta_{40} + \beta_{43}; \text{ and} \\
\boldsymbol{\Xi}_i &= e^{\varphi_i} \boldsymbol{\Xi}_1 \text{ for } i \geq 2.
\end{aligned}$$

The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 0$, $\dim(\boldsymbol{\tau}) = 0$, $\dim(\boldsymbol{\varphi}) = g$, $\dim(\boldsymbol{\beta}) = 44$, and $d = g + 44$.

7.3 Null is Model 6, Alternative is Model 3

To generate model 3 from model 6 use operation (iii) to change $\mathbf{r}_i$ from

$$\mathbf{r}_1 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ to } \mathbf{r}_1 = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \text{ and from } \mathbf{r}_i = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ to } \mathbf{r}_i = \begin{pmatrix} 2 \\ 7 \end{pmatrix} \text{ for } i \geq 2.$$

The new multiplicity vectors induced by operation (iii) are

$$\mathbf{m}_{11} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{m}_{i2} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ for } i \geq 2.$$

Use operation (ii) to expand the new multiplicity vector $\mathbf{m}_{11}$ from

$$\mathbf{m}_{11} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ into } \mathbf{m}_{11} = \mathbf{1}_5.$$

To satisfy Theorem 1, model 3 can be reparameterized as

$$
\begin{aligned}
\mathbf{r}_1 &= \begin{pmatrix} 5 \\ 4 \end{pmatrix}; \quad \mathbf{r}_i = \begin{pmatrix} 2 \\ 7 \end{pmatrix} \text{ for } i \geq 2; \\
\mathbf{T}_{11} &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}; \quad \mathbf{T}_{1i} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ for } i \geq 2; \\
\mathbf{T}_{21} &= \left(\mathbf{I}_2 \quad \mathbf{0}_{2\times 3(g-1)} \right); \quad \mathbf{T}_{2i} = \left\{ \mathbf{0}_{3\times 2} \quad (\mathbf{e}_{i-1}^{g-1} \otimes \mathbf{I}_3)' \right\}; \\
\mathbf{m}_{i2} &= \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ for } i \geq 2; \quad t_{i,2} = i - 1 \text{ for } i \geq 2; \\
\boldsymbol{\Xi}_1 &= \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \beta_2 & \beta_6 & \beta_7 & \beta_8 & \beta_9 \\ \beta_3 & \beta_7 & \beta_{10} & \beta_{11} & \beta_{12} \\ \beta_4 & \beta_8 & \beta_{11} & \beta_{13} & \beta_{14} \\ \beta_5 & \beta_9 & \beta_{12} & \beta_{14} & -\beta^* \end{pmatrix} \oplus \mathbf{0}_{4\times 4}; \\
\beta^* &= \beta_1 + \beta_6 + \beta_{10} + \beta_{13}; \text{ and} \\
\boldsymbol{\Xi}_i &= \mathbf{0} \text{ for } i \geq 2.
\end{aligned}
$$

The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 26$, $\dim(\boldsymbol{\tau}) = 12(g-1)$, $\dim(\boldsymbol{\varphi}) = 3g - 1$, $\dim(\boldsymbol{\beta}) = 14$, and $d = 15g + 27$.

7.4 Null is Model 6, Alternative is Model 2

To generate model 2 from model 6 use operation (*iii*) to change $\mathbf{r}_i$ from

$$
\mathbf{r}_1 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ to } \mathbf{r}_1 = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \text{ and from } \mathbf{r}_i = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ to } \mathbf{r}_i = \begin{pmatrix} 2 \\ 7 \end{pmatrix} \text{ for } i \geq 2.
$$

The new multiplicity vectors induced by operation (*iii*) are

$$
\mathbf{m}_{11} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{m}_{i2} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ for } i \geq 2.
$$

Use operation (*ii*) to expand the new multiplicity vectors from

$$\mathbf{m}_{11} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ into } \mathbf{m}_{11} = \mathbf{1}_5 \text{ and from } \mathbf{m}_{i2} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ into } m_{i2} = \mathbf{1}_7 \text{ for } i \geq 2.$$

To satisfy Theorem 1, model 2 can be reparameterized as

$$\begin{aligned}
\mathbf{r}_1 &= \begin{pmatrix} 5 \\ 4 \end{pmatrix}; \quad \mathbf{r}_i = \begin{pmatrix} 2 \\ 7 \end{pmatrix} \text{ for } i \geq 2; \\
\mathbf{T}_{11} &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}; \quad \mathbf{T}_{1i} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ for } i \geq 2; \\
\mathbf{T}_{21} &= \begin{pmatrix} \mathbf{I}_2 & \mathbf{0}_{2\times 2(g-1)} \end{pmatrix}; \quad \mathbf{T}_{22} = \left\{ \mathbf{0}_{2\times 2} \quad (\mathbf{e}_{i-1}^{g-1} \otimes \mathbf{I}_2)' \right\}; \\
\boldsymbol{\Xi}_1 &= \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \beta_2 & \beta_6 & \beta_7 & \beta_8 & \beta_9 \\ \beta_3 & \beta_7 & \beta_{10} & \beta_{11} & \beta_{12} \\ \beta_4 & \beta_8 & \beta_{11} & \beta_{13} & \beta_{14} \\ \beta_5 & \beta_9 & \beta_{12} & \beta_{14} & -\beta^* \end{pmatrix} \oplus \mathbf{0}_{4\times 4}; \\
\beta^* &= \beta_1 + \beta_6 + \beta_{10} + \beta_{13}; \text{ and} \\
\boldsymbol{\Xi}_i &= \mathbf{0}_{2\times 2} \oplus \mathbf{B}_{i2} \text{ for } i \geq 2;
\end{aligned}$$

where $\mathbf{B}_{i2}$ is a 7×7 symmetric matrix and satisfies $\text{tr}(\mathbf{B}_{i2}) = 0$. The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 26$, $\dim(\boldsymbol{\tau}) = 0$, $\dim(\boldsymbol{\varphi}) = 2g$, $\dim(\boldsymbol{\beta}) = 27g - 13$, and $d = 29g + 13$.

7.5 Null is Model 6, Alternative is Model 1

Corollary 1 verifies that model 1 can be reparameterized in such a manner as to satisfy Theorem 1. A suitable reparameterization is described in Corollary 1.

8 Examples from Analysis of Vole Data

Descriptions of models for the Vole data are given in Table 13. These descriptions are identical to those in Table 5, but two partial CPC models have been added.

Likelihood ratio test statistics for comparing the vole models are displayed in Table 15.

Suppose that Model 6 in Table 13 is the null model. This model can be parameterized as described in Table 14 and using the eigen-value parameterization in (3), where

$$\mathbf{T}_{11} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \mathbf{T}_{13} = \mathbf{T}_{14} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$\mathbf{T}_{21} = \begin{pmatrix} \mathbf{I}_4 & \mathbf{0}_{4\times 6} \end{pmatrix}; \quad \mathbf{T}_{22} = \begin{pmatrix} \mathbf{0}_{3\times 4} & \mathbf{I}_3 & \mathbf{0}_{3\times 3} \end{pmatrix}; \text{ and } \mathbf{T}_{23} = \mathbf{T}_{24} = \begin{pmatrix} \mathbf{0}_{3\times 7} & \mathbf{I}_3 \end{pmatrix}.$$

Use of Corollary 2 to generate alternative models and use of the extended model in (27) to reparameterize the alternative models is illustrated below. The reparameterizations also can be used when the null model is 7, 8, 3, 4, or 5. In all cases, eigen-values are parameterized using (3).

