RIGIDITY IN ONE-DIMENSIONAL TILING SPACES

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Abstract. Suppose $\mathcal{T}_\varphi$ and $\mathcal{T}_\theta$ are tiling spaces arising from primitive nonperiodic substitutions φ and θ. Suppose F_φ and F_θ denote the corresponding inflation and substitution maps on the respective tiling spaces. We prove that $\mathcal{T}_\varphi$ and $\mathcal{T}_\theta$ are homeomorphic if and only if there exist positive integers m and n such that F_φ^n and F_θ^m are topologically conjugate.

1. One-Dimensional Tiling Spaces

Let $\mathcal{A} = \{1, 2, \ldots, |\mathcal{A}|\}$ and $\mathcal{B} = \{1, 2, \ldots, |\mathcal{B}|\}$ be finite alphabets; $\mathcal{A}^*$ will denote the collection of finite nonempty words with letters in $\mathcal{A}$. Given a map $\tau : \mathcal{A} \to \mathcal{B}^*$, there is an associated transition matrix $A_\tau = (a_{ij})_{i\in\mathcal{B}, j\in\mathcal{A}}$ in which a_{ij} is the number of occurrences of i in the word $\tau(j)$. A map $\tau : \mathcal{A} \to \mathcal{B}^*$ extends naturally to $\tau : \mathcal{A}^* \to \mathcal{B}^*$. A **substitution** is a map $\varphi : \mathcal{A} \to \mathcal{A}^*$; φ is **primitive** if $\varphi^n(i)$ contains j for all $i, j \in \mathcal{A}$ and sufficiently large n. Equivalently, φ is primitive if and only if the matrix A_φ is aperiodic, in which case A_φ has an eigenvalue λ_φ larger in modulus than its remaining eigenvalues called the Perron-Frobenius eigenvalue of A_φ (and φ).

The substitutions $\varphi : \mathcal{A} \to \mathcal{A}^*$ and $\chi : \mathcal{B} \to \mathcal{B}^*$ are **weakly equivalent**, $\varphi \sim_w \chi$, if there are sequences of positive integers $\{n_i\}$, $\{m_i\}$, and maps

$$\sigma_i : \mathcal{A} \to \mathcal{B}^*, \quad \tau_i : \mathcal{B} \to \mathcal{A}^* \quad i = 1, 2, \ldots,$$

such that the following infinite diagram commutes:

$$\begin{tikzcd}
& \mathcal{A}^* \arrow[dl, "\sigma_1"'] & & \mathcal{A}^* \arrow[ll, "\varphi^{m_1}"'] \arrow[dl, "\sigma_2"'] & & \cdots \\
\mathcal{B}^* & & \mathcal{B}^* \arrow[ll, "\chi^{n_1}"] \arrow[ul, "\tau_1"'] & & \mathcal{B}^* \arrow[ll, "\chi^{n_2}"] \arrow[ul, "\tau_2"'] &
\end{tikzcd}$$

Weak equivalence of substitutions implies weak equivalence of the corresponding substitution matrices. It is known that two matrices are weakly equivalent iff there is a positive isomorphism between their induced dimension groups (see e.g. Swanson-Volkmer [4]).

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Consider a primitive substitution $\varphi : \mathcal{A} \to \mathcal{A}^*$ with matrix A_φ, Perron-Frobenius eigenvalue λ_φ, and associated positive (left) eigenvector $\vec{v}_\varphi$ with entries $\lambda_1, \ldots, \lambda_{card(\mathcal{A})}$. The intervals $P_i = [0, \lambda_i]$, $i = 1, \ldots, card(\mathcal{A})$, are called **prototiles** (consider P_i to be distinct from P_j for $i \neq j$ even if $\lambda_i = \lambda_j$). A **tiling** T of $\mathbf{R}$ by the prototiles is a collection $T = \{T_i\}_{i=-\infty}^{\infty}$ of tiles T_i for which $\bigcup_{i=-\infty}^{\infty} T_i = \mathbf{R}$, each T_i is a translate of some P_j, and $T_i \cap T_j$ is a singleton for each $i \neq j$. We will generally assume that the indexing is such that if $i < j$, then T_i is to the left of T_j and that $0 \in T_0 \setminus T_1$.

