INVARIANTS OF WEAK EQUIVALENCE IN PRIMITIVE MATRICES

RICHARD SWANSON AND HANS VOLKMER

Abstract. Weak equivalence of primitive matrices is a known invariant arising naturally from the study of inverse limit spaces. Several new invariants for weak equivalence are described. It is proved that a positive dimension group isomorphism is a complete invariant for weak equivalence. For the transition matrices corresponding to periodic kneading sequences, the discriminant is proved to be an invariant when the characteristic polynomial is irreducible. The results have direct application to the topological classification of one dimensional inverse limit spaces.

1. Introduction

Weak equivalence is a powerful invariant for classifying one dimensional inverse limit spaces. Barge and Diamond ([BD]) show that weak equivalence is necessary for homeomorphic inverse limit structures corresponding to admissible periodic kneading sequences of unimodal interval maps. Most recently, J. Jacklitch ([J]) has extended the methods of [BD]) and some results of R.F. Williams ([Wi]) to obtain the following important theorem: Suppose $f : M \to M$ and $g : N \to N$ are diffeomorphisms on manifolds with one-dimensional connected orientable hyperbolic attractors X and Y respectively. Let A and B denote the transition matrices of the induced maps on the underlying branched one-manifolds whose inverse limits correspond to X and Y. If X is homeomorphic to Y, then A and B are weakly equivalent.

The latter affords ample motivation for studying weak equivalence for arbitrary primitive matrices and not just those that match unimodal kneading sequences. The latter are analyzed in Sections 4, 5, 6 of this paper.

The plan of the paper is to progress from a very broad setting to a very specific one. In Section 2, we define weak equivalence and derive some of its consequences in the general setting of primitive matrices. These include the reduced discriminant invariant and common eigenvector ideals. Finally in Theorem 2.8, we give a complete invariant: the dimension groups are isomorphic, and the isomorphism preserves the "positive" semigroup.

Next, we consider specialized cases such as A and B being unimodular of the same size (Section 3). In Section 4, we show that matrices which arise from unimodal cyclic permutations are unimodularly similar to their companion matrices. When the characteristic polynomial is irreducible, and A and B are weakly equivalent, we show that the discriminants of the characteristic polynomials are identical (Theorem 4.4), having equal Perron integral rings $\boldsymbol{Z}[\alpha] = \boldsymbol{Z}[\beta]$ (Corollary 4.5), which improves on [BD].

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In section 5, we study weak equivalence in the specific pool of matrices generated by the family of unimodal kneading sequences. We show, for instance, that the last two periodic orbits to emerge, of the same period, can never correspond to weakly equivalent transition matrices (arising from the unimodal kneading sequences). As a result, the inverse limit structures are not homeomorphic for each period. Finally, in Section 6, a Maple symbolic computation shows that through period 15, no pair of admissible transition matrices is weakly equivalent.

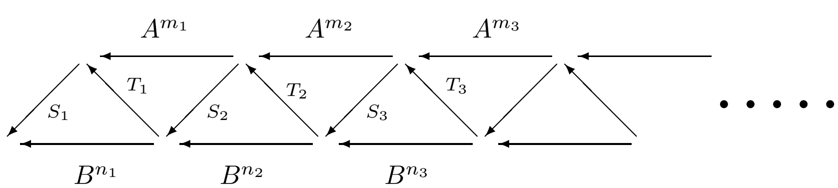
Let us review some definitions and results that are important for this paper. A *primitive* matrix A is a square nonnegative matrix (that is, its entries are nonnegative) for which some power A^n has only positive entries. A primitive matrix A has a (positive) eigenvalue α which equals its spectral radius. This eigenvalue α is called the *Perron eigenvalue* of A. An eigenvector belonging to the Perron eigenvalue α is called a *Perron eigenvector* of A. Such an eigenvector is simple and can be chosen to have positive entries, which we will always assume.

There are similarities between weak equivalence and shift equivalence of integral matrices. Shift equivalence has been a useful invariant in the study of conjugate subshifts of finite type. Shift equivalence is a consequence of the *conjugacy* of the resulting inverse limit structures and implies weak equivalence, as we show in Proposition 2.2. Because shift equivalence forces A and B to have the same Perron eigenvalue, weak equivalence is is usually strictly weaker. In Theorem 2.8 we show that weak equivalence is equivalent to a positive dimension group isomorphism. For shift equivalence this isomorphism is also a conjugacy of the induced actions of A and B (e.g. [LM, Cor. 7.5.9].) In Theorem 3.5, we show that for unimodular matrices, when the characteristic polynomials are equal and irreducible, weak equivalence *implies* shift equivalence.

2. Properties of Weak Equivalence

In this section we make the standing assumption that the matrices $A : \boldsymbol{R}^k \to \boldsymbol{R}^k$ and $B : \boldsymbol{R}^\ell \to \boldsymbol{R}^\ell$ (acting from the left) are primitive and integral of arbitrary sizes.

The matrices A and B are *weakly equivalent* if, for all $i \in \boldsymbol{N}$, there exist nonnegative integral matrices S_i, T_i, and positive integers m_i, n_i such that there is an infinite commuting diagram of the form:



The $\ell \times k$ and $k \times \ell$ matrices S_i and T_i will be called *connecting matrices.* While most applications of weak equivalence involve primitive matrices, the notion extends naturally to nonnegative integral matrices. The following result, though straightforward, is an immediate consequence of Theorem 2.8.

Proposition 2.1. *Weak equivalence is an equivalence relation on the set of all nonnegative square integral matrices.*

Shift equivalence ([Wi]), mentioned above, bears a simple relation to weak equivalence. Recall that A and B are *shift equivalent* (with *lag* ℓ) if there exist a pair $\{U, V\}$ of nonnegative integral matrices satisfying the relations $AU = UB$, $VA = BV$, $A^\ell = UV$, and $B^\ell = VU$. We have the following proposition (which is true for nonnegative integral matrices) and omit the easy proof.

Proposition 2.2. *If there are positive integers r and s for which A^r and B^s are shift equivalent, then A and B are weakly equivalent.*

Many results of this paper are based on the following groundbreaking theorem of Barge and Diamond [BD, Cor. 3.5].

Theorem 2.3. *Suppose the (primitive) matrices A and B are weakly equivalent with connecting matrices S_i and T_i. Let α and β be the Perron eigenvalues of A and B, respectively. Let v and w be Perron eigenvectors of A and B, respectively. Then*

(a) *S_i maps v to a positive multiple of w, and T_i maps w to a positive multiple of v for all i.*
(b) $\boldsymbol{Q}(\alpha) = \boldsymbol{Q}(\beta)$.