8.1 Null is Model 6, Alternative is Model 5

To generate model 5 from model 6 use operation (*iii*) to change $\mathbf{r}_3$ and $\mathbf{r}_4$ from

$$\mathbf{r}_3 = \mathbf{r}_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ to } \mathbf{r}_3 = \mathbf{r}_4 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The new multiplicity vectors induced by operation (*iii*) are

$$\mathbf{m}_{32} = \mathbf{m}_{42} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Model 5 (H_a) and model 6 (H_0) satisfy the conditions of theorem 1 without reparameterizing model 5 because the eigen-value model is identical under H_0 and H_a.

8.2 Null is Model 6, Alternative is Model 4

To generate model 4 from model 6 use operation (*iii*) to change $\mathbf{r}_3$ and $\mathbf{r}_4$ from

$$\mathbf{r}_3 = \mathbf{r}_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ to } \mathbf{r}_3 = \mathbf{r}_4 = \begin{pmatrix} 4 \end{pmatrix}.$$

The new multiplicity vectors induced by operation (*iii*) are

$$\mathbf{m}_{31} = \mathbf{m}_{41} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Model 4 (H_a) and model 6 (H_0) satisfy the conditions of theorem 1 without reparameterizing model 4 because the eigen-value model is identical under H_0 and H_a.

8.3 Null is Model 6, Alternative is Model 3

To generate model 3 from model 6 use operation (i) to expand $\mathbf{r}_2$ from

$$\mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \text{ to } \mathbf{r}_2 = \mathbf{1}_4.$$

The new multiplicities induced by operation (i) are

$$\mathbf{m}_{1j} = 1 \text{ for all } j.$$

Use operation (iii) to change $\mathbf{r}_3$ and $\mathbf{r}_4$ from

$$\mathbf{r}_3 = \mathbf{r}_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ to } \mathbf{r}_3 = \mathbf{r}_4 = (\,4\,)\,.$$

The new multiplicities induced by operation (iii) are

$$\mathbf{m}_{31} = \mathbf{m}_{41} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Also, use operation (ii) to expand $\mathbf{m}_{31}$ and $\boldsymbol{\mu}_{41}$ from

$$\mathbf{m}_{31} = \mathbf{m}_{41} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ to } \mathbf{m}_{31} = \mathbf{m}_{41} = \mathbf{1}_4.$$

Model 3 (H_a) and model 6 (H_0) satisfy the conditions of theorem 1 without reparameterizing model 3, but exercise caution because some eigen-values are spherical under H_0. Suitable $\mathbf{T}_{1i}$ and $\mathbf{T}_{2i}$ eigen-value design matrices are

$$\begin{aligned} \mathbf{T}_{1i} &= \mathbf{I}_p \text{ for all } i; \text{ and} \\ \mathbf{T}_{2i} &= (\mathbf{e}_i^g \otimes \mathbf{I}_4)' \text{ for all } i. \end{aligned}$$

8.4 Null is Model 6, Alternative is Model 8

To generate model 8 from model 6 use operation (i) to expand $\mathbf{r}_2$–$\mathbf{r}_4$ from

$$\mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \text{ to } \mathbf{r}_2 = \mathbf{1}_4 \text{ and from } \mathbf{r}_3 = \mathbf{r}_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ to } \mathbf{r}_3 = \mathbf{r}_4 = \mathbf{1}_4.$$

The new multiplicity vectors induced by operation (i) are

$$\mathbf{m}_{ij} = 1 \text{ for all } ij.$$

Now use operation (iii) to change $\mathbf{r}_2$–$\mathbf{r}_4$ from

$$\mathbf{r}_i = \mathbf{1}_4 \text{ to } \mathbf{r}_i = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \text{ for } i \geq 2.$$

The new multiplicity vectors induced by operation (iii) are

$$\mathbf{m}_{i3} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ for } i \geq 2.$$

To satisfy Theorem 1, model 8 can be reparameterized as

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{1}_4; \quad \mathbf{r}_i = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \text{ for } i \geq 2; \\
\mathbf{T}_{11} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{T}_{1i} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ for } i \geq 2; \\
\mathbf{T}_{21} &= \begin{pmatrix} \mathbf{I}_4 & \mathbf{0}_{4\times 9} \end{pmatrix}; \quad \mathbf{T}_{2i} = \left\{ \mathbf{0}_{3\times 4} \quad \left(\mathbf{e}_{i-1}^3 \otimes \mathbf{I}_3\right)' \right\} \text{ for } i \geq 2; \\
\boldsymbol{\Xi}_1 &= \mathbf{0}; \quad \boldsymbol{\Xi}_2 = \mathbf{0}_{2\times 2} \oplus \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & -\beta_1 \end{pmatrix}; \\
\boldsymbol{\Xi}_3 &= \mathbf{0}_{2\times 2} \oplus \begin{pmatrix} \beta_3 & \beta_4 \\ \beta_4 & -\beta_2 \end{pmatrix}; \text{ and } \boldsymbol{\Xi}_4 = \mathbf{0}_{2\times 2} \oplus \begin{pmatrix} \beta_5 & \beta_6 \\ \beta_6 & -\beta_5 \end{pmatrix}.
\end{aligned}$$

The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 6$, $\dim(\boldsymbol{\tau}) = 0$, $\dim(\boldsymbol{\varphi}) = 13$, $\dim(\boldsymbol{\beta}) = 6$, and $d = 25$.

8.5 Null is Model 6, Alternative is Model 7

To generate model 7 from model 6 use operation (*i*) to expand $\mathbf{r}_2$–$\mathbf{r}_4$ from

$$\mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \text{ to } \mathbf{r}_2 = \mathbf{1}_4 \text{ and from } \mathbf{r}_3 = \mathbf{r}_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ to } \mathbf{r}_3 = \mathbf{r}_4 = \mathbf{1}_4.$$

The new multiplicity vectors induced by operation (*i*) are

$$\mathbf{m}_{ij} = 1 \text{ for all } ij.$$

Now use operation (*iii*) to change $\mathbf{r}_2$–$\mathbf{r}_4$ from

$$\mathbf{r}_i = \mathbf{1}_4 \text{ to } \mathbf{r}_i = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ for } i \geq 2.$$

The new multiplicity vectors induced by operation (*iii*) are

$$\mathbf{m}_{i2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ for } i \geq 2.$$

To satisfy Theorem 1, model 7 can be reparameterized as

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{1}_4; \quad \mathbf{r}_i = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ for } i \geq 2; \\
\mathbf{T}_{11} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{T}_{1i} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ for } i \geq 2; \\
\mathbf{T}_{21} &= \begin{pmatrix} \mathbf{I}_4 & \mathbf{0}_{4\times 6} \end{pmatrix}; \quad \mathbf{T}_{2i} = \left\{ \mathbf{0}_{2\times 4} \quad \left(\mathbf{e}_{i-1}^3 \otimes \mathbf{I}_2\right)' \right\} \text{ for } i \geq 2; \\
\boldsymbol{\Xi}_1 &= \mathbf{0}; \quad \boldsymbol{\Xi}_2 = 0 \oplus \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_2 & \beta_4 & \beta_5 \\ \beta_3 & \beta_5 & -\beta_1^* \end{pmatrix}; \\
\boldsymbol{\Xi}_3 &= 0 \oplus \begin{pmatrix} \beta_6 & \beta_7 & \beta_8 \\ \beta_7 & \beta_9 & \beta_{10} \\ \beta_8 & \beta_{10} & -\beta_2^* \end{pmatrix}; \quad \boldsymbol{\Xi}_4 = 0 \oplus \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{12} & \beta_{14} & \beta_{15} \\ \beta_{13} & \beta_{15} & -\beta_3^* \end{pmatrix}; \\
\beta_1^* &= \beta_1 + \beta_4; \quad \beta_2^* = \beta_6 + \beta_9; \text{ and } \beta_3^* = \beta_{11} + \beta_{14}.
\end{aligned}$$

The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 6$, $\dim(\boldsymbol{\tau}) = 0$, $\dim(\boldsymbol{\varphi}) = 10$, $\dim(\boldsymbol{\beta}) = 15$, and $d = 31$.