If $\varphi(i) = i_1 i_2 \ldots i_n$, then $\lambda_\varphi \lambda_i = \sum_{j=1}^{n} \lambda_{i_j}$. Thus $|\lambda_\varphi P_i| = \sum_{j=1}^{n} |P_{i_j}|$, and $\lambda_\varphi P_i$ is tiled by $\{T_j\}_{j=1}^{n}$, where $T_j = P_{i_j} + \sum_{k=1}^{j-1} \lambda_{i_k}$. This process is called inflation and substitution and extends to a map F_φ taking a tiling $T = \{T_i\}_{i=-\infty}^{\infty}$ of $\mathbf{R}$ by prototiles to a new tiling, $F_\varphi(T)$, of $\mathbf{R}$ by prototiles defined by inflating, substituting, and suitably translating each T_i. More precisely, for $w = w_1 \ldots w_n \in \mathcal{A}^*$, define

$$\mathcal{P}_w + t = \{P_{w_1} + t, P_{w_2} + t + |P_{w_1}|, \ldots, P_{w_n} + t + \Sigma_{i<n} |P_{w_i}|\}.$$

Then $F_\varphi(P_i + t) = \mathcal{P}_{\varphi(i)} + \lambda_\varphi t$ and $F_\varphi(\{P_{k_i} + t_i\}_{i \in \mathbf{Z}}) = \cup_{i \in \mathbf{Z}} (\mathcal{P}_{\varphi(k_i)} + \lambda_\varphi t_i)$.

There is a natural topology on the collection Σ_φ of all tilings of $\mathbf{R}$ by prototiles ($\{T_i\}_{i=-\infty}^{\infty}$ and $\{T_i'\}_{i=-\infty}^{\infty}$ are "close" if there is an ϵ near 0 so that $\{T_i\}_{i=-\infty}^{\infty}$ and $\{T_i' + \epsilon\}_{i=-\infty}^{\infty}$ are identical in a large neighborhood of 0 (see [1] for details)). The space Σ_φ is compact and metrizable with this topology and $F_\varphi : \Sigma_\varphi \to \Sigma_\varphi$ is continuous. The **tiling space associated with** φ, $\mathcal{T}_\varphi$, is defined as the collection of tilings with the following property: $T \in \mathcal{T}_\varphi$ if whenever P is any segment of T with compact support, then there are $n \in \mathbf{N}$, $i \in \mathcal{A}$ and $t \in \mathbf{R}$ such that $P \subseteq F_\varphi^n(P_i + t)$. There is a natural flow (translation) given by $(\{T_i\}_{i=-\infty}^{\infty}, t) = \{T_i - t\}_{i=-\infty}^{\infty}$. This flow is minimal and each $T \in \mathcal{T}_\varphi$ is uniformly recurrent. It follows that $\mathcal{T}_\varphi$ is a continuum. Finally, $F_\varphi : \mathcal{T}_\varphi \to \mathcal{T}_\varphi$ is a homeomorphism.

Let W_φ denote the set of **allowed bi-infinite words** for φ. That is, $w \in W_\varphi$ if and only if for each finite subword w' of w, there are $i \in \mathcal{A}$ and $n \in \mathbf{N}$ such that w' is a subword of $\varphi^n(i)$. We identify the 0^{th} coordinate in a bi-infinite word w by either an indexing, as in $w = \ldots w_{-1} w_0 w_1 \ldots$, or by use of a decimal point. The substitution $\varphi : \mathcal{A} \to \mathcal{A}^*$ extends to $\varphi : W_\varphi \to W_\varphi$ where

$$\varphi(\ldots w_{-1} w_0 w_1 \ldots) = \ldots \varphi(w_{-1}) \,.\, \varphi(w_0) \varphi(w_1) \ldots.$$

The word w is **periodic** for φ, or $\varphi-$periodic, if for some $m \in \mathbf{N}$,

$$\varphi^m(w) = \ldots \varphi^m(w_{-1}).\varphi^m(w_0)\varphi^m(w_1) \ldots = \ldots w_{-1}.w_0 w_1 \ldots.$$

Each substitution has at least one allowed periodic bi-infinite word which is necessarily uniformly recurrent under the shift $\cdots w_{-1}.w_0 w_1 \cdots \mapsto \cdots\cdots w_{-1} w_0.w_1 \cdots$. (For instance, if ij is a subword of $\varphi(k)$ for some i, j, k, then as $n \to \infty$, the finite words $\varphi^n(i).\varphi^n(j)$ converge to a cycle of allowed $\varphi-$periodic, bi-infinite words that are uniformly recurrent under the shift.) A primitive substitution φ is **nonperiodic** if at least one (equivalently, each) $\varphi-$periodic bi-infinite word is not periodic under the shift. (Some authors use the term **aperiodic** where we use **nonperiodic** in referring to substitutions, and **primitive** rather than **aperiodic** in referring to matrices.)