In the rest of this section we use Theorem 2.3 to derive several additional necessary conditions for weak equivalence based on the notions of norms, ideals, Galois groups, and discriminants. Finally we show that dimension groups provide a *complete* invariant.

Theorem 2.4. *Suppose A and B are weakly equivalent. Then the norm of α (taken with respect to $\boldsymbol{Q}(\alpha) = \boldsymbol{Q}(\beta) : \boldsymbol{Q}$) has the same prime factors as the norm of β.*

Proof. There exist connecting matrices S, T and positive integers m, n such that $SA^mT\,w = B^n\,w$. By Theorem 2.3(a), there are real numbers s, t with $Sv = sw$ and $Tw = tv$, which yields $(st)\alpha^m = \beta^n$. The number st is an eigenvalue of the integral matrix ST and so is an algebraic integer. Hence, the prime factors of the norm of α divide the norm of β, and conversely, by symmetry of weak equivalence. □

Theorem 2.5. *Suppose A and B are weakly equivalent. Then there are Perron eigenvectors $v = (v_1, \dots, v_k)$ of A and $w = (w_1, \dots, w_l)$ of B with components in $\boldsymbol{Z}[\alpha]$ such that the ideals $\langle v_1, \dots, v_k\rangle$ and $\langle w_1, \dots, w_l\rangle$ computed with respect to the ring $R = \boldsymbol{Z}[1/\alpha]$ are equal. A similar statement holds in $R = \boldsymbol{Z}[1/\beta]$ if the roles of α and β are reversed.*

Proof. Let v denote a Perron eigenvector of A so that $Av = \alpha v$. We can choose v so that its components are in the ring $\boldsymbol{Z}[\alpha]$. Define $\langle v\rangle$ to be the ideal $\langle v_1, v_2, \dots, v_k\rangle$ in $\boldsymbol{Z}[1/\alpha]$. Now define $w := S_2 v$. By Theorem 2.3, weak equivalence of A and B implies that w is a Perron eigenvector of B, that is, $Bw = \beta w$. Since S_2 is integral, the vector w has components in $\boldsymbol{Z}[\alpha]$ and $\langle w\rangle \subset \langle v\rangle$ in $\boldsymbol{Z}[1/\alpha]$. On the other hand, for the choice $t = m_1$ in the weak equivalence diagram,

$$v = (\alpha^{-t})A^t v = (\alpha^{-t})T_1 S_2 v = (\alpha^{-t})T_1 w$$

implies $\langle v\rangle \subset \langle w\rangle$, so $\langle v\rangle = \langle w\rangle$. □

We note that if α is not a unit, then the inclusion $\boldsymbol{Z}[\alpha] \subset \boldsymbol{Z}[1/\alpha]$ will be proper. Also, there are primitive matrices A and B such that $\alpha = \beta$, yet A and B are not weakly equivalent, as in the following matrices:

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}.$$

The reader can verify that there are no Perron eigenvectors v of A and w of B that generate the same ideals in $\boldsymbol{Z}[\alpha] = \boldsymbol{Z}[1/\alpha]$. This observation can also be expressed by saying that the Perron eigenvectors of A and B generate different ideal classes in $\boldsymbol{Z}[\alpha]$. Two ideals I, J in $\boldsymbol{Z}[1/\alpha]$ are in the same *ideal class* iff there is a real number t such that $I = tJ$. The ideal class is a complete invariant for weak equivalence when the characteristic polynomials are irreducible and equal, as in the above matrix pair (Theorem 3.5).

Let the matrices A and B be weakly equivalent and let p and q denote the minimal polynomials of the Perron eigenvalues of A and B, respectively. By Theorem 2.3, the polynomials p and q have the same degree and their splitting fields are equal. Hence the Galois groups of p and q agree. However, Galois groups are notoriously difficult to compute. We would like to have an invariant of weak equivalence that is easier to compute. Such an invariant can be based on discriminants.

Let p denote a monic polynomial of degree k with integer coefficients. Suppose the (distinct) roots of p are $\alpha_1, \alpha_2, \dots, \alpha_k$. The *discriminant* of p is defined as

$$\operatorname{disc} p := \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

There are many other equivalent formulations of the discriminant, and we shall see the Vandermonde matrix version [R, p. 134] in Section 4.

The discriminant is an integer that we may factor into prime powers:

$$\operatorname{disc} p = \pm \pi_1^{e_1} \pi_2^{e_2} \cdots \pi_s^{e_s}.$$

Reduce each number e_i to $f_i := e_i \bmod 2$. So,

$$\operatorname{rdisc} p := \pm \pi_1^{f_1} \pi_2^{f_2} \cdots \pi_s^{f_s}.$$

Note that $\operatorname{rdisc} p = \operatorname{rdisc} q$ iff $(\operatorname{rdisc} p)(\operatorname{rdisc} q)$ is a perfect square.

Theorem 2.6. *Let p, q denote, respectively the minimal polynomials of the Perron eigenvalues of A and B. If A and B are weakly equivalent, then*

$$\operatorname{rdisc} p = \operatorname{rdisc} q.$$

Proof. As mentioned above, the Galois groups G_p and G_q of p, q are equal. The fixed field of the alternating subgroup of G_p is $\boldsymbol{Q}(\sqrt{\operatorname{disc} p})$; see [R, p. 133]. Thus $\boldsymbol{Q}(\sqrt{\operatorname{disc} p}) = \boldsymbol{Q}(\sqrt{\operatorname{disc} q})$ which easily establishes the claim. □

Remark 2.7. *In* [BD] *the authors use MAPLE to show the three period 5 orbits in the tent family do not have weakly equivalent transition matrices A, B and C. The minimal polynomials are $p = x^4 - x^3 - x^2 - x - 1$, $q = x^4 - x^3 - x^2 - x + 1$ and $r = x^4 - x^3 - x^2 + x - 1$ Computing splitting fields (with MAPLE) to distinguish pairs is not currently feasible for large polynomials. Reduced discriminants can be readily calculated from the polynomial* *coefficients. In this case,* $\operatorname{rdisc} p = -563$, $\operatorname{rdisc} q = -3$ *and* $\operatorname{rdisc} r = -331$. *In Section 6, this method is extended to much higher periods.*

In general, matrices can be weakly equivalent yet have different associated discriminants. For instance, put

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.$$

Since $B = A^2$, A and B are weakly equivalent. The discriminants of the characteristic polynomials, p and q, of A and B respectively, are $5 \neq 45$ but we have $\operatorname{rdisc} p = \operatorname{rdisc} q$. Also, for this example, $\boldsymbol{Z}[\alpha] \neq \boldsymbol{Z}[\alpha^2]$.