8.6 Null is Model 6, Alternative is Model 2

To generate model 2 from model 6 use operation (*iii*) to change $\mathbf{r}_2$–$\mathbf{r}_4$ from

$$\mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \text{ to } \mathbf{r}_2 = 4 \text{ and from } \mathbf{r}_3 = \mathbf{r}_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ to } \mathbf{r}_3 = \mathbf{r}_4 = 4.$$

The multiplicity vectors induced by operation (*iii*) are

$$\mathbf{m}_{21} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \text{ and } \mathbf{m}_{i1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ for } i \geq 3.$$

Model 2 (H_a) and model 6 (H_0) satisfy the conditions of theorem 1 without reparameterizing model 2 because the eigen-value model is identical under H_0 and H_a.

8.7 Null is Model 6, Alternative is Model 1

Corollary 1 verifies that model 1 can be reparameterized in such a manner as to satisfy Theorem 1. A suitable reparameterization is described in Corollary 1.

9 Examples from Analysis of Sparrow Data

Descriptions of models for the sparrow data are given in Table 16. These descriptions are identical to those in Table 9, but five models have been added. Parameterization of the models is described in Table 17. Alternative parameterizations for selected sparrow models is given in Table 18. Likelihood ratio test statistics for comparing the sparrow models are displayed in Table 19.

9.1 Likelihood Ratio Tests Assuming Normality

Suppose that Model 4 in Table 16 is the null model. This model can be parameterized as described in Table 17. The eigen-value design matrices, using the parameterization in (3), are

$$\mathbf{T}_{11} = \mathbf{T}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \text{ and } \mathbf{T}_{21} = \mathbf{T}_{22} = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Use of Corollary 2 to generate alternative models and use of the extended model in (27) to reparameterize the alternative models is illustrated below. The reparameterizations also can be used when the null model is 3, 9, or 8. Reparameterization is not necessary if the null model is 7, 2, 6, or 5 because there are no spherical components in these models. In all cases, eigen-values are parameterized using (3).

9.1.1 Null is Model 4, Alternative is Model 5

To generate model 5a in Table 18 from model 4 use operation (*iii*) to change $\mathbf{r}_i$ from

$$\mathbf{r}_1 = \mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \text{ to } \mathbf{r}_1 = \mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

The new multiplicity vectors induced by operation (*iii*) are

$$\mathbf{m}_{13} = \mathbf{m}_{23} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Use operation (*ii*) to expand the new multiplicity vectors from

$$\mathbf{m}_{13} = \mathbf{m}_{23} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ into } \mathbf{m}_{13} = \mathbf{m}_{23} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

To satisfy Theorem 1, model 5 can be reparameterized as

$$\begin{aligned}
\mathbf{r}_1 &= \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}; \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}; \\
\mathbf{T}_{11} &= \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \\
\mathbf{T}_{21} &= \left(\mathbf{I}_4 \quad \mathbf{0}_{4\times 4} \right); \quad \mathbf{T}_{22} = \left(\mathbf{0}_{4\times 4} \quad \mathbf{I}_4 \right); \\
\boldsymbol{\Xi}_1 &= \mathbf{0}_{2\times 2} \oplus \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & -\beta_1 \end{pmatrix} \oplus 0; \text{ and} \\
\boldsymbol{\Xi}_2 &= \mathbf{0}_{2\times 2} \oplus \begin{pmatrix} \beta_3 & \beta_4 & \beta_5 \\ \beta_4 & -\beta_3 & \beta_6 \\ \beta_5 & \beta_6 & 0 \end{pmatrix}.
\end{aligned}$$

The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 9$, $\dim(\boldsymbol{\tau}) = 0$, $\dim(\boldsymbol{\varphi}) = 8$, $\dim(\boldsymbol{\beta}) = 6$, and $d = 23$.

9.1.2 Null is Model 4, Alternative is Model 6

Model 6 cannot be generated by the operations in Corollary 2, even though model 6 is nested within model 4. It is conjectured that the likelihood ratio test statistic does not have an asymptotic χ^2 distribution.

9.1.3 Null is Model 4, Alternative is Model 2

To generate model 2a in Table 18 from model 4 use operation (*ii*) to expand m_{13} and m_{23} from

$$m_{13} = m_{23} = 2 \text{ into } \mathbf{m}_{13} = \mathbf{m}_{23} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To satisfy proportionality, the rotation matrices in group 1 and group 2 must be identical. The design matrices for model 2 in Table 17 are

$$\mathbf{T}_{11} = \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{T}_{21} = (\,\mathbf{I}_5 \quad \mathbf{0}_{5\times1}\,); \text{ and } \mathbf{T}_{22} = (\,\mathbf{I}_5 \quad \mathbf{1}_5\,).$$

To satisfy Theorem 1, model 2 can be reparameterized as

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}; \quad \mathbf{T}_{11} = \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \\
\mathbf{T}_{21} &= (\,\mathbf{I}_4 \quad \mathbf{0}_{4\times1}\,); \quad \mathbf{T}_{22} = (\,\mathbf{I}_4 \quad \mathbf{1}_4\,); \\
\boldsymbol{\Xi}_1 &= \mathbf{0}_{2\times2} \oplus \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & -\beta_1 \end{pmatrix} \oplus 0; \quad \boldsymbol{\Xi}_2 = e^{\varphi_5}\boldsymbol{\Xi}_1; \\
\mathbf{w}_1 &= \mathbf{0}_{5\times1} \text{ and } \mathbf{w}_2 = \mathbf{e}_1^5.
\end{aligned}$$

The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 9$, $\dim(\boldsymbol{\tau}) = 0$, $\dim(\boldsymbol{\varphi}) = 5$, $\dim(\boldsymbol{\beta}) = 2$, and $d = 16$.

9.1.4 Null is Model 4, Alternative is Model 7

To generate model 7a in Table 18 from model 4 use operation (*ii*) to expand m_{13} and m_{23} from

$$m_{13} = m_{23} = 2 \text{ into } \mathbf{m}_{13} = \mathbf{m}_{23} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To satisfy proportionality, the rotation matrices in group 1 and group 2 must be identical. The design matrices for model 7 in Table 17 are

$$\mathbf{T}_{11} = \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{T}_{21} = \mathbf{T}_{22} = \mathbf{I}_5.$$

To satisfy Theorem 1, model 7 can be reparameterized as

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}; \quad \mathbf{T}_{11} = \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \\ \mathbf{T}_{21} &= \mathbf{T}_{22} = \mathbf{I}_4; \\ \boldsymbol{\Xi}_1 &= \boldsymbol{\Xi}_2 = \mathbf{0}_{2\times 2} \oplus \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & -\beta_1 \end{pmatrix} \oplus 0; \end{aligned}$$

The dimensions of the reparameterized model are $\dim(\boldsymbol{\mu}) = 9$, $\dim(\boldsymbol{\tau}) = 0$, $\dim(\boldsymbol{\varphi}) = 4$, $\dim(\boldsymbol{\beta}) = 2$, and $d = 15$.

