Math 225-01 FINAL EXAM SOLUTIONS Spring 2008

Show your work! Use extra paper if necessary. **Name:**

1. For the ODE system

$$\frac{dx_1}{dt} = -2x_1 + x_2$$
$$\frac{dx_2}{dt} = x_1 - 2x_2$$

a. Find the general solution.

b. Give the behavior of the solution $\vec{x}(t) = (x_1(t), x_2(t))$ as $t \to \infty$ (does it converge to the zero vector? blow up? oscillate? ...)

The system can be rewritten as $\vec{x}' = A\vec{x}$, where

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

If the eigenvalues λ_1, λ_2 of A are real, then the general solution can be expressed as

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_1 e^{\lambda_2 t}\vec{v}_2,$$

where c_1, c_2 are arbitrary constants. To compute the eigenvalues, we solve the characteristic equation

$$0 = \det(A - \lambda I) = (-2-\lambda)^2 - 1 \cdot 1 = \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1)$$

to get $\lambda_1 = -3$ and $\lambda_2 = -1$. To get an eigenvector corresponding to $\lambda_1 = -3$, we solve $(A - \lambda_1 I)\vec{v} = (A + 3I)\vec{v} = \vec{0}$. The augmented system is

$$[A + 3I|\vec{0}] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

The row operation Row 2 $\leftarrow$ Row 2 - Row 1 zeros out the second row. If we let ξ_1, ξ_2 denote the components of the eigenvector $\vec{v}$, then the first row tells us that

$$\xi_1 + \xi_2 = 0.$$

Thus $\xi_1 = -\xi_2$, where ξ_2 is arbitrary but nonzero. Setting $\xi_2 = 1$ gives us $\xi_1 = -1$ and an eigenvector $\vec{v}_1 = (-1, 1)$ corresponding to $\lambda_1 = -3$. Similarly, to get an eigenvector $\vec{v}_2$ corresponding to $\lambda_2 = -1$, we obtain the augmented system

$$[A + I|\vec{0}] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

and we can choose $\vec{v}_2 = (1, 1)$. The general solution is

$$\vec{x}(t) = c_1 e^{-3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -c_1 e^{-3t} + c_2 e^{-t} \\ c_1 e^{-3t} + c_2 e^{-t} \end{bmatrix}$$

Part b. Because of the negative exponentials, the solution decays to the zero vector as $t \to \infty$.

2. Consider the 2nd order ODE

$$\frac{d^2y}{dt^2} + 4y = \sin(2t)$$

Find the general solution to this ODE. Use whatever method you feel is appropriate, but **state which method you use** and show important details.

The characteristic equation for the homogeneous ODE is $r^2 + 4 = 0$, which has roots $\pm i2$ and corresponding solutions $\sin(2t)$, $\cos(2t)$. Since the sine term is also on the right-hand-side, the choice of the form of the particular solution for the method of undetermined coefficients is a bit messy. I think something like $y_p(t) = A\sin(2t) + Bt\sin(2t) + C\cos(2t) + Dt\cos(2t)$ should work. Implementation of Laplace transforms is also messy, for the same reason.

To apply variation of parameters, we assume the solution takes the form

$$\vec{x}(t) = k_1(t)\vec{x}_1(t) + k_2(t)\vec{x}_2(t)$$

where $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are linearly independent solutions to the homogeneous system $\vec{x}' = A\vec{x}$ which arises when we convert the scalar 2nd order ODE to system form

$$\frac{d\vec{x}}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \sin(2t) \end{bmatrix}}_{\vec{g}(t)}$$

Since $x_1(t) = y(t)$ and $x_2(t) = y'(t)$, given solutions $\sin(2t)$, $\cos(2t)$ to the homogeneous scalar ODE, we immediately get solutions to the homogeneous vector ODE $\vec{x}' = A\vec{x}$,

$$\vec{x}_1(t) = \begin{bmatrix} \sin(2t) \\ \frac{d}{dt}\sin(2t) \end{bmatrix} = \begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}, \qquad \vec{x}_2(t) = \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}$$