Let A be a k by k integral matrix. The *eventual range* $R(A)$ of A consists of all row vectors $x \in \boldsymbol{Q}^k$ which can be written as $x = yA^m$, with $y \in \boldsymbol{Q}^k$, for all $m \in \boldsymbol{N}$. The *eventual kernel* $K(A)$ of A consists of all $y \in \boldsymbol{Q}^k$ for which there exists $m \in \boldsymbol{N}$ with $yA^m = 0$. It is known that $R(A)$ and $K(A)$ are linear subspaces of $\boldsymbol{Q}^k$ whose direct sum is $\boldsymbol{Q}^k$. Both $R(A)$ and $K(A)$ are invariant subspaces of the map $x \mapsto xA$. This map is bijective from $R(A)$ onto $R(A)$ and nilpotent from $K(A)$ to $K(A)$. Note that $R(A) = \boldsymbol{Q}^k A^k$ and $xA^k = 0$ for $x \in K(A)$. The *dimension group* $\operatorname{Dim}(A)$ of A consists of all $x \in R(A)$ for which there is $m \in \boldsymbol{N}$ with $xA^m \in \boldsymbol{Z}^k$. This is a subgroup of $\boldsymbol{Q}^k$. If A is nonnegative, then the *dimension semigroup* $\operatorname{Dim}^+(A)$ of A consists of all $x \in R(A)$ for which there exists $m \in \boldsymbol{N}$ with $xA^m \in (\boldsymbol{Z}^+)^k$ ($\boldsymbol{Z}^+$ denotes the nonnegative integers.) If A is primitive, then [LM, Lemma 7.3.8] shows that $\operatorname{Dim}^+(A) = \{0\} \cup \{x \in \operatorname{Dim}(A) : x \cdot v > 0\}$, where v is a Perron eigenvector of A.

Theorem 2.8. *If A, B are primitive integral matrices, the following are equivalent:*

1) *The matrices A, B are weakly equivalent.*
2) *There is a group isomorphism $\phi : \operatorname{Dim}(A) \to \operatorname{Dim}(B)$ such that $\phi(\operatorname{Dim}^+(A)) = \operatorname{Dim}^+(B)$.*
3) *There is an integral matrix T and eigenvectors v, w belonging to the Perron eigenvalues of A, B, respectively, such that $Tw = v$ and $x \mapsto xT$ defines a group isomorphism of* $\operatorname{Dim}(A)$ *onto* $\operatorname{Dim}(B)$.

Proof. 1) implies 2): Let A and B be weakly equivalent. Then there is a weak equivalence diagram involving positive powers m_i, n_i and connecting nonnegative integral matrices S_i, T_i. Choose i so large that $n := n_2 + \cdots + n_i \geq \ell$. Set $m := m_1 + \cdots + m_{i-1}$. We claim that $x \in R(A)$ implies $xT_i \in R(B)$. To show this let $x \in R(A)$. There is $y \in \boldsymbol{Q}^k$ with $x = yA^m$. Then $xT_i = yA^mT_i = yT_1B^n$. Since $n \geq \ell$ this proves that $xT_i \in R(B)$. Similarly, we show that $x \in R(B)$ implies $xS_i \in R(A)$ if i is sufficiently large. We now omit S_i and T_i for small i from the weak equivalence diagram to obtain a new diagram with all S_i and T_i having the above shown property.

We now claim that $(\operatorname{Dim}(A))T_i \subset \operatorname{Dim}(B)$ for all i. If $x \in \operatorname{Dim}(A)$, we know that $xT_i \in R(B)$. There is an $m_0 \in \boldsymbol{N}$ such that $xA^{m_0} \in \boldsymbol{Z}^k$. Choose $j > i$ so large that $m := m_i + \cdots + m_{j-1} \geq m_0$. Then $y := xA^m \in \boldsymbol{Z}^k$. Setting $n := n_{i+1} + \cdots + n_j$, we have $xT_iB^n = xA^mT_j = yT_j \in \boldsymbol{Z}^k$. This proves that $xT_i \in \operatorname{Dim}(B)$. This proof also shows $x \in \operatorname{Dim}^+(A)$ implies $xT_i \in \operatorname{Dim}^+(B)$.

Similarly, $x \in \mathrm{Dim}(B)$ implies that $xS_i \in \mathrm{Dim}(A)$ and $x \in \mathrm{Dim}^+(B)$ implies that $xS_i \in \mathrm{Dim}^+(A)$. Therefore, restricting all maps $x \mapsto xA$, $x \mapsto xB$, $x \mapsto xS_i$, $x \mapsto xT_i$ to the dimension groups, we obtain a new commutative diagram of group homomorphisms. Since A and B induce bijections, connecting homomorphisms induced by S_i (if $i \geq 2$) and T_i must be as well. So for every fixed i, $\phi(x) := xT_i$ defines a group isomorphism ϕ from $\mathrm{Dim}(A)$ onto $\mathrm{Dim}(B)$, such that $\phi(\mathrm{Dim}^+(A)) = \mathrm{Dim}^+(B)$.

2) implies 1): By [LM, Prop. 7.5.6], ϕ extends uniquely to a vector space isomorphism from $R(A)$ onto $R(B)$, also called ϕ. Let e_i be the standard ith basis (row) vector in $\boldsymbol{Q}^\ell$. Let $e_i = f_i + g_i$ be the unique decomposition with $f_i \in R(B)$ and $g_i \in K(B)$. Since $e_iB^\ell = f_iB^\ell$ we have $f_i \in \mathrm{Dim}^+(B)$. Hence $\phi^{-1}(f_i) \in \mathrm{Dim}^+(A)$ for all i. Choose $m_0 \in \boldsymbol{N}$ such hat $\phi^{-1}(f_i)A^{m_0} \in (\boldsymbol{Z}^+)^k$ for all i. Define an ℓ by k nonnegative integral matrix S_1 by $e_iS_1 = \phi^{-1}(f_i)A^{m_0}$ for $i = 1, \ldots, \ell$. One has $xS_1 = \phi^{-1}(x)A^{m_0}$ for $x \in R(B)$, and $xS_1 = 0$ for $x \in K(B)$.