9.1.5 Null is Model 4, Alternative is Model 8

No operations are required to generate model 8 in Table 17 from model 4. The design matrices for model 8 are

$$\mathbf{T}_{11} = \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{T}_{21} = \begin{pmatrix} \mathbf{I}_4 & \mathbf{0}_{4\times 1} \end{pmatrix}; \text{ and } \mathbf{T}_{22} = \begin{pmatrix} \mathbf{I}_4 & \mathbf{1}_4 \end{pmatrix}.$$

It is not necessary to reparameterize model 8 in order to satisfy Theorem 1 because the eigen-value multiplicities are identical under H_0 and H_a.

9.1.6 Null is Model 4, Alternative is Model 9

No operations are required to generate model 9 in Table 17 from model 4. The design matrices for model 9 are

$$\mathbf{T}_{11} = \mathbf{T}_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{T}_{21} = \mathbf{T}_{22} = \mathbf{I}_4.$$

It is not necessary to reparameterize model 9 in order to satisfy Theorem 1 because the eigen-value multiplicities are identical under H_0 and H_a.

9.1.7 Null is Model 4, Alternative is Model 3

No operations are required to generate model 3 in Table 17 from model 4. The design matrices for model 3 are

$$\mathbf{T}_{11} = \mathbf{T}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \mathbf{T}_{21} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{pmatrix}; \text{ and } \mathbf{T}_{22} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 4 \end{pmatrix}.$$

It is not necessary to reparameterize model 3 in order to satisfy Theorem 1 because the eigen-value multiplicities are identical under H_0 and H_a.

9.1.8 Null is Model 4, Alternative is Model 1

Corollary 1 verifies that model 1 can be reparameterized in such a manner as to satisfy Theorem 1. A suitable reparameterization is described in Corollary 1.

9.2 Tests Assuming Elliptical Distributions

Tables 20 and 21 display the adjusted test statistic and the estimated adjustment factors assuming that the data have been sampled from elliptical distributions. The adjustment factors in Table 20 were computed under the assumption that the two populations share a common kurtosis parameter. The pooled estimate is $\widehat{\kappa}_p = -0.235$. The adjustment factors in Table 21 were computed under the assumption that each population has its own kurtosis parameter. The separate estimates are $\widehat{\kappa}_1 = -0.298$ and $\widehat{\kappa}_2 = -0.189$. The approximation in (28) was used for both tables.

10 Brief User's Guide to the Matlab Programs

10.1 Modifications Required from User

The following steps are required to fit spectral models using the Matlab programs.

1. Build a model generating file that reads the data, computes the sample covariance matrices, and specifies the parameterization of the spectral models of interest. The models are numbered 1, 2, etc. By default, model 1 is the saturated model. A parameterization for model 1 need not be specified. Examine *vole_models.m* and *sparrow_models.m* for examples of model generating files.

2. Modify subprogram *spectral.m* by
    - (a) equating the name of the model generating file created in step 1 to the character variable *data.*;
    - (b) specifying the null and alternative models of interest;
    - (c) setting the variable *Bartlett* to 1 if a Bartlett correction is desired or to zero if the correction is not desired; and
    - (d) setting the variable *elliptical_correction* to 1 if the elliptical correction to the test statistic is desired or to zero if the correction is not desired.
3. Execute program *spectral.m.*

10.2 Description of Model Generating File

Files *vole_models.m* and *sparrow_models.m* illustrate the manner in which the model parameterizations are specified. These two files have a parallel structure to perform the following steps.

1. Read the data, compute the sample covariance matrices, and construct a vector containing the degrees of freedom associated with the sample covariances.

2. Initially, equate the various design matrices ($\mathbf{T}_{1i}$, $\mathbf{T}_{2i}$, $\mathbf{T}_{3i}$, and $\mathbf{w}_i$) to empty matrices. This step increases the efficiency of the programs.

3. Initially, equate the variable *permute_eigenvalues* to zero and equate *expand* and t to empty matrices. The variable *permute_eigenvalues* should be reset to one if the sample eigen-values need to be sorted in an order other than decreasing when computing initial guesses. If *permute_eigenvalues* $= 1$, then the program expects vectors *perm*$\{i\}$ for $i = 1, \ldots, g$, where *perm*$\{i\}$ is a permutation of the numbers $1, 2, \ldots, p$. In this case, sample eigen-values in the i^{th} group are sorted in the order given in *perm*$\{i\}$.

4. For each model to be specified, create an array of matrices *multi*$\{i\}$ for $i = 1, \ldots, g$. The j^{th} column of *multi*$\{i\}$ is the multiplicity vector $\mathbf{m}_{ij}$, possible with zeros appended at the end as place holders. The sum of the rows of *multi*$\{i\}$ is $\mathbf{r}_i'$. The matrix *multi*$\{i\}$ has d_i columns and as many rows as required to specify all $\mathbf{m}_{ij}$ vectors. For example, consider model 5 in Table 17. For this model,

$$\begin{aligned} \textit{multi}\{1\} &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} \text{ and} \\ \textit{multi}\{2\} &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

In group 1, $\mathbf{r}_1 = \mathbf{1}_5$ and $m_{1j} = 1$ for $j = 1, \ldots, 5$. Accordingly, $\boldsymbol{\Psi}_{1\cdot} = \mathbf{I}_5$, and

$$\boldsymbol{\Sigma}_1 = \boldsymbol{\Gamma}_0 \boldsymbol{\Lambda}_1 \boldsymbol{\Gamma}_0',$$

where each diagonal entry in $\boldsymbol{\Lambda}_1$ is distinct. In group 2,

$$\mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad m_{21} = m_{22} = 1; \text{ and } \mathbf{m}_{23} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Accordingly,

$$\begin{aligned} \boldsymbol{\Sigma}_2 &= \boldsymbol{\Gamma}_0 \boldsymbol{\Psi}_{2\cdot} \boldsymbol{\Lambda}_2 \boldsymbol{\Psi}_{2\cdot}' \boldsymbol{\Gamma}_0', \text{ where} \\ \boldsymbol{\Psi}_{i\cdot} &= \mathbf{I}_2 \oplus \boldsymbol{\Psi}_1; \end{aligned}$$

$\boldsymbol{\Psi}_1$ is a 3×3 orthogonal matrix; and each diagonal entry in $\boldsymbol{\Lambda}_2$ is distinct.

5. The diagonal blocks of

$$\boldsymbol{\Psi}_{i\cdot} = \bigoplus_{j=1}^{d_i} \boldsymbol{\Psi}_{t_{ij}}$$

   for $i = 1, \ldots, g$ are assigned index numbers $t(i, j)$ beginning with 1, 2, etc. Shared components are given the same index number. Components for which $r_{ij} = m_{ij}$ are identity matrices and are not numbered. The index numbers are stored in the matrix variable *t*.