Turning the variation of parameters crank, we need to solve the symbolic linear system

$$\begin{bmatrix} \sin(2t) & \cos(2t) \\ 2\cos(2t) & -2\sin(2t) \end{bmatrix} \begin{bmatrix} k_1' \\ k_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sin(2t) \end{bmatrix}$$

Applying Cramer's rule,

$$k_1' = \frac{\det\begin{bmatrix} 0 & \cos(2t) \\ \sin(2t) & -2\sin(2t) \end{bmatrix}}{\det\begin{bmatrix} \sin(2t) & \cos(2t) \\ 2\cos(2t) & -2\sin(2t) \end{bmatrix}} = \frac{-\sin(2t)\cos(2t)}{-2} = \frac{\sin(2t)\cos(2t)}{2}$$

$$k_2' = \frac{\det\begin{bmatrix} \sin(2t) & 0 \\ 2\cos(2t) & \sin(2t) \end{bmatrix}}{\det\begin{bmatrix} \sin(2t) & \cos(2t) \\ 2\cos(2t) & -2\sin(2t) \end{bmatrix}} = \frac{\sin^2(2t)}{-2}$$

The next step is to integrate,

$$k_1(t) = \int k_1'(t)\,dt = \frac{1}{2}\frac{\sin^2(2t)}{4} + C_1,$$

$$k_2(t) = \int k_2'(t)\,dt = -\frac{1}{2}\left(\frac{t}{2} - \frac{\sin(4t)}{8} + C_2\right)$$

Finally, we substitute back into the above form. The solution to the original scalar ODE is the first component,

$$\begin{aligned} y(t) = x_1(t) &= (\sin^2(2t)/8 + C_1)\sin(2t) + (-t/4 + \sin(4t)/16 + C_2)\cos(2t) \\ &= \underbrace{\sin^3(2t)/8 - t\cos(2t)/4 + \cos(2t)\sin(4t)/16}_{\textit{particular solution}} + \underbrace{C_1\sin(2t) + C_2\cos(2t)}_{\textit{homogeneous solution}}. \end{aligned}$$

3. The 3 plots below show phase portraits for linear systems $\frac{d}{dt}\vec{x} = A\vec{x}$ with the origin $\vec{0}$ as the only fixed point. For each of the 3 plots, provide the following information:

(i) Classify the origin (i.e., source, sink, ...)

(ii) Describe the eigenvalues of the matrix A (i.e., both real and positive, or complex with negative real parts, ...),

(iii) State whether the system $\frac{d}{dt}\vec{x} = A\vec{x}$ is asymptotically stable, stable but not asymptotically stable, or unstable.

(iv) Write down a scalar, 2nd order, homogeneous, constant coefficient ODE whose phase portrait is qualitatively similar to the given phase portrait.

A. The origin is a sprial source; the eigenvalues are complex with positive real part and the system is unstable. $y'' - y' + y = 0$ works since its characteristic equation $r^2 - r + 1 = 0$ has roots $1/2 \pm i\sqrt{3}/2$

B. The origin is a sink; the eigenvalues are both real and negative; and the system is aysmptotically stable. A model for an overdamped mass-spring, $y'' + 3y' + y = 0$, works.

C. The origin is a saddle; the eigenvalues are both real, but one is positive and the other is negative; the system is unstable. $y'' - y = 0$ works because its characteristic equation has roots ± 1.

4. Consider the nonlinear first order ODE

$$\frac{dy}{dt} = 1 - y^2$$

a. Find all the fixed points (equilibrium solutions) for this ODE and classify them as stable, asymptotically stable, or unstable, and sketch the direction field.