2. Main Result

Theorem 2.1 (Main Result). *Suppose that φ and θ are primitive nonperiodic substitutions with induced inflation and substitution homeomorphisms F_φ and F_θ acting on the associated tiling spaces $\mathcal{T}_\varphi$ and $\mathcal{T}_\theta$. Then $\mathcal{T}_\varphi$ and $\mathcal{T}_\theta$ are homeomorphic if and only if there exist positive integers m and n such that F_φ^n and F_θ^m are topologically conjugate.*

If $w = w_0w_1\cdots$ is a word fixed by the primitive and nonperiodic substitution τ and u is a prefix of w, then a factor, i.e. subword, v of w is called a **return word to** u provided

(a) vu is a factor of w,
(b) u is a prefix of vu and
(c) u occurs exactly twice in vu.

The collection $\mathcal{R}_u$ of all return words to u is finite ([3]), and for each $v \in \mathcal{R}_u$, the word $\tau(u)$ can be uniquely factored as a product of elements of $\mathcal{R}_u$. This defines the substitution $\tau_u : \mathcal{R}_u \to \mathcal{R}_u^*$.

Lemma 2.2 (F. Durand [3]). *Let τ and σ be two primitive substitutions having the same nonperiodic infinite word w as a fixed point. There exist a prefix u of w and integers i and j such that $\tau_u^i = \sigma_u^j$.*

Lemma 2.3. *If τ is a primitive and nonperiodic substitution fixing the right infinite word w and u is a nonempty prefix of w, then the systems $(\mathcal{T}_\tau, F_\tau)$ and $(\mathcal{T}_{\tau_u}, F_{\tau_u})$ are topologically conjugate.*

Proof. If consecutive tiles $T_n, T_{n+1}, \ldots, T_{n+k}$ of $T \in \mathcal{T}_\tau$ are of type $i_0, i_1, \ldots, i_k$, with $v = i_0i_1\ldots i_k$ a return word for u, the union of these tiles is a tile of type v for τ_u. As the allowed bi-infinite word for τ determined by $T = \{T_i\}_{i=-\infty}^{\infty}$ factors uniquely as a product of return words to u, the above process produces a well defined $T' \in \mathcal{T}_{\tau_u}$ for each $T \in \mathcal{T}_\tau$. This process is clearly reversible and yields a conjugacy between F_τ and F_{τ_u}. □

Two distinct tilings T, T' of a tiling space $\mathcal{T}_\varphi$ are said to be **forward** (**backward**) **asymptotic** provided the distance between $T - t$ and $T' - t$ goes to zero as $t \to \infty$ ($t \to -\infty$). If T and T' are asymptotic, we will also say that their composants $\mathcal{C} = \{T - t : t \in \mathbf{R}\}$ and $\mathcal{C}' = \{T' - t : t \in \mathbf{R}\}$ are **asymptotic composants**. Every tiling space of a primitive nonperiodic substitution has a finite nonzero number of pairs of forward and of backward asympotic composants ([2]), and every composant $\mathcal{C}$ that is asymptotic to some other composant $\mathcal{C}'$ has on it a tiling T that is periodic under the inflation and substitution homeomorphism. Now, given any tiling T' of $\mathcal{T}_\varphi$, we mark the tiles of T' as follows. For each $T = \{T_i\}_{i=-\infty}^{\infty}$ that is periodic under F_φ and that lies on an asymptotic composant, let j be the type of T_0 and let x_T be such that $T_0 = [0, \lambda_j] - x_T$. For every tile T_n of T' that is of type j, say $T'_n = [0, \lambda_j] + t_n$, put a mark in $T'_n \subset \mathbf{R}$ at position $t_n + x_T$. The collection of all such marks breaks $\mathbf{R}$ into a set of translates of intervals of finitely many distinct lengths (these intervals constitute a new set of prototiles) and of finitely many types (the types being determined by the types, and the order, of the tiles of T' that the translates meet). The inflation and substitution induced by φ then determines the **derived substitution** φ^*, whose alphabet may be taken to be the new set of prototiles. The substitution φ^* is primitive and nonperiodic, and it is easy to see that the systems $(\mathcal{T}_\varphi, F_\varphi)$ and $(\mathcal{T}_{\varphi^*}, F_{\varphi^*})$ are topologically conjugate. See [2] for more details.

