MATH 224-02 Sample Test 5

Chapter 16

(1) (15 pts) Sketch the vector field $\vec{F}(x, y) = y\vec{i} - x\vec{j}$

(2) (20 pts) Find the work done by the force $\vec{F} = (y + 3x^2)\vec{i} + (x - 3y^2)\vec{j}$ along a parabola joining $(3, 4)$ to $4, 3)$. Hint: Use the fundamental theorem of line integrals.

(3) (15 pts) Use Green's Theorem to find the work done by the force $\vec{F}(x, y) = (-y/2)\vec{i} + (x/2)\vec{j}$ going around the curve $C = C_1 + C_2 + C_3$. Suppose C_1 is the line segment of length 1 starting at the origin, making a 30 degree angle with the x-axis. C_2 is a counter-clockwise circular arc centered at $(0, 0)$ from the end of C_1 to the start of C_3. C_3 is the line segment running back to $(0, 0)$ and making a 60 degree angle with the x-axis. The whole curve looks like a small piece of pi(e).

(4) (15 pts) Give the cartesian parameterization for the upper hemisphere S: $z = \sqrt{4 - x^2 - y^2}$ of radius 2. Find the surface element dS in terms of x, y for the hemisphere S.

(5) (20 pts) $\vec{F} = -y\vec{i} + x\vec{j} + z^2\vec{k}$. S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$ and is oriented upward. Suppose C is the boundary of S. Use Stoke's Theorem and polar coordinates to set up an equivalent integral to

$$\oint_C \vec{F} \cdot d\vec{r}$$

(6) (15 pts) Suppose that $\vec{F} = z \arctan y^2 \vec{i} + z^3 \ln(x^2 + 1)\vec{j} + z\vec{k}$. Use the divergence theorem and cylindrical coordinates to set up an integral for the flux of $\vec{F}$ across the part of the paraboloid $z = 2 - x^2 + y^2$ that lies above $z = 1$ and is oriented upward.

SOLUTIONS: On the next page. Let me know if you find errors!

(1) Sketch is just like the "eye of the hurricane" model (ex. 1 page 1057) except this one rotates clockwise.

(2) Check whether $\vec{F}$ conservative: $Q_x = 1 = P_y$, so $\vec{F}$ is conservative and we can integrate to find the potential function: $f(x, y) = yx + x^3 - y^3$. By the fundamental theorem the work done is

$$W = \oint_C \vec{F} \cdot d\vec{r} = f(4, 3) - f(3, 4) = (12 + 64 - 27) - (12 + 27 - 64) = 74$$

(3) Since we are using Green's Theorem, we need to integrate $Q_x - P_y = (1/2) - (-1/2) = 1$ over the pie-shaped region. So we just get the area of a pie-shaped sector of angle 30 degrees or $\pi/6$. Since $\pi/6$ is 1/12 of the entire circle, the answer is $\pi/12$.

(4) The parameterization is $\vec{r}(x, y) = \langle x, y, \sqrt{4 - x^2 - y^2} \rangle$ for (x, y) inside the disc of radius 2. Since the surface is a graph, we need to calculate $dS = \sqrt{1 + z_x^2 + z_y^2}\, dA$ for $z = \sqrt{4 - x^2 - y^2}$. After simplifying, we get $dS = 2/\sqrt{4 - x^2 - y^2}\, dx\, dy$.

(5) The boundary of S is given by the intersection $9 - x^2 - y^2 = 5$ so that $x^2 + y^2 = 4$, a circle of radius 2. We want to find the flux of the curl of $\vec{F}$ across this surface. But the curl is just $2\vec{k}$. Also, instead of using the original surace, we can use the disc of radius 2 in the plane $z = 5$. This disc has area element $\vec{k}, dA$, so curl $\vec{F} \cdot \vec{k}\, dA = 2\, dA$. If we integrate this over the disc we twice the area of the disc or $2(4\pi) = 8\pi$.

(6) The divergence is just $2z$. The solid lies below the paraboloid $z = 2 - x^2 - y^2$ and above $z = 1$ which intersect at $x^2 + y^2 = 1$. Put this together to get

$$\int_0^{2\pi} \int_0^1 \int_1^{2-r^2} 2z\, r\, dz dr d\theta$$