Now let e_i be the standard ith basis vector in $\boldsymbol{Q}^k$. Let $e_i = f_i + g_i$ be the unique decomposition with $f_i \in R(A)$ and $g_i \in K(A)$. Thus, $f_i \in \mathrm{Dim}^+(A)$ for all i. Since $x \mapsto xA$ is bijective on $\mathrm{Dim}^+(A)$, there are uniquely determined $h_i \in \mathrm{Dim}^+(A)$ such that $f_i = h_iA^{m_0}$. Now $\phi(h_i) \in \mathrm{Dim}^+(B)$ for all i. So there is $n_1 \in \boldsymbol{N}$ with $n_1 \geq \ell$ and $\phi(h_i)B^{n_1} \in (\boldsymbol{Z}^+)^\ell$ for all i. Define a k by ℓ nonnegative integral matrix T_1 by $e_iT_1 = \phi(h_i)B^{n_1}$ for all i. We see that $yT_1 = \phi(h)B^{n_1}$ with $h \in R(A)$ determined by $hA^{m_0} = y$ for all $y \in R(A)$. This is used to show that $B^{n_1} = S_1T_1$.

Continuing inductively in this way, we obtain positive powers $m_1, n_2, \ldots$ and connecting nonnegative integral matrices $S_2, T_2, \ldots$ for a commutative diagram of the form needed to establish weak equivalence of A and B.

1) implies 3): If i is sufficiently large, then the proof that 1) implies 2) shows that $x \mapsto xT_i$ is a group isomorphism from $\mathrm{Dim}(A)$ onto $\mathrm{Dim}(B)$. By Theorem 2.3, $T := T_i$ satisfies 3).

3) implies 2): By replacing T by $-T$, if necessary, we can assume that $v = Tw$ with Perron vectors v, w. Define ϕ by $\phi(x) := xT$. Then, for $0 \neq x \in \mathrm{Dim}(A)$, $x \in \mathrm{Dim}^+(A)$ iff $xv > 0$ iff $xTw > 0$ iff $\phi(x) = xT \in \mathrm{Dim}^+(B)$. □

Note that the equivalence of 1) and 2) holds for nonnegative matrices. Also, the matrix T in statement 3) can be chosen nonnegative. We will revisit this result in the next section.

3. Weak equivalence of matrices of the same size

For arbitrary nonnegative matrices A and B the shift equivalence classes of A and B contain matrices of the same sizes. The same must obtain for weak equivalence, and, in this section, we investigate weak equivalence of matrices A and B which are of the same size.

Lemma 3.1. *Let A and B be weakly equivalent k by k nonnegative integral matrices. Then A and B are either both singular or* $\det A$ *and* $\det B$ *have the same prime factors.*

Proof. Clearly, A is singular if and only if B is. So, let A and B be invertible. Let p be a prime factor of $\det A$. Since $(\det A)^{m_1} = \det T_1 \det S_2$, we see that p is a prime factor of $\det T_1$ or $\det S_2$. Hence $(\det B)^{n_1} = \det S_1 \det T_1$ and $(\det B)^{n_2} = \det S_2 \det T_2$ show that p is also a prime factor of $\det B$. Similarly, we obtain that every prime factor of $\det B$ is also a prime factor of $\det A$. □

If $k = 1$, weak equivalence implies the entries have the same prime factors, which can be used to obtain a result of Watkins [Wa].

The following theorem is simply a restatement of Theorem 2.8 in the invertible case, that is easier to apply sometimes.

Theorem 3.2. *Let A and B be k by k invertible primitive integral matrices. Then A and B are weakly equivalent if and only if there exists a k by k invertible integral matrix S which satisfies the following three properties:*

1) *S maps eigenvectors belonging to the Perron eigenvalue of A to eigenvectors belonging to the Perron eigenvalue of B.*
2) *For each $n \in \boldsymbol{N}$, there is $m \in \boldsymbol{N}$ such that $B^{-n}SA^m$ is an integral matrix.*
3) *For each $m \in \boldsymbol{N}$, there is $n \in \boldsymbol{N}$ such that $A^{-m}S^{-1}B^n$ is an integral matrix.*

We now investigate weak equivalence of unimodular matrices, where the dimension group is $\boldsymbol{Z}^k$. Note that if A is a unimodular matrix which is weakly equivalent to B, then B is also unimodular by Lemma 3.1. If A and B are unimodular, then property 2) of Theorem 3.2 is automatic whereas property 3) is equivalent to the statement that S is unimodular. Since S can be replaced by SA^n with large n, S can be chosen nonnegative. So we obtain the following theorem.

Theorem 3.3. *Let A and B be unimodular and primitive. Then A, B are weakly equivalent if and only if there is a (nonnegative) unimodular matrix S which satisfies property 1) of Theorem 3.2.*

Theorem 3.4. *Let A and B be k by k unimodular primitive matrices with irreducible characteristic polynomials. Then the following are equivalent:*

(a) *A and B are weakly equivalent.*
(b) *There is a nonnegative unimodular matrix S such that $S^{-1}BS$ commutes with A.*
(c) *There is a nonnegative unimodular matrix S and a polynomial $r \in \boldsymbol{Q}[x]$ of degree less than k such that $S^{-1}BS = r(A)$.*

Proof. (a) implies (b): Suppose that A, B are weakly equivalent. By Theorem 3.3, there is a nonnegative unimodular matrix S such that $w = Sv$ for Perron eigenvectors v, w of A, B, respectively. Let the components of v, w be in $\boldsymbol{Q}(\alpha) = \boldsymbol{Q}(\beta)$, where α, β denote the Perron eigenvalues of A, B, respectively.

The field extension $\boldsymbol{Q}(\alpha)$ has degree k over $\boldsymbol{Q}$. Let $\chi_1 = \mathrm{id}, \chi_2, \dots, \chi_k$ be the conjugate embeddings of $\boldsymbol{Q}(\alpha) = \boldsymbol{Q}(\beta)$, with $\alpha_i = \chi_i(\alpha)$. Apply the embeddings to the components of v and w to obtain vectors $v = v_1, v_2, \dots, v_k$, $w = w_1, w_2, \dots, w_k$ with components in $\boldsymbol{Q}(\alpha_i)$ such that $Av_i = \alpha_i v_i$, $Bw_i = \beta_i w_i$, and $\beta_i = \chi_i(\beta)$. Form the matrices V, W with columns $v_1, \dots, v_k$ and $w_1, \dots, w_k$, respectively. Then $Sv = w$ implies $SV = W$. Since

$$AV = V \operatorname{diag}(\alpha_1, \dots, \alpha_k), \quad BW = W \operatorname{diag}(\beta_1, \dots, \beta_k),$$

$V^{-1}AV$ and $W^{-1}BW$ are diagonal matrices, so they commute. Since $SV = W$, we infer that $S^{-1}BS$ commutes with A.