6. If the extended parameterization is needed for computing either the Bartlett correction and/or the elliptical correction, then a $1 \times d_i$ matrix, *expand*$\{i\}$, is specified for $i = 1, \ldots, g$. The j^{th} entry of *expand*$\{i\}$ corresponds to the j^{th} column of *multi*$\{i\}$. Each entry in *expand*$\{i\}$ must be either 0, 1, or -1. A value of zero means that the extended parameterization (using β) should not be used to parameterize the corresponding $\boldsymbol{\Psi}_{t_{ij}}$. A value of 1 or -1 means that the extended parameterization (using β) should be used to parameterize the corresponding $\boldsymbol{\Psi}_{t_{ij}}$. If the value for the j^{th} entry in *expand*$\{i\}$ is 1, then a matrix variable named *block*$\{i, j\}$ must be defined. The matrix *block*$\{i, j\}$ consists of two columns. The first column specifies the dimension of a partitioning of r_{ij}. The second column specifies the type of expansion for the j^{th} partition. The program currently supports four types of expansions.

   Type 1 expansion: Create block diagonal symmetric matrices having no other restrictions.

   Type 2 expansion: Create block diagonal symmetric matrices for which the trace of each block is zero.

Type 3 expansion: Create block diagonal symmetric matrices for which the sum of the traces of the blocks is zero.

Type 4 expansion: Create a partitioned symmetric matrix for which the trace of each block on the diagonal is zero

For example, consider the reparameterization of model 5 in Table 17 (see §9.1.1 of this supplement). For this model,

$$\begin{aligned} \mathit{multi}\{1\} &= \begin{pmatrix} 1 & 1 & 2 & 1 \end{pmatrix}; \quad \mathit{multi}\{2\} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}; \\ \mathit{expand}\{1\} &= \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}; \quad \mathit{expand}\{2\} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}; \\ \mathit{block}\{1,3\} &= \begin{pmatrix} 2 & 2 \end{pmatrix}; \text{ and } \mathit{block}\{2,3\} = \begin{pmatrix} 2 & 4 \\ 1 & 4 \end{pmatrix}. \end{aligned}$$

The type 2 expansion in group 1 creates a 2 × 2 block having trace equal to zero. Because there is only a single block, a type 3 expansion would create the same structure. The type 4 expansion in group 2 creates a 3 × 3 partitioned matrix such that the first 2 × 2 block on the diagonal has trace zero and the next 1 × 1 block on the diagonal has trace zero.

If the j^{th} entry in *expand*$\{i\}$ is -1, then *block*$\{i,j\}$ is a one by two matrix with entries i', j'. The expansion of the j^{th} component of $\boldsymbol{\Psi}_{i\cdot}$ is shared with the expansion of the j'^{th} component of $\boldsymbol{\Psi}_{i'\cdot}$. This device is useful when specifying proportional models. For example, see the extended parameterization of model 7 in Table 17 (§9.1.4 of this supplement).

7. Values for the design matrices $\mathbf{T}_{1i}$, $\mathbf{T}_{2i}$, and $\mathbf{w}_i$ (if needed) are specified in any convenient manner. See the files *vole_models.m* and *sparrow_models.m* for examples.

10.3 Output

The program automatically displays the test statistics and p-values. Several other quantities that may be of interest also are available. Most of the quantities are stored as sparse matrices and all quantities refer to the null model. It is assumed that the model is parameterized as in equation 1 rather than as in equation (27). The eigen-values are parameterized as in equation (3).

1. Bias_lam: *Bias_lam*$\{i\}$ = estimated bias of the maximum likelihood estimator of $\boldsymbol{\lambda}_i$.

2. D_G: $\mathbf{D}_G^{(1)}$.

3. D_J: $D\_J\{t_{ij}\} = \mathbf{D}^{(1)}_{J_{t_{ij}}}$.
4. D_Jt: $D\_Jt\{i\} = \mathbf{D}^{(1)}_{J_{i\cdot}}$.
5. D_lam: $D\_lam\{i\} = \mathbf{D}^{(1)}_{\lambda_i}$.
6. di: $g \times 1$ vector, $di(i) = d_i = \dim(\mathbf{r}_i)$.
7. F1: $F1\{i\} = \mathbf{F}^{(1)}_i$ = matrix of first derivatives of $\operatorname{vec}\boldsymbol{\Sigma}_i$ with respect to $\boldsymbol{\theta}'$.
8. Gam: $Gam\{i\}$ = maximum likelihood estimate of $\boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}_0\boldsymbol{\Psi}_{i\cdot}$.
9. Gam0: maximum likelihood estimate of $\boldsymbol{\Gamma}_0$.
10. Ibar_33i: maximum likelihood estimate of $\overline{\mathbf{I}}^{-1}_{\boldsymbol{\theta},33}$.
11. I_theta: maximum likelihood estimate of $\mathbf{I}_{\boldsymbol{\theta},\infty}$.
12. Ibar_theta: maximum likelihood estimate of $\overline{\mathbf{I}}_{\boldsymbol{\theta},\infty}$.
13. Ibar_thetai: maximum likelihood estimate of $\overline{\mathbf{I}}^{-1}_{\boldsymbol{\theta},\infty}$.
14. Ig: $\mathbf{I}_g$.
15. Ip: $\mathbf{I}_p$.
16. Lam: $Lam\{i\}$ = maximum likelihood estimate of $\boldsymbol{\Lambda}_i$.
17. lnlik: maximized log likelihood function for the null model.
18. lnlik0: maximized log likelihood function for the saturated model.
19. M: $M\{i\} = \mathbf{M}_i$ of equation (18).
20. m_tilde: $\widetilde{\mathbf{m}}$.
21. model: model number.
22. n: $g \times 1$ vector containing degrees of freedom for the sample covariance matrices.
23. nbar: arithmetic average of the elements of $\mathbf{n}$.
24. Np: $p^2 \times p^2$ symmetric idempotent matrix given by $(\mathbf{I}_{p^2} + \mathbf{I}_{(p,p)})/2$.
25. nu: 3×1 vector containing $\dim(\boldsymbol{\mu})$, $\dim(\boldsymbol{\tau})$, and $\dim(\boldsymbol{\varphi})$.
26. Omega3: maximum likelihood estimate of $\boldsymbol{\Omega}_3$, the asymptotic covariance matrix of $\sqrt{\bar{n}}(\widehat{\boldsymbol{\lambda}} - \boldsymbol{\lambda})$. See equation (21).

27. p: dimension of $\mathbf{\Sigma}_i$.
28. phi: maximum likelihood estimate of $\boldsymbol{\varphi}$.
29. Psi_dot: *Psi_dot*$\{i\}$ = maximum likelihood estimate of $\mathbf{\Psi}_{i\cdot}$.
30. Psi_t: *Psi_t*$\{t_{ij}\}$ = maximum likelihood estimate of $\mathbf{\Psi}_{t_{ij}}$.
31. ri: *ri*$\{i\} = \mathbf{r}_i$, the $d_i \times 1$ vector of rotation dimensions for group i.
32. S: *S*$\{i\}$ = sample covariance matrix for group i.
33. SE_lam: $p \times g$ matrix containing the standard errors of the maximum likelihood estimators of $\boldsymbol{\lambda}_i$ for $i = 1, \ldots, g$.
34. Sig: *Sig*$\{i\}$ = maximum likelihood estimate of $\mathbf{\Sigma}_i$.
35. Sigi: *Sigi*$\{i\}$ = maximum likelihood estimate of $\mathbf{\Sigma}_i^{-1}$.
36. t: matrix containing $\{t_{ij}\}$ values.
37. T1: *T1*$\{i\} = \mathbf{T}_{1i}$.
38. T2: *T2*$\{i\} = \mathbf{T}_{2i}$.
39. useD_J: $g \times 1$ vector containing ones and zeros. The i^{th} entry is a one if $\mathbf{\Psi}_{i\cdot} \neq \mathbf{I}_p$.
40. Wsi: *Wsi*$\{i\}$ = maximum likelihood estimate of $\mathbf{\Sigma}_i^{-1} \otimes \mathbf{\Sigma}_i^{-1}$.

