Math 225-01 Handout   Method of Variation of Parameters   Spring 2008

Variation of parameters (VP) is an alternative to the method of integrating integrating factors, which is presented in section 2.1. Although VP is not quite as general as the integrating factor approach, it is simpler and the underlying ideas extend to linear systems of ODEs. See section 4.4.

Derivation: Consider the general linear first order ODE in "standard form",

$$\frac{dy}{dt} + p(t)y = g(t). \tag{1}$$

The corresponding linear homogeneous equation is obtained by setting $g(t) = 0$:

$$\frac{dy}{dt} + p(t)y = 0. \tag{2}$$

The general solution to the homogeneous problem (2) is

$$y(t) = k\,h(t) \tag{3}$$

where $h(t) = \exp(-\int p(t)dt)$ and k is an arbitrary constant, or "parameter". This formula can be derived by the method of separation of variables. In the special case that $p(t) = p =$ constant, the homogeneous solution reduces to $h(t) = e^{-pt}$.

The key idea underlying VP is to let the "parameter" k in eqn (3) be "variable" in the sense that it depends on the independent variable t. Substituting

$$y(t) = k(t)h(t) \tag{4}$$

into the original nonhomogeneous ODE (1) gives

$$\underbrace{\frac{d}{dt}(k(t)h(t))}_{k'(t)h(t)+k(t)h'(t)} \;+\; p(t)k(t)h(t) \;=\; g(t). \tag{5}$$

But $h'(t) = -p(t)h(t)$, so $k(t)h(t)$ cancels with $p(t)k(t)h(t)$ in eqn (5), and we obtain

$$k'(t)h(t) = g(t).$$

We want to determine $k(t)$, so we first divide by $h(t)$ to get

$$k'(t) = \frac{g(t)}{h(t)}. \tag{6}$$

Since the right-hand-side depends on t but not k, direct integration gives us

$$k(t) = \int \frac{g(t)}{h(t)}dt + C. \tag{7}$$

Assuming we can do the integration, we substitute $k(t)$ back into eqn (4) to obtain the general solution to the original ODE (1). The constant C can be determined if an initial condition is given.

Example 1. The equation of motion for a falling object with linear drag force is

$$m\frac{dv}{dt} = -mg - \gamma v.$$

We rewrite this in standard form,

$$\frac{dv}{dt} + pv = -g, \tag{8}$$

where $p = \gamma/m$ is constant. Note that g is also constant. The homogenous solution (obtained by solving for v when the right-hand-side is zero) is

$$h(t) = e^{-pt}.$$

Subsituting $v(t) = k(t)h(t)$ into (8), differentiating, and cancelling terms (or simply using formula (6) with $g(t) = -g$) gives

$$k'(t) = \frac{-g}{h(t)} = -ge^{pt}.$$

Then by direct integration,

$$k(t) = -g\int e^{pt}dt + C = -\frac{g}{p}e^{pt} + C.$$

The general solution to the ODE is then

$$v(t) = k(t)h(t) = \left(-\frac{g}{p}e^{pt} + C\right)e^{-pt} = -\frac{g}{p} + Ce^{-pt} = -\frac{mg}{\gamma} + Ce^{-\frac{\gamma}{m}t}.$$

To obtain the lase equality, we substituted $p = \gamma/m$. This agrees with the solution obtained using separation of variables in the previous handout.

Example 4 on p. 38, except right-hand-side is $2t$. First put eqn in standard form

$$\frac{dy}{dt} + \underbrace{\frac{t}{2}}_{p(t)} y = \underbrace{t}_{g(t)} \tag{9}$$

The homogeneous solution (set $g(t) = 0$ in eqn (9) and solve) is

$$h(t) = \exp\left(-\int \frac{t}{2}dt\right) = \exp(-t^2/4).$$

The solution to (9) is

$$y(t) = k(t)h(t), \tag{10}$$

where

$$k'(t) = \frac{g(t)}{h(t)} = t\exp(t^2/4)$$

Integrate by substitution $u = t^2/4$, $du = t/2$ to get $k(t) = 2\exp(t^2/4) + C$ and put this in eqn (10) to get

$$y(t) = (2\exp(t^2/4) + C)\exp(-t^2/4) = 2 + C\exp(-t^2/4).$$