(b) implies (c): By assumption, the characteristic polynomial of A equals the minimal polynomial (A is "nonderogatory") and $S^{-1}BS$ commutes with A. A well known theorem (e.g. [HJ, Thm. 3.2.4.2]) implies that there is a polynomial $r \in \boldsymbol{Q}[x]$ of degree less than k such that $S^{-1}BS = r(A)$.

(c) implies (a): Assume $S^{-1}BS = r(A)$ as in c). If v is a Perron eigenvector of A, then $BSv = r(\alpha)Sv$. Since Sv is a positive vector, this implies that $\beta = r(\alpha)$ and Sv is a Perron eigenvector of B. Statement (a) now follows from Theorem 3.3. □

Theorem 3.5. *Suppose the primitive unimodular matrices A and B have the same irreducible characteristic polynomial. Then the following are equivalent:*

1. *A, B are weakly equivalent.*
2. *The ideal classes of the Perron eigenvectors of A, B in $\boldsymbol{Z}[\alpha] = \boldsymbol{Z}[\beta]$ are the same.*
3. *There exists a unimodular matrix S such that $S^{-1}BS = A$, i.e., A is unimodularly similar to B.*
4. *A, B are shift equivalent.*

Proof. (1) implies (2) by Theorem 2.5. The equality of (2) and (3) is the content of Taussky ([T]). It is shown in [LM, Prop. 7.3.10] that 3) implies 4). Finally, shift equivalence implies weak equivalence by Proposition 2.2 above. □

For example, consider the notorious "Kollmer example", e.g., [LM, p. 249]

$$A := \begin{pmatrix} 19 & 5 \\ 4 & 1 \end{pmatrix}, \qquad B := \begin{pmatrix} 19 & 4 \\ 5 & 1 \end{pmatrix}.$$

Since A and B are not unimodularly similar, neither are they weakly equivalent, by Theorem 3.5.

We remark that (2) shows there are only finitely many weak equivalence classes in this situation. Property (4) implies some power of A is *strong shift equivalent* to a like power of B and the induced subshift powers are actually *conjugate* ([LM]). This shows that, mysteriously, some inverse limit homeomorphisms force conjugacies of underlying subshifts.

4. Transition matrices for cyclic unimodal permutations

In this and the following section, we study weak equivalence of the special class of matrices (as in [BD]) arising from periodic kneading sequences.

Let $K := \{0, 1, \dots, k-1\}$ with $k \in \boldsymbol{N}$. A permutation $\sigma : K \to K$ is called *cyclic* (of order k) if there is $i \in K$ such that $K = \{i, \sigma(i), \sigma^2(i), \dots, \sigma^{k-1}(i)\}$. A permutation $\sigma : K \to K$ is called *unimodal* if $0 \le i < j \le c$ implies $\sigma(i) < \sigma(j)$ and $c \le i < j \le k-1$ implies $\sigma(i) > \sigma(j)$ where $c := \sigma^{-1}(k-1)$. Note that if σ is cyclic and unimodal, then $\sigma(k-1) = 0$. If $k \ge 3$, then $0 < c < k-1$, and $\sigma^{k-1}(0) = k-1, \sigma^{k-2}(0) = c$.

We associate a continuous function $f : [0, k-1] \to [0, k-1]$ to a given permutation $\sigma : K \to K$ as follows. Let $f(i) := \sigma(i)$ and define f to be linear on each interval $[i-1, i]$.

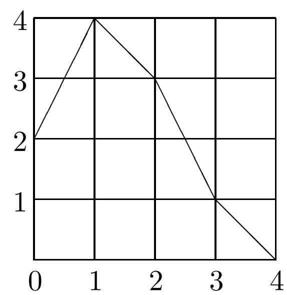
Let A be the $k-1$ by $k-1$ transition matrix for f. Every entry of A is 0 or 1. In the mth row and nth column the entry is 1 iff

$$\min\{\sigma(n-1),\sigma(n)\} < m \leq \max\{\sigma(n-1),\sigma(n)\}.$$

As an example consider $k=5$ and

$$\sigma := \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 & 0 \end{pmatrix}$$

This permutation is cyclic and unimodal with $c=1$. The graph of f is shown below:



The corresponding transition matrix A is given by

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

For $i=0,\ldots,k-1$ we define the vector $v_i := (0\ldots,0,1,\ldots,1)^t$ with $k-1$ components whose first i components are 0 and the remaining components are equal to 1. Note that v_{k-1} is the zero vector.

For the rest of this section let $\sigma: K \to K$ be a cyclic unimodal permutation with corresponding transition matrix A. The reader should be forewarned that this transition matrix is the *transpose* of what is often called the transition matrix. The theory is unchanged, and there are some advantages to the present formulation. We omit the easy proof of the following lemma.

Lemma 4.1. *We have*

$$Av_i = \begin{cases} v_0 + v_{\sigma(i)} & \text{if } 0 \leq i \leq c \\ v_0 - v_{\sigma(i)} & \text{if } c \leq i \leq k-1 \end{cases}$$

We define

$$\delta_j := \begin{cases} 1 & \text{if } \sigma^{j-1}(0) < c \\ -1 & \text{if } \sigma^{j-1}(0) > c \end{cases}$$

for $j=0,\ldots,k-2$, $\epsilon_0 := 1$ and $\epsilon_j := \epsilon_{j-1}\delta_{j-1}$ for $j=1,\ldots,k-1$. So $\epsilon_j = \delta_{j-1}\delta_{j-2}\ldots\delta_0$.

Lemma 4.2. *The vectors* $f_j := -\epsilon_{k-j} v_{\sigma^{k-j-1}(0)}$, $j = 1, \dots, k-1$, *form a basis of* $\boldsymbol{R}^{k-1}$. *The matrix representation of* A *with respect to this basis is the companion matrix*

$$\tilde{A} := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\epsilon_{k-1} & -\epsilon_{k-2} & -\epsilon_{k-3} & \dots & -\epsilon_1 \end{pmatrix}$$

In particular, the polynomial

$$p(x) := \sum_{j=0}^{k-1} \epsilon_j x^{k-1-j}$$

is the minimal and characteristic polynomial of A.