10.4 List of Matlab Subprograms

1. *commute.m:* Computes the permutation matrix $\mathbf{I}_{(a,b)}$.
2. *compute_params.m:* Given *multi*$\{i\}$ for $i = 1, \ldots, g$ and values for all non-zero t_{ij} index values, this subprogram computes $\{\mathbf{r}_i\}_{i=1}^{g}$; $\{d_i\}_{i=1}^{g}$; $\mathbf{m}_{ij} = \mathbf{m}(\boldsymbol{\lambda}_{ij})$; $\mathbf{m}_t$; $\widetilde{\mathbf{m}}$; $\dim(\boldsymbol{\mu})$; and $\dim(\boldsymbol{\tau})$. The subprogram also assigns zeros to entries of $\{t_{ij}\}$ for which $\mathbf{\Psi}_{t_{ij}}$ is an identity matrix and it creates an index matrix to partition $\boldsymbol{\tau}$ into $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2, \ldots,$ where $\boldsymbol{\tau}_{t_{ij}}$ is used to parameterize $\mathbf{\Psi}_{t_{ij}}$.
3. *dup.m:* Computes the duplication matrix, $\mathbf{D}_p$.
4. *FG_algorithm:* Computes maximum likelihood estimates for the CPC model using the F-G algorithm (Flury and Gautschi, 1986). This program is not called by any other program.
5. *kappa_eig.m:* Estimates the eigen-values $\{\boldsymbol{\omega}\}_{i=1}^{d_\Delta}$ that are required for the elliptical correction to the test statistic.

6. *kappa_estimate.m:* Estimates the kurtosis parameters assuming independent elliptical distributions. It is assumed that the raw data are stored in $N_i \times p$ matrices $\mathbf{Y}\{i\}$ for $i = 1, \ldots, g$ and that the only explanatory variable is an $N_i \times 1$ vector of ones.

7. *kron3.m:* Computes (right) Kronecker products of three or more input matrices.

8. *m_inv.m:* Computes the $\mathbf{m}^{-1}$ function.

9. *major.m:* Use a majorization algorithm (Kiers and ten Berge, 1992) to obtain initial guesses for $\boldsymbol{\Gamma}_0$ and $\{\boldsymbol{\Psi}_{i\cdot}\}_{i=1}^{g}$.

10. *make_E.m:* Constructs the elementary matrix $\mathbf{E}_{i,\boldsymbol{\nu}}$.

11. *make_T3.m:* Constructs the design matrices $\{\mathbf{T}_{3i}\}_{i=1}^{g}$ for the extended parameterization $\operatorname{vec}\boldsymbol{\Xi}_i = \mathbf{T}_{3i}\boldsymbol{\beta}e^{\mathbf{w}_i'\boldsymbol{\varphi}}$.

12. *multiplicity.m:* Computes the adjacent multiplicity function, $\mathbf{m}$.

13. *oplus.m:* Computes the direct sum of two or more matrices.

14. *oplus2.m:* Computes the direct sum of two matrices.

15. *rho.m:* Computes the index function $\boldsymbol{\rho}$.

16. *solve_eta.m:* Given a solution for $\boldsymbol{\mu}$, this function solves for $\boldsymbol{\eta}$ subject to $\mathbf{G} \approx \mathbf{I}_p$ and $\mathbf{G}'\mathbf{G} = \mathbf{I}_p$.

17. *sparrow.dat:* Contains the raw data for the sparrow example. This is a data file rather than a Matlab file.

18. *sparrow_models.m:* Contains the parameterizations of the models for the sparrow data.

19. *spectral.m:* Front end program that must be modified by the user.

20. *spectral_Bart.m:* Computes the Bartlett correction. The procedure is computationally intensive and may take a while.

21. *spectral_cpc.m:* Computes maximum likelihood estimates of $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda}_i$ in the CPC model using the Fisher scoring algorithm. This subprogram is a specialized version of *spectral_drive.m* and was used to compare the efficiency of the scoring algorithm to the FG algorithm.

22. *spectral_D2G.m:* Computes the matrices of second and third derivatives of $\operatorname{vec}\mathbf{G}$ with respect to $\boldsymbol{\mu}$.

23. *spectral_DG.m:* Computes the matrix of first derivatives of $\operatorname{vec}\mathbf{G}$ with respect to $\boldsymbol{\mu}$.

24. *spectral_drive.m:* Uses the Fisher scoring algorithm to compute the maximum likelihood estimates of $\boldsymbol{\Sigma}_i$ for $i = 1, \ldots, g$.

25. *spectral_guess.m:* Computes initial guesses for $\boldsymbol{\Gamma}_0$, $\{\boldsymbol{\Psi}_{i\cdot}\}_{i=1}^{g}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\beta}$.

26. *spectral_guess_Gam.m:* Computes initial guesses for $\boldsymbol{\Gamma}_0$. This subprogram is called by subprogram *spectral_guess.m*.

27. *spectral_guess_Lam.m:* Computes initial guesses for $\boldsymbol{\varphi}$ and $\boldsymbol{\beta}$. This subprogram is called by subprogram *spectral_guess.m*.

28. *spectral_guess_Psi.m:* Computes initial guesses for $\{\boldsymbol{\Psi}_{i\cdot}\}_{i=1}^{g}$. This subprogram is called by subprogram *spectral_guess.m*.

29. *spectral_model.m:* Calls *spectral_drive.m* to fit null and alternative models. This program writes a temporary file named *spectral_temp.mat*. The file is automatically deleted. It may be necessary to replace the Unix command

    ```
    !rm spectral_temp.mat
    ```

    with a windows command to delete the file.

30. *vec.m:* Stacks the columns of a matrix into a single column vector.

31. *vole_models.m:* Contains the parameterizations of the models for the vole data.

Table 12: Alternate Parameterizations of Models in Table 1

					Dimensions			
Model	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{m}_{ij}$	$\widetilde{\mathbf{m}}$	$\boldsymbol{\mu}$	$\boldsymbol{\tau}$	$\boldsymbol{\varphi}$	d
1. Saturated	10	10	$(\,\mathbf{1}_{10}\,)_{i=1,j=1}$ $(\,\mathbf{1}_{10}\,)_{i=2,j=1}$	10	90	0	20	110
2. Com. Space, Simul. Spher.	$\begin{pmatrix}7\\3\end{pmatrix}$	$\begin{pmatrix}7\\3\end{pmatrix}$	$(\,\mathbf{1}_{7}\,)_{i=1,j=1}$ $(\,\mathbf{1}_{7}\,)_{i=2,j=1}$	$\begin{pmatrix}7\\3\end{pmatrix}$	21	42	16	79
3. CPC	10	10	$(\,\mathbf{1}_{10}\,)_{i=1,j=1}$ $(\,\mathbf{1}_{10}\,)_{i=2,j=1}$ $t(1,1)$ $= t(2,1)$	10	0	45	20	65
4. Com. Space, Part. Spher.	$\begin{pmatrix}7\\3\end{pmatrix}$	$\begin{pmatrix}7\\3\end{pmatrix}$	$(\,\mathbf{1}_{7}\,)_{i=1,j=1}$ $\begin{pmatrix}3\\4\end{pmatrix}_{i=2,j=1}$	$\begin{pmatrix}7\\3\end{pmatrix}$	21	33	11	65
5. CPC, Simul. Spher.	$\begin{pmatrix}7\\3\end{pmatrix}$	$\begin{pmatrix}7\\3\end{pmatrix}$	$(\,\mathbf{1}_{7}\,)_{i=1,j=1}$ $(\,\mathbf{1}_{7}\,)_{i=2,j=1}$ $t(1,1)$ $= t(2,1)$	$\begin{pmatrix}7\\3\end{pmatrix}$	21	21	16	58

