Sample test B for Chapter 14

(1) (15 pts) Match the function $z = f(x, y)$ on the left with the correct description of the contour plot on the right: No justification required.

_____ (a) $z = (x - y^2)^3$ (i) A family of hyperbolas

_____ (b) $z = (x^2 - y^2)^{-1}$ (ii) A family of parabolas

_____ (c) $z = e^{y-x}$ (iii) A family of straight lines

(iv) None of these.

(2) (20 pts) Find an equation for the tangent line to the level curve

$$xy(x + y)^2 = 4,$$

at the point $(1, 1)$. Be sure to verify that the point really lies on the level curve.

(3) (15 pts) Suppose $w = \dfrac{xz^2}{y}$, $x = u^2 - uv$, $z = \tan(1/u)$, $y = \cos(uv)$.

Find $\dfrac{\partial w}{\partial v}$ at $v = 0$.

(4) (15 pts) Suppose that $f(x, y) = x^2 + e^{xy}$.

(a) Find the maximum rate of change of f at the point $(1, 0)$

(b) Find the direction (vector) in which the directional derivative at the point $(1, 0)$ has the minimum possible value.

(c) Find a direction in which the directional derivative at the point $(1, 0)$ has the value 1.

(5) (20 pts) Determine whether each critical point of $f(x, y) = xy + \dfrac{1}{x} + \dfrac{1}{y}$ is a local maximum, local minimum or a saddle point.

(6) (15 pts)

(a) (10 points) Use the method of lagrange multipliers to find the absolute minimum value of $f(x, y) = x^2 + y^2$ on the line $x - 2y = 5$.

(b) (5 points) Give a simple geometric interpretation of the square root of the minimum value found in part (a).

Solutions:

(1) (a) matches with (ii). (b) matches with (iv) "None of these". (c) matches with (iii).

(2) Substitute $x = 1$ and $y = 1$ into $xy(x+y)^2$ to get 4, which checks. The tangent line to the level curve $z = 4$ satisfies $dz = 0$. Therefore,

$$\begin{aligned} dz &= \frac{\partial z}{\partial x}(1,1)dx + \frac{\partial z}{\partial y}(1,1)dy \\ &= 8\,(x-1) + 8\,(y-1) = 0 \end{aligned}$$

So the tangent line is $(x-1)+(y-1) = 0$ which gives $y = -x+2$.

(3) Using "tree method", the variable w depends on x, y, z, but only x and y depend on v. So we get

$$\frac{\partial w}{\partial v} = (w_x)(x_v) + (w_y)(y_v) = (z^2/y)(-u) - (xz^2/y^2)(-\sin(uv)(u))$$

Set $v = 0$ to get

$$\frac{\partial w}{\partial v} = -uz^2/y + 0 = -u(\tan^2(1/u)/(\cos(0)) = -u\tan^2(1/u)$$

(4) (a) The maximum rate of change is the length of the gradient vector at $(1,0)$: $|\overrightarrow{\nabla f(1,0)}| = |\langle 2,1\rangle| = \sqrt{5}$

(b) The direction for minimum directional derivative is minus the gradient. We get $-\langle 2,1\rangle = \langle -2,-1\rangle$.

(c) We need a direction $\langle a,b\rangle$, *of unit length*, so that $\overrightarrow{\nabla f(1,0)} \cdot \langle a,b\rangle = 1$. This means $\langle 2,1\rangle \cdot \langle a,b\rangle = 2a+b = 1$. Since $a^2+b^2 = 1$ and $b^2 = (1-2a)^2$, we find $a^2 + (1-2a)^2 = 1$. Simplify to get $5a^2 - 4a = a(5a-4) = 0$. We have $a = 0,\ b = 1$ or $a = 4/5,\ b = 3/5$. One solution is $\overrightarrow{\jmath}$

(5) Find critical points: First solve $f_x = 0 = f_y$

$$y - 1/x^2 = 0 \tag{1}$$

$$x - 1/y^2 = 0 \tag{2}$$

(1) gives $y = 1/x^2$ which we substitute into (2) and simplify to obtain $x - x^4 = x(1-x^3) = 0$. Solutions $x = 0$ and $x = 1$. Since $x \neq 0$ (why?), The only possibility is $x = 1$. Using (1): $y = 1/x^2 = 1/1^2 = 1$. The only critical point is $(1,1)$. We need to apply the second derivative test to this critical point

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2/x^3 & 1 \\ 1 & 2/y^3 \end{pmatrix}_{x=1,y=1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$D = 3 > 0$ and $f_{xx}(1,1) = 2 > 0$ so $(1,1)$ is a local minimum.

(6) (a) The lagrange multiplier equations become

$$2x = \lambda \tag{3}$$

$$2y = -2\lambda \tag{4}$$

$$x - 2y = 5. \tag{5}$$

So $2x = \lambda = -y$ and (5) $x = 5+2y$ imply $y = -2(5+2y) = -10-4y$. And so $5y = -10,\ y = -2,\ x = 1$. The absolute minimum value is $(-2)^2 + 1^2 = 5$.

(b) The number $\sqrt{5}$ is the distance between the line $x - 2y = 5$ and the origin..