Proof. It is clear that the indicated vectors form a basis of $\boldsymbol{R}^{k-1}$. By Lemma 4.1 we have

$$\begin{aligned} Af_j &= -\epsilon_{k-j}(v_0 + \delta_{k-j} v_{\sigma^{k-j}(0)}) \\ &= -\epsilon_{k-j} v_0 - \epsilon_{k-j+1} v_{\sigma^{k-j}(0)} \\ &= -\epsilon_{k-j} f_{k-1} + f_{j-1}, \end{aligned}$$

where $f_0 := 0$. This establishes the claim. □

Let $e_1, \dots, e_{k-1}$ denote the standard basis of $\boldsymbol{R}^{k-1}$. We define matrices P and T by

$$Pe_i := -\epsilon_{k-i} e_{\sigma^{k-1-i}(0)+1}, \quad Te_i := v_{i-1}.$$

It is easy to see that except for the sign factors $-\epsilon_{k-i}$ the matrix P is a permutation matrix. Hence, P is a unimodular matrix. The matrix T has entries 0 above the main diagonal and entries 1 on and below the main diagonal. So T is also unimodular. Clearly, $f_j = TPe_j$ for all $j = 1, \dots, k-1$. Hence TP is the transformation matrix for the change of basis $e_1, \dots, e_{k-1}$ to $f_1, \dots, f_{k-1}$. Therefore, we obtain the following result.

Theorem 4.3. *The matrices* $\tilde{A}$ *and* A *are connected by*

$$\tilde{A} = (TP)^{-1} ATP.$$

In particular, A *and* $\tilde{A}$ *are unimodularly similar.*

It is well known that every square rational matrix M is similar to its Frobenius normal form N by a rational invertible transformation T: $N = T^{-1}MT$. But if M is integral then all possible transformation matrices T may be nonintegral as in the Kollmer example after Theorem 3.5. A special feature of our transition matrices A is that they *are* unimodularly similar to their Frobenius normal form.

For the example at the start of this section, the characteristic polynomial of A is $p(x) = x^4 - x^3 - x^2 + x - 1$. The matrices P and T are

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The following theorem improves Theorem 2.6 in the special case that A and B are transition matrices of cyclic unimodal permutations.

Theorem 4.4. *Let σ and τ be two cyclic unimodal permutations of $\{0, 1, \ldots, k-1\}$ with weakly equivalent transition matrices A and B, respectively. Assume that the characteristic polynomials p of A and q of B are irreducible over $\boldsymbol{Q}$. Then* $\operatorname{disc} p = \operatorname{disc} q$.

Proof. We remark that A and B are irreducible nonnegative matrices because their characteristic polynomials are irreducible. Moreover, since the traces of A and B are nonzero (see Lemma 4.2) we also know that A and B are primitive matrices (see [HJ, Thm. 8.5.10]).

Let $\tilde{A}$ and $\tilde{B}$ be the companion matrices for A and B as introduced in Lemma 4.2. According to Theorem 4.3 we can write

$$\tilde{A} = (TP)^{-1}ATP, \quad \tilde{B} = (TQ)^{-1}BTQ. \tag{4.1}$$

The matrix Q is defined for τ as P was defined for σ. Since $\tilde{A}$ and $\tilde{B}$ are companion matrices, there are eigenvectors v belonging to the Perron eigenvalue α of A and w belonging to the Perron eigenvalue β of B such that

$$v = TPz(\alpha), \quad w = TQz(\beta),$$

where $z(\alpha) := (1, \alpha, \alpha^2, \ldots, \alpha^{k-2})^t$. By Theorem 3.3, there is a unimodular matrix S and a nonzero real number γ such that $Sv = \gamma w$. Thus we obtain

$$STPz(\alpha) = \gamma TQz(\beta).$$

The matrix $U := (TQ)^{-1}STP$ is unimodular and satisfies

$$Uz(\alpha) = \gamma z(\beta). \tag{4.2}$$

It follows that $\gamma \in \boldsymbol{Z}[\alpha]$ and $\beta \in \boldsymbol{Q}(\alpha)$. Similarly, $1/\gamma \in \boldsymbol{Z}[\beta]$ and $\alpha \in \boldsymbol{Q}(\beta)$ (so $\boldsymbol{Q}(\alpha) = \boldsymbol{Q}(\beta)$ what we already know.) The numbers α and β are algebraic integers. Hence also γ and $1/\gamma$ are algebraic integers. Hence γ is a unit in the ring of algebraic integers in $\boldsymbol{Q}(\alpha)$. Since we assumed that p is irreducible, $\boldsymbol{Q}(\alpha)$ has dimension $k-1$ over $\boldsymbol{Q}$. Let $id = \chi_1, \chi_2, \ldots, \chi_{k-1}$ be the conjugate embeddings of $\boldsymbol{Q}(\alpha)$. Let $\alpha_i := \chi_i(\alpha)$, $\beta_i := \chi_i(\beta)$ and $\gamma_i := \chi_i(\gamma)$. Let $V(\alpha)$ be the Vandermonde matrix whose ith column is $z(\alpha_i)$. If we apply the embeddings to each component of each side of equation (4.2), then we obtain

$$UV(\alpha) = V(\beta)\operatorname{diag}(\gamma_1, \ldots, \gamma_{k-1}). \tag{4.3}$$

We take determinants and square. Recall that U is unimodular and that $\gamma_1\gamma_2\ldots\gamma_{k-1} = \pm 1$ because γ is a unit. Then we obtain

$$(\det V(\alpha))^2 = (\det V(\beta))^2.$$

Since $\operatorname{disc} p = (\det V(\alpha))^2$ and $\operatorname{disc} q = (\det V(\beta))^2$, the proof is complete. □

Corollary 4.5. *Under the assumptions of the previous theorem, the Perron eigenvalues α of A and β of B satisfy $\boldsymbol{Z}[\alpha] = \boldsymbol{Z}[\beta]$.*

Proof. We combine equation (4.3) with

$$\tilde{A}V(\alpha) = V(\alpha)\operatorname{diag}(\alpha_1,\dots,\alpha_{k-1}) \quad \tilde{B}V(\beta) = V(\beta)\operatorname{diag}(\beta_1,\dots,\beta_{k-1}).$$

This yields

$$\tilde{B}UV(\alpha) = UV(\alpha)\operatorname{diag}(\beta_1,\dots,\beta_{k-1}).$$

Let $\beta = \sum_{i=0}^{k-2} c_i\alpha^i$ with $c_i \in \boldsymbol{Q}$. Then we find

$$U^{-1}\tilde{B}U = V(\alpha)\sum_{i=0}^{k-2} c_i \operatorname{diag}(\alpha_1^i,\dots\ \alpha_{k-1}^i)V(\alpha)^{-1}.$$

Since

$$\tilde{A}^i = V(\alpha)\operatorname{diag}(\alpha_1^i,\dots,\alpha_{k-1}^i)V(\alpha)^{-1}$$

we can simplify to

$$U^{-1}\tilde{B}U = \sum_{i=0}^{k-2} c_i\tilde{A}^i. \tag{4.4}$$

The first row of the matrix on the right hand side is $(c_0,\dots,c_{k-2})$. Since U is unimodular, (4.4) shows that the numbers c_i are integers. So $\beta \in \boldsymbol{Z}[\alpha]$, and similarly, $\alpha \in \boldsymbol{Z}[\beta]$. □

5. Special classes of unimodal cyclic permutations

In this section we investigate weak equivalence of transition matrices of the form considered in the previous section.