Table 13: Descriptions for Models of Vole Data

Model	Description	d
1	Saturated	40
2	Partial sphericity & eigen-value homogeneity within species ochrogaster	31
7	Partial CPC, first component is common	31
8	Partial CPC, first two components are common	25
3	CPC	22
4	CPC by species, partial sphericity & eigen-value homogeneity within species ochrogaster	21
5	Partial CPC, eigen-vectors shared within species, first component shared by all groups, partial sphericity & eigen-value homogeneity within species ochrogaster	18
6	CPC, partial sphericity & eigen-value homogeneity within species ochrogaster	16

Table 14: Parameterization of Models for Vole Data

Model	$\mathbf{r}_i$	$\mathbf{m}_{ij}$	$\widetilde{\mathbf{m}}$	Dim $\begin{pmatrix}\boldsymbol{\mu}\\ \boldsymbol{\tau}\\ \boldsymbol{\varphi}\end{pmatrix}$	d
1.	$(4)_{i\geq1}$	$(\mathbf{1}_4)_{i\geq1}$	4	$\begin{pmatrix}0\\24\\16\end{pmatrix}$	40
2.	$(\mathbf{1}_4)_{i=1}$ $(4)_{i\geq2}$	$\begin{pmatrix}1\\1\\2\end{pmatrix}_{i=2,j=1}$ $\begin{pmatrix}1\\2\\1\end{pmatrix}_{i\geq2,j=1}$	$\mathbf{1}_4$	$\begin{pmatrix}6\\15\\10\end{pmatrix}$	31
7.	$(\mathbf{1}_4)_{i=1}$ $\begin{pmatrix}1\\3\end{pmatrix}_{i\geq2}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}_{i\geq2,j=2}$	$\mathbf{1}_4$	$\begin{pmatrix}6\\9\\16\end{pmatrix}$	31
8.	$(\mathbf{1}_4)_{i=1}$ $\begin{pmatrix}1\\1\\2\end{pmatrix}_{i\geq2}$	$\begin{pmatrix}1\\1\end{pmatrix}_{i\geq2,j=3}$	$\mathbf{1}_4$	$\begin{pmatrix}6\\3\\16\end{pmatrix}$	25
3.	$(\mathbf{1}_4)_{i\geq1}$		$\mathbf{1}_4$	$\begin{pmatrix}6\\0\\16\end{pmatrix}$	22
4.	$(\mathbf{1}_4)_{i=1}$ $\begin{pmatrix}1\\1\\2\end{pmatrix}_{i=2}$ $(4)_{i\geq3}$	$(2)_{i=2,j=3}$ $\begin{pmatrix}1\\2\\1\end{pmatrix}_{i\geq3,j=1}$ $t_{31}=t_{41}=1$	$\mathbf{1}_4$	$\begin{pmatrix}6\\5\\10\end{pmatrix}$	21

Table 14 is continued on page 47.

Table 14 Continued

Model	$\mathbf{r}_i$	$\mathbf{m}_{ij}$	$\widetilde{\mathbf{m}}$	Dim $\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\tau} \\ \boldsymbol{\varphi} \end{pmatrix}$	d
5.	$\left(\mathbf{1}_4 \right)_{i=1}$ $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}_{i=2}$ $\begin{pmatrix} 1 \\ 3 \end{pmatrix}_{i\geq 3}$	$\left(2 \right)_{i=2,j=3}$ $\begin{pmatrix} 2 \\ 1 \end{pmatrix}_{i\geq 3,j=2}$ $t_{32} = t_{42} = 1$	$\mathbf{1}_4$	$\begin{pmatrix} 6 \\ 2 \\ 10 \end{pmatrix}$	18
6.	$\left(\mathbf{1}_4 \right)_{i=1}$ $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}_{i=2}$ $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}_{i\geq 3}$	$\left(2 \right)_{i=2,j=3}$ $\left(2 \right)_{i\geq 3,j=2}$	$\mathbf{1}_4$	$\begin{pmatrix} 6 \\ 0 \\ 10 \end{pmatrix}$	16

Table 15: Likelihood Ratio Tests on Vole Models

Table entries are $\begin{pmatrix} d_\Delta \\ Q \\ Q_c \end{pmatrix}$. Bartlett corrected statistics are **boldface** if reparameterization is required.

Model							
	H_0						
H_a	6	5	4	3	8	7	2
5	2 10.34 9.98						
4	5 14.93 14.50	3 4.59 4.48					
3	6 3.94 3.81	Not Nested	Not Nested				
8	9 8.89 **8.64**	Not Nested	Not Nested	3 4.951 4.857			
7	15 19.95 **19.34**	13 9.62 **9.33**	Not Nested	9 16.02 15.55	6 11.07 10.69		
2	15 28.46 27.40	13 18.12 17.49	10 13.53 13.02	Not Nested	Not Nested	Not Nested	
1	24 32.70 **31.81**	22 22.36 **21.76**	19 17.77 **17.28**	18 28.76 28.04	15 23.81 23.18	9 12.75 12.47	9 4.25 **4.17**

Table 16: Descriptions for Models of Sparrow Data

Model	Description	d
1	Saturated	30
5	Partial CPC, first two components common	23
6	CPC	20
2	Proportional with unrestricted eigen-values in group 1	16
7	Complete homogeneity, $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$	15
8	Proportional, partial sphericity (3^{rd} and 4^{th} components)	14
9	Complete homogeneity, partial sphericity	13
3	Proportional, partial sphericity & exponential trend	12
4	Homogeneity, partial sphericity & exponential trend	11