To get the fixed points, set the right-hand-side, $f(y) = 1 - y^2$ equal to zero and solve to get $y = -1$ and $y = 1$. Applying linearized perturbation analysis about the fixed points, we evaluate $\lambda = f'(p)$ at the fixed points and look at the sign of λ. $f'(y) = -2y$. At $p = -1$, $\lambda = f'(p) = 2$ is positive, so $p = -1$ is unstable. At $p = 1$, $\lambda = f'(p) = -1$ is negative, so $p = 1$ is asymptotically stable. The direction field looks qualitatively similar to that of the logistic equation on p. 81 in the text, except the fixed points are shifted.

b. Compute the analytical solution to the ODE with the initial condition $y(0) = 0$. You may find it useful to apply partial fractions.

First apply separation of variables, so

$$\frac{dy}{1 - y^2} = dt.$$

Next, apply partial fractions to get

$$\frac{1}{1 - y^2} = \frac{1}{(1 - y)(1 + y)} = \frac{A}{1 - y} + \frac{B}{1 + y} = \frac{(A - B)y + A + B}{(1 - y)(1 + y)}.$$

Equating coefficients gives the linear system $A + B = 0$, $A - B = 1$, which has $A = B = 1/2$ as its solution. Then

$$-\frac{1}{2}\frac{dy}{y - 1} + \frac{1}{2}\frac{dy}{1 + y} = dt$$

Note the sign switch to change $1 - y$ to $y - 1$. This makes integration easier, so

$$-\frac{1}{2}\ln|y - 1| + \frac{1}{2}\ln|y + 1| = t + C$$

We next need to solve for y. First isolate the logs by multiplying through by -2, and then use the fact that $\ln a - \ln b = \ln(a/b)$ to get

$$\ln\frac{|y - 1|}{|y + 1|} = -2t + B, \quad B = -2C = const.$$

Then exponentiate both sides and use the fact that $|a| = \pm a$ to get

$$\frac{y - 1}{y + 1} = Ke^{-2t}, \quad K = \pm e^{B}.$$

At this point we can apply the initial condition, $y = 0$ when $t = 0$, to get $K = -1$. Finally, we isolate y.

$$y - 1 = -e^{-2t}(y + 1) = -e^{-2t}y - e^{-2t} \quad \Rightarrow \quad (1 + e^{-2t})y = 1 - e^{-2t}$$

so the analytical solution is

$$y(t) = \frac{1 - e^{-2t}}{1 + e^{-2t}}$$

c. Give the value of $\lim_{t\to\infty} y(t)$, where $y(t)$ is the solution in part b.

The stability analysis and direction field from part a show that $y(t)$ converges to the fixed point $y = 1$ as $t \to \infty$. We get the same result by taking the limit of the function from part b.

5. Consider the nonlinear system of autonomous ODE's

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = -x_1 + x_1^2 - x_2.$$

a. Find the fixed points, or equilibrium solutions, for this system.

Set $f_1(x_1, x_2) = x_2 = 0$, $f_2(x_1, x_2) = -x_1 + x_1^2 - x_2 = 0$ and solve simultaneously to get $x_2 = 0$, $x_1(-1 + x_1) = 0$, so the fixed points are $(0, 0)$ and $(1, 0)$.

b. Apply linearized perturbation analysis to determine the stability of the fixed point $(1, 0)$. The Jacobian (matrix of partial derivatives) for this system is

$$\vec{F}'(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + 2x_1 & -1 \end{bmatrix}.$$

Evaluation at the fixed point $\vec{p} = (1, 0)$ gives the matrix

$$A = \vec{F}'(\vec{p}) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

Next, compute the eigenvalues of A. The characteristic equation is

$$0 = \det(A - \lambda I) = (-\lambda)(-1 - \lambda) - 1 \cdot 1 = \lambda^2 + \lambda - 1.$$

This gives

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4(-1)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} \approx .618, \ -1.618$$

At least one eigenvalue is positive so (1,0) is an unstable fixed point.