Second Order Linear Constant Coefficient Ordinary Differential Equations.

“If we couldn’t laugh, we would all go insane.” Jimmy Buffett

A second order linear constant coefficient ordinary differential equation takes the form

$$\frac{d^2 f(t)}{dt^2} = a\frac{df(t)}{dt} + bf(t), \quad t \geq 0. \tag{1}$$

The second order means that the highest derivative of the dependent variable ($f(t)$) appearing in (1) is two. Linear refers to the fact that the dependent variable $f(t)$ and its derivatives $\frac{d^2 f(t)}{dt^2}$ and $\frac{df(t)}{dt}$ appear to the first power. Constant coefficient means that a and b are numbers (they do not depend on t). If (1) is equipped with the initial conditions

$$f(0) = f_0 \;\; f'(0) = f_1 \tag{2}$$

then the problem (1) combined with (2) is called an initial value problem.

The procedure outlined for solving the Redneck difference equation in your text can also be used for this problem. Here is how this goes. Make the change of variable

$$\frac{df(t)}{dt} = u(t) \;\text{ and }\; f(t) = v(t)$$

and rewrite the difference equation in (1) and (2) as

$$\begin{aligned} \frac{du(t)}{dt} &= \frac{d^2 f(t)}{dt^2} = au(t) + bv(t) & \qquad u(0) = \frac{df(0)}{dt} = f_1 \\ \frac{dv(t)}{dt} &= \frac{df(t)}{dt} = u(t) & \qquad v(0) = f(0) = f_0. \end{aligned}$$

Set $\vec{w}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ and $\vec{w}'(t) = \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix}$ so the last display takes the form

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \text{ or in terms of } \vec{w}\text{: } \vec{w}'(t) = A\vec{w}(t). \tag{3}$$

Recall that the solution of the scalar problem $y'(t) = ay(t)$ satisfying $y(0) = y_0$ is given by $y(t) = e^{at}y_0$ (plug it in, it works). Forgetting the vector notation for a minute, the solution of (3) (think of the matrix A as a scalar) takes the form

$$\vec{w}(t) = e^{At}w(0) \text{ or written out: } \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = e^{At}\begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = e^{At}\begin{bmatrix} f_1 \\ f_0 \end{bmatrix}. \tag{4}$$

Booyah, we’re done time to quit. There is the minor question of what we may mean by the symbol e^{At}. This little sticky point replaces the A^n sticky point in the redneck difference equation problem. The route around this sticky point is provided by the same similarity transformation which was used in Exercise (5.4) of your text.

Here is a brief review of this procedure. The eigenvalues of A are found from the equation $\det(A-\lambda I)=\lambda^2-a\lambda-b=0$ which gives

$$\lambda_{\pm}=\frac{a\pm\sqrt{a^2+4b}}{2}. \tag{5}$$

These eigenvalues satisfy $\lambda_+\pm\lambda_-=\begin{cases} a \\ \\ \sqrt{a^2+b^2}\end{cases}$ and $\lambda_+\lambda_-=-b$. There are two cases to consider:

$$(i)\ \ a^2+4b=0\ \ and\ \ (ii)\ \ a^2+4b\neq 0. \tag{6}$$

In case (ii) there are two distinct roots $\lambda_+=\dfrac{a+\sqrt{a^2+4b}}{2}\neq\lambda_-=\dfrac{a-\sqrt{a^2+4b}}{2}$. The independent eigenvectors $\vec{p}_+=\begin{bmatrix}\lambda_+\\1\end{bmatrix}$ and $\vec{p}_-=\begin{bmatrix}\lambda_-\\1\end{bmatrix}$ (you find these) corresponding to λ_+ and λ_- give the similarity matrix

$$P=[\ \vec{p}_+\ \vec{p}_-\ ]=\begin{bmatrix}\lambda_+ & \lambda_-\\ 1 & 1\end{bmatrix}\text{ so that }P^{-1}AP=\Lambda=\begin{bmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{bmatrix}.$$

Write this equation in the form

$$\begin{aligned}A &= \begin{bmatrix}a & b\\ 1 & 0\end{bmatrix}=P\Lambda\left\{P^{-1}\right\}\\ &= \begin{bmatrix}\lambda_+ & \lambda_-\\ 1 & 1\end{bmatrix}\begin{bmatrix}\lambda_- & 0\\ 0 & \lambda_-\end{bmatrix}\left\{\left(\frac{1}{\lambda_+-\lambda_-}\right)\begin{bmatrix}1 & -\lambda_-\\ -1 & \lambda_+\end{bmatrix}\right\}.\end{aligned} \tag{7}$$