If φ is a substitution with $i \mapsto i_1 \cdots i_n$, then the **reverse** of φ is the substitution with $i \mapsto i_n \cdots i_1$.

Lemma 2.4 (Barge and Diamond [2]). *For primitive nonperiodic substutions φ and ψ, $\mathcal{T}_\varphi$ and $\mathcal{T}_\psi$ are homeomorphic if and only if the derived substitutions φ^* and θ^* (or φ^* and the reverse of θ^*) are weakly equivalent.*

Proof. (of Theorem 2.1) Suppose that $\mathcal{T}_\varphi$ and $\mathcal{T}_\theta$ are homeomorphic, and let φ^* and θ^* be the derived substitutions defined above. By Lemma 2.4, either φ^* and θ^* are weakly equivalent or φ^* and the reverse of θ^* are weakly equivalent. Since the reverse of θ^* is the derived substitution of the reverse of θ, we may assume without loss of generality that φ^* and θ^* are weakly equivalent. Because $(\mathcal{T}_{\sigma^n}, F_{\sigma^n}) = (\mathcal{T}_\sigma, F^n_\sigma)$, we may further suppose, without loss of generality, that all right-infinite words, periodic under φ^* or θ^*, are actually fixed.

Let σ_i and τ_i be maps, and m_i, n_i natural numbers so that

$$\sigma_i \tau_i = (\theta^*)^{n_i} \text{ and } \tau_i \sigma_{i+1} = (\varphi^*)^{m_i}, \quad i = 1, 2, \ldots$$

For a right-infinite word w, fixed by φ^*, one sees that

$$\sigma_1(w) = \sigma_1\left((\varphi^*)^{\sum_{i=1}^{k} m_i}(w)\right) = (\theta^*)^{\sum_{i=1}^{k} n_i} \sigma_{k+1}(w), \quad \text{for all } k$$

must be fixed by θ^*. Similarly, if v is a right-infinite word fixed by θ^*, then $\tau_1(v)$ must be fixed by φ^*. Thus, the substitution $\tau_1\sigma_1$ carries the finite collection of right-infinite words fixed by φ^* into itself, and there must be a natural number k and a right-infinite fixed word w of φ^* such that $(\tau_1\sigma_1)^k(w) = w$. Since $\sigma_1\tau_1 = (\theta^*)^{n_1}$ is primitive, so is $\tau_1\sigma_1$ and, hence, also $(\tau_1\sigma_1)^k$.

By Durand's result (Lemma 2.2), there is a nonempty prefix u of w and natural numbers i, j for which

$$(\varphi^*)^i_u = \left((\tau_1\sigma_1)^k\right)^j_u = \left((\tau_1\sigma_1)^{kj}\right)_u.$$

For convenience, put $\gamma := (\tau_1\sigma_1)^{kj}$, and we arrive at

$$(\mathcal{T}_\varphi, F^i_\varphi) \simeq (\mathcal{T}_{\varphi^*}, F^i_{\varphi^*}) \simeq (\mathcal{T}_{(\varphi^*)^i_u}, F_{(\varphi^*)^i_u}) \simeq (\mathcal{T}_{\gamma_u}, F_{\gamma_u}) \simeq (\mathcal{T}_\gamma, F_\gamma),$$

where $\simeq$ denotes topological conjugacy.

Now let $\sigma := \sigma_1(\tau_1\sigma_1)^{kj-1}$ and let $\tau := \tau_1$. Then $\gamma = \tau\sigma$ while $\sigma\tau = (\sigma_1\tau_1)^{kj} = (\theta^*)^{n_1kj}$. That is, γ is shift equivalent with $(\theta^*)^{n_1kj}$ and from ([2])

$$(\mathcal{T}_\gamma, F_\gamma) \simeq (\mathcal{T}_{(\theta^*)^{n_1kj}}, F_{(\theta^*)^{n_1kj}}).$$

