Math 225-01 Handout Elementary Solution Methods Spring 2008

Solution by Direct Integration. This works **ONLY** for first order ODEs of the form

$$\frac{dy}{dt} = f(t), \tag{1}$$

i.e., the right-hand-side depends **ONLY** on the independent variable t. Integrating both sides with respect to t gives

$$y(t) = \int f(t)\, dt + C,$$

where C is an arbitrary constant, which can be determined from an initial condition.

Example. Equation of motion for a falling object in a vacuum.

$$\frac{dv}{dt} = -g,$$

where $v(t)$ represents velocity and g is the acceleration of gravity (9.8 meters/second/second in MKS units). Integrating both sides w.r.t. t gives

$$v(t) = -gt + C.$$

Given initial velocity,

$$v(0) = v_0,$$

we obtain $v_0 = -g \cdot 0 + C$, so $C = v_0$. Thus

$$v(t) = -gt + v_0.$$

Example. Model for nothing in particular.

$$\frac{dy}{dt} = e^{4t}$$

with initial condition

$$y(0) = 2.$$

Integrating both sides of the ODE w.r.t. independent variable t gives

$$y(t) = \int e^{4t} dt + C = \frac{1}{4}e^{4t} + C.$$

From the initial condition and the fact that $e^{4 \cdot 0} = e^0 = 1$, we obtain $2 = y(0) = \frac{1}{4} \cdot 1 + C$, so $C = \frac{7}{4}$, and

$$y(t) = \frac{1}{4}e^{4t} + \frac{7}{4}.$$

Solution by Separation of Variables. This works for equations of the form

$$\frac{dy}{dt} = f(y), \tag{2}$$

i.e., the right-hand-side depends only on the **dependent** variable y. For a more complicated version of this method, see section 2.2. Note that direct integration **will not work** because we don't know y, so $\int f(y(t))\,dt$ is also unknown.

First we "separate variables". Formally multiplying both sides of the ODE (2) by dt and dividing by $f(y)$ gives

$$\frac{dy}{f(y)} = dt. \tag{3}$$

Now suppose we find an antiderivative $G(y)$ for the function $g(y) = \frac{1}{f(y)}$. Integrating both sides of (3) gives

$$G(y) = t + C.$$

Next we need to solve for y, obtaining

$$y(t) = G^{-1}(t + C),$$

where G^{-1} denotes the inverse function for G. Finally, we apply the initial condition to determine C or some derived parameter that is related to C.

Example. Radioactive decay model.

$$\frac{dm}{dt} = -rm \tag{4}$$

Here $m = m(t)$ is the dependent variable and $f(m) = -rm$, where the decay rate parameter r is a positive constant. We separate variables to get

$$-\frac{1}{r}\frac{dm}{m} = dt,$$

and then we integrate both sides and use the fact that $\ln|m|$ is an antiderivative of $\frac{1}{m}$ to get

$$\underbrace{-\frac{1}{r}\ln|m|}_{G(m)} = t + C.$$

Next we need to solve for m. First multiply both sides by $-r$ to get

$$\ln|m| = -rt + B, \quad \text{where } B = -rC.$$

Then exponentiate both sides and use the fact that $|m| = \pm m$ to get

$$m = \pm\exp(-rt + B).$$

Finally, we use the fact that $\exp(A + B) = e^A e^B$ to get the general solution to the ODE (4),

$$m(t) = \underbrace{\pm e^B}_{K} e^{-rt} = Ke^{-rt}.$$

in terms of the derived parameter K. From the initial condition

$$m(0) = m_0$$

we obtain $m_0 = m(0) = Ke^{-r \cdot 0} = K \cdot 1$, so

$$m(t) = m_0 e^{-rt}.$$

Example. Equation of motion for falling object with drag.

$$\frac{dv}{dt} = -g - \frac{\gamma}{m} v \tag{5}$$

where the positive parameter γ is the drag coefficient. To simplify notation, set $a = \gamma/m$, and then separate variables to get

$$-\frac{dv}{g + av} = dt. \tag{6}$$

Note that

$$\int \frac{dv}{g + av} = \frac{1}{a} \int \frac{a\,dv}{g + av} = \frac{1}{a} \int \frac{du}{u}, \quad \text{where } u = g + av.$$

Integrating both sides of (6) then gives

$$-\frac{1}{a} \ln |g + av| = t + C.$$

Multiplying both sides by $-a$ and then exponentiating gives

$$|g + av| = \exp(-at + B), \quad B = -aC,$$

so by the same reasoning used in the radioactive decay model, $g + av = Ke^{-at}$. Thus the general solution to the ODE (5) is

$$v(t) = -\frac{g}{a} + \frac{K}{a} e^{-at}.$$

Finally, we replace a by γ/m and set $L = K/a$ to get

$$v(t) = -\frac{gm}{\gamma} + Le^{-\frac{\gamma t}{m}}.$$

From the initial condition $v(0) = v_0$, we obtain $L = v_0 + \frac{gm}{\gamma}$, so

$$v(t) = -\frac{gm}{\gamma} + \left(v_0 + \frac{gm}{\gamma}\right) e^{-\frac{\gamma t}{m}}.$$

"Asymptotic" (long-time) behavior. As $t \to +\infty$, $e^{-at} \to 0$. Consequently, in the limit as time t goes to infinity,

$$v(t) \to -\frac{gm}{\gamma}.$$

We could confirm this without even solving the ODE (5). If v tends to a constant (the "terminal velocity" of the object), then $\frac{dv}{dt}$ tends to 0. Substituting $\frac{dv}{dt} = 0$ into (5) and then solving for v gives

$$v = -\frac{gm}{\gamma} \stackrel{\text{def}}{=} v_{terminal\ velocity}.$$