First consider the unimodal permutation of the set $\{0,1,\dots,k-1\}$, $k \geq 4$, given by the cycle

$$(0,1,2\dots,k-1) \tag{5.1}$$

of order k. Its transition matrix A_k has the distinctive characteristic polynomial

$$p_k(x) = x^{k-1} - x^{k-2} - \cdots - x - 1.$$

This polynomial is irreducible over $\boldsymbol{Q}$ for all k; see [B, Theorem 2]. Moreover, Komatsu has shown in [K] that the Galois group of p_k is the symmetric group S_{k-1} for all k.

Now consider the unimodal cyclic permutation given by

$$(0,1,2,\dots,k-5,k-4,k-2,k-3,k-1). \tag{5.2}$$

¿From Lemma 4.2, the characteristic polynomial of the associated transition matrix B_k is

$$q_k(x) = x^{k-1} - x^{k-2} - \cdots - x^2 - x + 1 = p_k(x) + 2.$$

We notice that q_k is self-reciprocal; that is, $q_k(x) = x^{k-1}q_k(1/x)$.

For a given period, these two kneading sequences are the last to emerge in the kneading sequence ordering. Every tent map periodic orbit corresponds to a nonrenormalizable kneading sequence in the horseshoe. If one builds train track models based on these two kneading sequences, then (5.1) corresponds to a finite order diffeomorphism of the disc; while (5.2) is the last pseudo-Anosov horseshoe orbit to emerge (see T. Hall, [H]).

Theorem 5.1. *The transition matrices* A_k *and* B_k *are not weakly equivalent for every* $k \geq 4$.

Proof. Let α_k, β_k be the Perron eigenvalues of A_k, B_k, respectively. Since p_k is irreducible over $\boldsymbol{Q}$ for all k, $[\boldsymbol{Q}(\alpha_k) : \boldsymbol{Q}] = k-1$. If q_k is reducible over $\boldsymbol{Q}$, then $[\boldsymbol{Q}(\beta_k) : \boldsymbol{Q}] < k-1$, so $\boldsymbol{Q}(\alpha_k)$ is different from $\boldsymbol{Q}(\beta_k)$. Hence, by Theorem 2.3, A_k, B_k are not weakly equivalent. Now assume that the polynomial q_k is irreducible over $\boldsymbol{Q}$. This implies that k is odd because $q_k(-1) = 0$ if k is even. The distinct zeros of q_k can be arranged in pairs δ_i, $1/\delta_i$ for $i = 1, \ldots, (k-1)/2$. Every element of the Galois group of q_k that keeps δ_i fixed must also keep $1/\delta_i$ fixed. Hence the Galois group of q_k does not contain a cycle of order $k-2$. Since the Galois group of p_k is the symmetric group S_{k-1}, we obtain again that $\boldsymbol{Q}(\alpha_k)$ and $\boldsymbol{Q}(\beta_k)$ are different. Now an application of Theorem 2.3 completes the proof. □

Using [BD, Thm. 3.4], we obtain

Corollary 5.2. *Let* f_k*,* g_k *be Markov maps induced by the permutations* (5.1)*,* (5.2)*, respectively, as defined in the previous section. Let* P_k*,* Q_k *be the inverse limit spaces induced by* f_k *and* g_k *as the only bonding map, respectively. Then* P_k *and* Q_k *are not homeomorphic.*

We remark that the proof of Theorem 5.1 shows that the statements of Theorem 5.1 and Corollary 5.2 remain true for every self-reciprocal characteristic polynomial induced by unimodal cyclic permutations of order k in place of q_k.

The cited result of [K] is based on a computation of the discriminant of p_k,

$$\operatorname{disc} p_k = \pm \frac{2^k(k-1)^{k-1} - k^k}{(k-2)^2},$$

where the + sign holds if and only if $k \equiv 1$ or $k \equiv 2 \mod 4$. Unfortunately, there appear to be no general formulas for the discriminants of all the characteristic polynomials induced by transition matrices of cyclic unimodal permutations.

However, the following result on the discriminants of self-reciprocal polynomials will be helpful. We omit the straightforward proof to save space.

Theorem 5.3. *Let* r *be a monic polynomial of degree* $n = 2m$*,* m *a positive integer, with integer coefficients. Let* r *be self-reciprocal. Let* u *be the monic polynomial of degree* m *with integer coefficients which is uniquely determined by* $r(x) = x^m u(y)$ *with* $y = x + 1/x$*. Then*

$$\operatorname{disc} r = (-1)^m r(1) r(-1) (\operatorname{disc} u)^2.$$

We now use Theorem 5.3 to identify infinitely many nonweakly equivalent pairs of matrices arising from unimodal cyclic permutations.

Let $k \geq 5$ be an odd integer. Then $q_k(1) = 4 - k$, $q_k(-1) = 3$. Hence

$$\operatorname{disc} q_k = \pm (k-4) 3\, b_k^2, \tag{5.3}$$

where b_k is an integer.

Now let $k \geq 7$ be an odd integer. We consider the unimodal permutation of the set $\{0, 1, 2 \ldots, k-1\}$ which maps $k-5$ to $k-1$, $k-4$ to $k-2$, $k-3$ to $k-5$, $k-2$ to 1 and $k-1$ to 0. This is a cyclic permutation of order k. For example, if $k = 7$, it is given by $(0, 3, 5, 1, 4, 2, 6)$. Let r_k be the characteristic polynomial of the transition matrix C_k induced by this permutation.

By Lemma 4.2,

$$r_k(x) = x^{k-1} - x^{k-2} - \cdots - x^{(k+1)/2} + x^{(k-1)/2} - x^{(k-3)/2} - \cdots - x + 1.$$

All coefficients of this polynomial are equal to -1 except the coefficients of x^{k-1}, $x^{(k-1)/2}$ and x^0. Since this polynomial is self-reciprocal, from Theorem 5.3 we obtain

$$\operatorname{disc} r_k = \pm(k-6)3\, c_k^2 \tag{5.4}$$

with an integer c_k if $(k-1)/2$ is even. If $(k-1)/2$ is odd, then

$$\operatorname{disc} r_k = \pm(k-6)\, c_k^2. \tag{5.5}$$

Suppose $k \geq 7$, and $(k-1)/2$ is even. Since $(k-4)(k-6)$ cannot be a perfect square, (5.3) and (5.4) show that $\operatorname{rdisc} q_k$ and $\operatorname{rdisc} r_k$ are different if they are nonzero.