Table 17: Parameterization of Models for Sparrow Data

Model	$\mathbf{r}_i$	$\mathbf{m}_{ij}$	$\widetilde{\mathbf{m}}$	Dim $\begin{pmatrix}\boldsymbol{\mu}\\ \boldsymbol{\tau}\\ \boldsymbol{\varphi}\end{pmatrix}$	d
1.	$(\,5\,)_{i\geq 1}$	$(\,\mathbf{1}_5\,)_{i\geq 1}$	5	$\begin{pmatrix}0\\ 20\\ 10\end{pmatrix}$	30
5.	$(\,\mathbf{1}_5\,)_{i=1}$ $\begin{pmatrix}1\\ 1\\ 3\end{pmatrix}_{i=2}$	$\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}_{i=2,j=3}$	$\mathbf{1}_5$	$\begin{pmatrix}10\\ 3\\ 10\end{pmatrix}$	23
6.	$(\,\mathbf{1}_5\,)_{i\geq 1}$		$\mathbf{1}_5$	$\begin{pmatrix}10\\ 0\\ 10\end{pmatrix}$	20
2.	$(\,\mathbf{1}_5\,)_{i\geq 1}$		$\mathbf{1}_5$	$\begin{pmatrix}10\\ 0\\ 6\end{pmatrix}$	16
7.	$(\,\mathbf{1}_5\,)_{i\geq 1}$		$\mathbf{1}_5$	$\begin{pmatrix}10\\ 0\\ 5\end{pmatrix}$	15
8.	$\begin{pmatrix}1\\ 1\\ 2\\ 1\end{pmatrix}_{i\geq 1}$	$(\,2\,)_{i\geq 1,j=3}$	$\begin{pmatrix}1\\ 1\\ 2\\ 1\end{pmatrix}$	$\begin{pmatrix}9\\ 0\\ 5\end{pmatrix}$	14
9.	$\begin{pmatrix}1\\ 1\\ 2\\ 1\end{pmatrix}_{i\geq 1}$	$(\,2\,)_{i\geq 1,j=3}$	$\begin{pmatrix}1\\ 1\\ 2\\ 1\end{pmatrix}$	$\begin{pmatrix}9\\ 0\\ 4\end{pmatrix}$	13
3.	$\begin{pmatrix}1\\ 1\\ 2\\ 1\end{pmatrix}_{i\geq 1,j=1}$	$(\,2\,)_{i\geq 1,j=3}$	$\begin{pmatrix}1\\ 1\\ 2\\ 1\end{pmatrix}$	$\begin{pmatrix}9\\ 0\\ 3\end{pmatrix}$	12
4.	$\begin{pmatrix}1\\ 1\\ 2\\ 1\end{pmatrix}_{i\geq 1,j=1}$	$(\,2\,)_{i\geq 1,j=3}$	$\begin{pmatrix}1\\ 1\\ 2\\ 1\end{pmatrix}$	$\begin{pmatrix}9\\ 0\\ 2\end{pmatrix}$	11

Table 18: Alternate Parameterizations of Models for Sparrow Data

Model	$\mathbf{r}_i$	$\mathbf{m}_{ij}$	$\widetilde{\mathbf{m}}$	Dim $\begin{pmatrix}\boldsymbol{\mu}\\ \boldsymbol{\tau}\\ \boldsymbol{\varphi}\end{pmatrix}$	d
5a.	$\begin{pmatrix}1\\1\\3\end{pmatrix}_{i\geq 1}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}_{i\geq 1, j=3}$	$\begin{pmatrix}1\\1\\3\end{pmatrix}$	$\begin{pmatrix}7\\6\\10\end{pmatrix}$	23
2a.	$\begin{pmatrix}1\\1\\2\\1\end{pmatrix}_{i\geq 1}$	$\begin{pmatrix}1\\1\end{pmatrix}_{i\geq 1, j=3}$ $t_{13} = t_{23} = 1$	$\begin{pmatrix}1\\1\\2\\1\end{pmatrix}$	$\begin{pmatrix}9\\1\\6\end{pmatrix}$	16
7a.	$\begin{pmatrix}1\\1\\2\\1\end{pmatrix}_{i\geq 1}$	$\begin{pmatrix}1\\1\end{pmatrix}_{i\geq 1, j=3}$ $t_{13} = t_{23} = 1$	$\begin{pmatrix}1\\1\\2\\1\end{pmatrix}$	$\begin{pmatrix}9\\1\\5\end{pmatrix}$	15

Table 19: Likelihood Ratio Tests on Sparrow Models Assuming Normality

Table entries are $\begin{pmatrix} d_\Delta \\ Q \\ Q_c \end{pmatrix}$. Bartlett corrected statistics are **boldface** if reparameterization is required.

Model								
	H_0							
H_a	4	3	9	8	7	2	6	5
3	1 1.95 1.94							
9	2 1.32 1.27	Not Nested						
8	3 3.59 3.49	2 1.64 1.58	1 2.27 2.26					
7	4 6.86 **6.54**	Not Nested	2 5.54 **5.26**	Not Nested				
2	5 9.13 **8.78**	4 7.18 **6.85**	3 7.81 **7.52**	2 5.54 **5.25**	1 2.26 2.25			
6	9 11.20 Not χ^2	8 9.25 Not χ^2	7 9.88 Not χ^2	6 7.60 Not χ^2	5 4.33 3.74	4 2.07 1.72		
5	12 15.30 **13.81**	11 13.35 **11.95**	10 13.98 **12.48**	9 11.71 **10.34**	8 8.44 7.42	7 6.17 5.34	3 4.11 3.73	
1	19 18.65 **16.92**	18 16.70 **15.07**	17 17.33 **15.62**	16 15.06 **13.49**	15 11.79 10.55	14 9.52 8.46	10 7.45 6.79	7 3.35 3.06

Table 20: Tests on Sparrow Models Assuming Elliptical Distributions I
Common Kurtosis Parameter

Table entries are $\begin{pmatrix} k_1 \\ k_2 \\ k_1 Q \end{pmatrix}$. Adjusted statistic, k_1Q is **boldface** if reparameterization is required.

Model								
	H_0							
H_a	4	3	9	8	7	2	6	5
3	5.66 1.00 11.04							
9	1.31 2.00 1.73	Not Nested						
8	1.42 2.42 5.10	1.31 2.00 2.15	5.66 1.00 12.87					
7	1.31 4.00 **8.98**	Not Nested	1.31 2.00 **7.25**	Not Nested				
2	1.36 4.42 **12.46**	1.31 4.00 **9.39**	1.42 2.42 **11.09**	1.31 2.00 **7.24**	5.66 1.00 12.82			
6	Not χ^2	Not χ^2	Not χ^2	Not χ^2	1.36 4.42 5.91	1.31 4.00 2.70		
5	1.33 11.41 **20.33**	1.31 11.00 **17.46**	1.33 9.41 **18.64**	1.31 9.00 **15.31**	1.34 7.41 11.31	1.31 7.00 8.07	1.31 3.00 5.37	
1	1.32 18.41 **24.63**	1.31 18.00 **21.84**	1.32 16.41 **22.91**	1.31 16.00 **19.69**	1.32 14.41 15.61	1.31 14.00 12.45	1.31 10.00 9.75	1.31 7.00 4.38

Table 21: Tests on Sparrow Models Assuming Elliptical Distributions II
Separate Kurtosis Parameters

Table entries are $\begin{pmatrix} k_1 \\ k_2 \\ k_1Q \end{pmatrix}$. Adjusted statistic, k_1Q is **boldface** if reparameterization is required.

	Model							
	H_0							
H_a	4	3	9	8	7	2	6	5
3	8.37 1.00 16.31							
9	1.31 2.00 1.73	Not Nested						
8	1.39 2.30 5.00	1.31 2.00 2.15	8.37 1.00 19.02					
7	1.31 4.00 **8.98**	Not Nested	1.31 2.00 **7.25**	Not Nested				
2	1.35 4.29 **12.33**	1.31 4.00 **9.39**	1.39 2.30 **10.88**	1.31 2.00 **7.24**	8.37 1.00 18.95			
6	Not χ^2	Not χ^2	Not χ^2	Not χ^2	1.38 4.30 5.98	1.34 4.00 2.76		
5	1.34 11.25 **20.45**	1.32 10.96 **17.63**	1.35 9.27 **18.82**	1.33 8.98 **15.53**	1.36 7.30 11.49	1.34 7.00 8.25	1.36 3.00 5.59	
1	1.34 18.25 **24.92**	1.33 17.96 **22.15**	1.34 16.27 **23.25**	1.33 15.98 **20.04**	1.35 14.30 15.90	1.34 14.00 12.72	1.35 10.00 10.03	1.34 7.00 4.48