Finally,

$$(\mathcal{T}_{(\theta^*)^{n_1kj}}, F_{(\theta^*)^{n_1kj}}) = (\mathcal{T}_{\theta^*}, F_{\theta^*}^{n_1kj}) \simeq (\mathcal{T}_\theta, F_\theta^{n_1kj}),$$

so that $(\mathcal{T}_\varphi, F_\varphi^i) \simeq (\mathcal{T}_\theta, F_\theta^{n_1kj})$. □

3. Nonrigidity in Solenoids and Higher Dimensional Tiling Spaces

If $f : M \to M$ is a diffeomorphism of the manifold M for which $A \subset M$ is a connected and orientable one-dimensional hyperbolic attractor, then the system (A, f_A) is topologically conjugate with either the shift homeomorphism on a classic solenoid or with $(\mathcal{T}_\varphi, F_\varphi)$ for some primitive nonperiodic substitution φ ([5],[1] and [2]). The solenoids (at least the n-adic solenoids for which n has a least two distinct prime factors) lack dynamical rigidity in the sense that two solenoids may be homeomorphic without any positive powers of their shift homeomorphisms being conjugate (Example 3.1). In light of this, our main theorem, that all one-dimensional substitution tiling spaces display such rigidity, is surprising.

Example 3.1. For $n \in \mathbf{N}$ the inverse limit of the map $z \mapsto z^n$, on the complex unit circle $\mathbf{T}^1$, given by

$$\mathbf{S}_n := \{(z_1, z_2, \dots) : \ z_i \in \mathbf{T}^1, \ z_{i+1}^n = z_i, \ \text{for all } i \in \mathbf{N}\},$$

with the product topology, is called the **n-adic solenoid**, and $\sigma_n : \mathbf{S}_n \to \mathbf{S}_n$ with $(z_1, z_2, \dots) \overset{\sigma_n}{\mapsto} (z_1^n, z_2^n, \dots)$ is the **shift homeomorphism**. The commuting diagram, below,

$$\begin{tikzcd}
\mathbf{T}^1 & & \mathbf{T}^1 \arrow[ll, "(6)"'] \arrow[dl, "(2)"'] & & \mathbf{T}^1 \arrow[ll, "(6^2)"'] \arrow[dl, "(6)"] & & {} \arrow[ll, "(6)"'] & \cdots \\
& \mathbf{T}^1 \arrow[ul, "(3)"] & & \mathbf{T}^1 \arrow[ll, "(12)"] \arrow[ul, "(6)"'] & & \mathbf{T}^1 \arrow[ll, "(12)"] \arrow[ul, "(2)"'] & & {} \arrow[ll]
\end{tikzcd}$$

in which $(n) : \mathbf{T}^1 \to \mathbf{T}^1$ denotes $z \mapsto z^n$, suggests a recipe for continuing indefinitely to the right, and determines a homeomorphism between $\mathbf{S}_6$ and $\mathbf{S}_{12}$. But there can be no stationary (i.e., periodic) diagram of the above sort: no nonzero power of σ_6 is conjugate to any power of σ_{12}. One way to see this is to observe that σ_n has topological entropy $h(\sigma_n) = \log n$ and that $\frac{\log 6}{\log 12}$ is irrational. Entropy, of course, is an invariant of topologicai conjugacy.

Example 3.2. This example, due to Lorenzo Sadun, shows that Theorem 2.1 does not extend to higher dimensional tiling spaces. Let τ denote the Thue-Morse substitution: $\tau(a) = ab$ and $\tau(b) = ba$. Let φ denote the Fibonacci substitution $\varphi(a) = ab$ and $\varphi(b) = a$.

The two-dimensional [1] tiling spaces $\mathcal{T}_{\tau^2\times\varphi}$ and $\mathcal{T}_{\tau\times\varphi}$ are identical, yet no powers of the the inflation and substitution homeomorphisms are conjugate. Again, checking entropy, $\frac{h(F_{\tau^2\times\varphi})}{h(F_{\tau\times\varphi})} = \frac{\log(4\lambda)}{\log(2\lambda)}$, for $\lambda = \frac{1+\sqrt{5}}{2}$, is not rational.

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[1] The prototiles of $\mathcal{T}_{\alpha\times\beta}$ are rectangles in which inflation and substitution $F_{\alpha\times\beta}$ acts as the product of F_α and F_β (see [1] for the precise definition). Consequently, $(\mathcal{T}_{\alpha\times\beta}, F_{\alpha\times\beta}) \simeq (\mathcal{T}_\alpha \times \mathcal{T}_\beta, F_\alpha \times F_\beta)$.