If, in addition, we know that q_k and r_k are irreducible over $\boldsymbol{Q}$, then Theorem 2.6 implies the transition matrices B_k and C_k associated with q_k and r_k are not weakly equivalent. Reducing mod 2 shows that there are infinitely many values of k for which q_k and r_k are irreducible; see [LN, Thm. 2.47]. These include $k = 5, 11, 13$, for example. Similarly, we can treat characteristic polynomials that are self-reciprocal under the substitution $x \to -1/x$.

6. A Maple program

In this section, we describe a MAPLE program which calculates and compares discriminants. Using the program, we conclude that pairs of transition matrices of unimodal cyclic permutations of $\{0, 1, \ldots, k-1\}$ are not weakly equivalent for all $k \leq 15$. We consider only those unimodal cyclic permutations whose transition matrices are primitive. These correspond to periodic kneading sequences that arise in the tent family with slope greater than $\sqrt{2}$. Two such transition matrices can only be weakly equivalent if they are of the same size ([BD, p. 172]).

For a given k the MAPLE program first computes all unimodal cyclic permutations whose transition matrices are primitive. In the next step the program computes the minimal polynomials of the Perron eigenvalues and the prime factorization of their discriminants. Then the program uses Theorems 2.6 and 4.4 to verify that the corresponding transition matrices are not weakly equivalent. In Table 1 we display the results for $k = 6$, $k = 7$ and $k = 8$. Each row represents one unimodal cyclic permutation. The first column gives an approximation of the Perron eigenvalue α. The second column gives the kneading sequence $\delta_0, \ldots, \delta_{k-2}$ from Section 4, with $R = -1, L = +1$. The third column gives the minimal polynomial p of α, and the fourth is the prime factorization of the discriminant of p.

Call a pair of unimodal cyclic permutations (with a primitive transition matrix) *critical* if the corresponding characteristic polynomials of the same degree are irreducible with the same discriminant or if the minimal polynomials have the same degree and the same reduced discriminant (so that Theorems 2.3, 2.6 and 4.4 cannot be applied.) For periods

6 and 7, the discriminant shows none of the corresponding transition matrix pairs is weakly equivalent.

However, for $k = 8$, the fourth and ninth entries comprise a critical pair. Let α and β be the corresponding Perron eigenvalues. Using MAPLE, the only zero of the minimal polynomial p of α in $\boldsymbol{Q}(\beta)$ is $-1/\beta$. This proves $\boldsymbol{Q}(\alpha) \neq \boldsymbol{Q}(\beta)$. Thus, no period 8 pair is weakly equivalent.

Using MAPLE to examine higher period cases, we have confirmed that there are no weakly equivalent transition matrices arising from cyclic unimodal permutations for all $k \leq 15$.

$k=6$			
1.792402	$RLLRR$	$x^5 - x^4 - x^3 - x^2 + x - 1$	$(2)^6(11)(23)$
1.883204	$RLLLR$	$x^4 - 2x^3 + x^2 - 2x + 1$	$-(2)^6(7)$
1.965948	$RLLLL$	$x^5 - x^4 - x^3 - x^2 - x - 1$	$(2)^4(599)$
$k=7$			
1.465571	$RLRRRR$	$x^3 - x^2 - 1$	$-(31)$
1.556030	$RLRRLR$	$x^6 - x^5 - x^4 + x^3 - x^2 - x + 1$	$(257)^2$
1.685926	$RLLRLR$	$x^6 - x^5 - x^4 - x^3 + x^2 + x - 1$	(125201)
1.754878	$RLLRRR$	$x^3 - 2x^3 + x - 1$	$-(23)$
1.823945	$RLLRRL$	$x^6 - x^5 - x^4 - x^3 + x^2 - x - 1$	$(5)^4(7)^2(17)$
1.855886	$RLLLRL$	$x^6 - x^5 - x^4 - x^3 - x^2 + x + 1$	(82793)
1.907342	$RLLLRR$	$x^6 - x^5 - x^4 - x^3 - x^2 + x - 1$	$(5)(136373)$
1.946856	$RLLLLR$	$x^6 - x^5 - x^4 - x^3 - x^2 - x + 1$	$(3)^4(107)^2$
1.983583	$RLLLLL$	$x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$	(205937)
$k=8$			
1.597209	$RLRRLRR$	$x^7 - x^6 - x^5 + x^4 - x^3 - x^2 + x - 1$	$-(2)^6(13399)$
1.648304	$RLLRLRR$	$x^5 - x^4 - 2x^2 + x - 1$	$(2)^4(677)$
1.715651	$RLLRLRL$	$x^7 - x^6 - x^5 - x^4 + x^3 + x^2 - x - 1$	$-(2)^6(42239)$
1.729119	$RLLRRRL$	$x^6 - x^4 - 2x^3 - x^2 - 2x - 1$	$(2)^8(7)(229)$
1.776889	$RLLRRRR$	$x^7 - x^6 - x^5 - x^4 + x^3 - x^2 + x - 1$	$-(2)^6(3)^3(53)(97)$
1.807092	$RLLRRLR$	$x^7 - x^6 - x^5 - x^4 + x^3 - x^2 - x + 1$	$(2)^6(286009)$
1.870943	$RLLLRLR$	$x^7 - x^6 - x^5 - x^4 - x^3 + x^2 + x - 1$	$-(2)^6(97)(2311)$
1.894945	$RLLLRRR$	$x^7 - x^6 - x^5 - x^4 - x^3 + x^2 - x + 1$	$(2)^6(23)(8887)$
1.918928	$RLLLRRL$	$x^6 - 2x^5 + x^4 - 2x^3 + x^2 - 1$	$(2)^8(7)(229)$
1.936323	$RLLLLRL$	$x^7 - x^6 - x^5 - x^4 - x^3 - x^2 + x + 1$	$(2)^6(13)^2(937)$
1.956813	$RLLLLRR$	$x^5 - x^4 - 2x^3 + x - 1$	$(2)^4(997)$
1.974819	$RLLLLLR$	$x^6 - 2x^5 + x^4 - 2x^3 + x^2 - 2x + 1$	$(2)^6(5)(37)^2$
1.991964	$RLLLLLL$	$x^7 - x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$	$-(2)^6(84223)$

TABLE 1. Unimodal cyclic permutations and their discriminants

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Richard Swanson, Department of Mathematical Sciences, Montana State University, Bozeman, MT 59717-0240

Hans Volkmer, Department of Mathematical Sciences, University of Wisconsin–Milwaukee, Milwaukee, WI 53201-0413