Substitute (7) into the right hand side of (3) and multiply both sides by P^{-1} to find

$$\begin{bmatrix}x'(t)\\ y'(t)\end{bmatrix}\equiv\Lambda\left\{P^{-1}\begin{bmatrix}u(t)\\ v(t)\end{bmatrix}\right\}\equiv\Lambda\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{bmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix} \tag{8}$$

where the change of variable

$$\left\{P^{-1}\right\}\begin{bmatrix}u(t)\\ v(t)\end{bmatrix}\equiv\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}\quad\text{if and only if}\quad\begin{bmatrix}u(t)\\ v(t)\end{bmatrix}\equiv P\begin{bmatrix}x(t)\\ y(t)\end{bmatrix} \tag{9}$$

has been introduced. The two component equations in (8) read

$$x'(t)=\lambda_+x(t)\ \text{ and }\ y'(t)=\lambda_-y(t)$$

whose solutions are (recall the sentence following (3))

$$x(t)=e^{\lambda_+t}x(0)\ \text{ and }\ y(t)=e^{\lambda_-t}y(0)$$

where each of $x(0)$ and $y(0)$ are defined through the variable change in (9). Substituting these scalar solutions into (9) and multiplying by P gives

$$\begin{aligned}\begin{bmatrix}u(t)\\ v(t)\end{bmatrix} &= P\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}\lambda_+ & \lambda_-\\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{\lambda_+t}x(0)\\ e^{\lambda_-t}y(0)\end{bmatrix}\\ &= \begin{bmatrix}\lambda_+e^{\lambda_+t}x(0)+\lambda_-e^{\lambda_-t}y(0)\\ e^{\lambda_+t}x(0)+e^{\lambda_-t}y(0)\end{bmatrix}.\end{aligned} \tag{10}$$

The values $x(0)$ and $y(0)$ are found from (9)

$$\begin{aligned}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = P^{-1}\begin{bmatrix} u(0) \\ v(0) \end{bmatrix} &= \left(\frac{1}{\lambda_+ - \lambda_-}\right)\begin{bmatrix} 1 & -\lambda_- \\ -1 & \lambda_+ \end{bmatrix}\begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \\ &= \left(\frac{1}{\lambda_+ - \lambda_-}\right)\begin{bmatrix} f_1 - \lambda_- f_0 \\ -f_1 + \lambda_+ f_0 \end{bmatrix}. \end{aligned} \tag{11}$$

The solution to the original problem (1) and (2) is found by substituting each of $x(0)$ and $y(0)$ into the second component of (10) so that

$$v(t) = f(t) = \left(\frac{1}{\lambda_+ - \lambda_-}\right)\left[e^{\lambda_+ t}\left(f_1 - \lambda_- f_0\right) + e^{\lambda_- t}\left(-f_1 + \lambda_+ f_0\right)\right]. \tag{12}$$

As a check one can substitute the right hand side of (12) into (1) to see that $f(t)$ is a solution (actually one only needs to check that $e^{\lambda_+ t}$ and $e^{\lambda_- t}$ satisfy (1) since the equation is linear). It is straightforward to see that the right hand of (12) has the correct initial values so (12) is the unique solution to the initial value problem in (1) and (2). Buried in the above is the answer to the question mentioned following (4), that is

$$e^{At} = Pe^{\Lambda t}P^{-1} = P\begin{bmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{bmatrix}P^{-1}. \tag{13}$$

For those who have had a course in elementary differential equations you may have solved the differential equation by assuming that $f(t) = e^{\lambda t}$, substituting this into (1) to find the auxiliary $\lambda^2 - a\lambda - b\lambda = 0$ whose solutions are the numbers $\lambda_\pm$ in (5). Then one writes $f(t) = c_+ e^{\lambda_+ t} + c_- e^{\lambda_- t}$ followed by finding $c_\pm$ using the initial conditions $f(0) = f_0$ and $f'(0) = f_1$. Indeed, enforcing these two equations leads to the system in (11) for the coefficients $c_\pm$. The solution is, of course, that in (12). If the solution procedure outlined in this paragraph is carried out, one may conclude that the matrix procedure is quite a bit more complicated. A bit like taking a cannon to shoot an ant. However, one might admit it's a pretty shiny cannon.

If $\lambda_+ = \lambda_- = \dfrac{a}{2} \equiv \lambda$ (case (i) in (6)), equation (7) shows that the matrix A factors

$$A = Q\,(R)\,Q^T = \frac{1}{\sqrt{\lambda^2+1}}\begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}\left(\begin{bmatrix} \lambda & -(\lambda^2+1) \\ 0 & \lambda \end{bmatrix}\right)\frac{1}{\sqrt{\lambda^2+1}}\begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix}.$$

All of the steps from (7) through (9) remain valid with Q^T replacing P^{-1}, Q replacing P and R replacing Λ. The new form of (7) reads

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} \equiv R\left(Q^T\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}\right) = \begin{bmatrix} \lambda & -(\lambda^2+1) \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{14}$$

with the change of variable

$$Q^T\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \equiv \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \text{if and only if} \quad \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \equiv Q\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{15}$$

The two component equations in (14) read

$$x'(t) = \lambda x(t) - (\lambda^2+1)y(t) \quad \text{and} \quad y'(t) = \lambda y(t).$$

The solution to the second is

$$y(t) = e^{\lambda t} y(0)$$

which when substituted in the first gives the non-homogeneous equation

$$x'(t) = \lambda x(t) - (\lambda^2+1)e^{\lambda t} y(0)$$

whose solution is

$$x(t) = \Big[x(0) - (\lambda^2+1)y(0)t\Big]\, e^{\lambda t}.$$

Substituting these scalar solutions into (15) and multiplying by Q gives

$$\begin{aligned} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} &= Q \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \left(\frac{1}{\sqrt{\lambda^2+1}}\right) \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} \begin{bmatrix} [x(0) - (\lambda^2+1)y(0)t]\, e^{\lambda t} \\ e^{\lambda t} y(0) \end{bmatrix} \\ &= \left(\frac{e^{\lambda t}}{\sqrt{\lambda^2+1}}\right) \begin{bmatrix} \lambda\, [x(0) - (\lambda^2+1)y(0)t] - y(0) \\ [x(0) - (\lambda^2+1)y(0)t] + \lambda y(0) \end{bmatrix} \end{aligned} \tag{16}$$

where the last step uses the calculation

$$\begin{aligned} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = Q^T \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} &= \left(\frac{1}{\sqrt{\lambda^2+1}}\right) \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{\lambda^2+1}}\right) \begin{bmatrix} \lambda f_1 + f_0 \\ -f_1 + \lambda f_0 \end{bmatrix} \end{aligned} \tag{17}$$

Now the solution to the original problem (1) and (2) is given by the second component of (16) with $x(0)$ and $y(0)$ given by (17)

$$\begin{aligned} v(t) = f(t) &= \left(\frac{e^{\lambda t}}{\sqrt{\lambda^2+1}}\right) \Big\{x(0) + (\lambda - (\lambda^2+1)t)y(0)\Big\} \\ &= \left(\frac{e^{\lambda t}}{\sqrt{\lambda^2+1}}\right) \left\{\left[\frac{\lambda f_1 + f_0}{\sqrt{\lambda^2+1}} + (\lambda - (\lambda^2+1)t)\frac{(-f_1 + \lambda f_0)}{\sqrt{\lambda^2+1}}\right]\right\} \\ &= \left(\frac{e^{\lambda t}}{\lambda^2+1}\right) \Big\{\lambda f_1 + f_0 + (\lambda - (\lambda^2+1)t)(-f_1 + \lambda f_0)\Big\} \\ &= e^{\lambda t}\left[f_0 + t\,(f_1 - \lambda f_0)\right]. \end{aligned} \tag{18}$$

By proceeding as in the lines following (13) one finds $f(t) = (c_0 + c_1 t)\, e^{\lambda t}$. The coefficients c_0 and c_1 are found using the initial conditions $f(0) = f_0$ and $f'(0) = f_1$ which leads directly to $c_0 = f(0) = f_0$ and $f'(0) = \lambda c_0 e^0 + c_1 = \lambda f_0 + c_1 = f_1$ so that $c_1 = f_1 - \lambda f_0$. Substitution of these values in $f(t) = (c_0 + c_1 t)\, e^{\lambda t}$ for c_0 and c_1 leads to (18). Kaboom, there goes that cannon again.