Math 225-01 Supplemental Notes Spring 2008

Examples of Laplace Transform Solutions to ODEs

Exam I, Problem 2, in standard form,

$$y' - 2y = -\frac{1}{3}t, \quad y(0) = 1$$

Apply Laplace transform to both sides of the ODE. On the left hand side (LHS) we get

$$\begin{aligned}
\mathcal{L}\{y' - 2y\} &= \mathcal{L}\{y'\} - 2\mathcal{L}\{y\}, \quad \text{from linearity of the Laplace transform} \\
&= s\mathcal{L}\{y\} - y(0) - 2\mathcal{L}\{y\}, \quad \text{from Table 6.2.1 entry no. 18 with } n = 1 \\
&= (s-2)\mathcal{L}\{y\} - 1, \quad \text{from the initial condition } y(0) = 1
\end{aligned}$$

On the right-hand-side (RHS) of the ODE we get

$$\begin{aligned}
\mathcal{L}\{-\frac{1}{3}t\} &= -\frac{1}{3}\mathcal{L}\{t\}, \quad \text{from linearity of the Laplace transform} \\
&= -\frac{1}{3}\frac{1}{s^2}, \quad \text{from Table 6.2.1 entry no. 3 with } n = 1
\end{aligned}$$

Equate transformed LHS to transformed RHS and solve for $\mathcal{L}\{y\}$. This gives

$$(s-2)\mathcal{L}\{y\} = 1 - \frac{1}{3}\frac{1}{s^2},$$

so

$$\mathcal{L}\{y\} = \frac{1}{s-2} - \frac{1}{3}\frac{1}{(s-2)s^2} \tag{1}$$

The next step is to turn the right-hand-side (RHS) into a superposition of elementary Laplace transforms (i.e., stuff that looks like the 2nd column in Table 6.2.1). The first term on the above RHS looks like table entry no. 2 with $a = 2$. The second term involves the **product** of table entries no. 2 and no. 3 with $n = 1$. We turn this product into a superposition by applying the partial fraction expansion

$$\frac{1}{(s-2)s^2} = \frac{A}{s-2} + \frac{B}{s} + \frac{C}{s^2} \tag{2}$$

We need to compute the coefficients A, B, C. To do this we find a common denominator on the right hand side and equate coefficients of like powers of s:

$$\begin{aligned}
\frac{0 \cdot s^2 + 0 \cdot s + 1}{(s-2)s^2} &= \frac{As^2}{(s-2)s^2} + \frac{Bs(s-2)}{(s-2)s^2} + \frac{C(s-2)}{(s-2)s^2} \\
&= \frac{As^2 + Bs^2 - 2Bs + Cs - 2C}{(s-2)s^2} \\
&= \frac{(A+B)s^2 + (-2B+C)s - 2C}{(s-2)s^2}
\end{aligned}$$

Equating constants gives $1 = -2C$, so $C = -1/2$. Equating terms multiplying s gives $0 = C - 2B$, so $B = C/2 = -1/4$. Equating terms multiplying s^2 gives $0 = A + B$, so $A = -B = 1/4$. We plug this back into the expansion to get

$$\frac{1}{(s-2)s^2} = \frac{1}{4}\frac{1}{s-2} - \frac{1}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2}$$

Finally, we plug this back into equation (1) to get

$$\begin{aligned}\mathcal{L}\{y\} &= \frac{1}{s-2} - \frac{1}{3}\left(\frac{1}{4}\frac{1}{s-2} - \frac{1}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2}\right) \\ &= \frac{11}{12}\frac{1}{s-2} + \frac{1}{12}\frac{1}{s} + \frac{1}{6}\frac{1}{s^2}.\end{aligned}$$

We now have a superposition of elementary Laplace transforms. To get the solution y to the ODE, we formally apply the inverse Laplace transform to both sides and use the fact that this operator is also linear, i.e.,

$$\mathcal{L}^{-1}\{c_1F_1(s) + c_2F_2(s)\} = c_1f_1(t) + c_2f_2(t)$$

whenever $f_1(t) = \mathcal{L}^{-1}\{F_1(s)\}$ and $f_2(t) = \mathcal{L}^{-1}\{F_2(s)\}$, or equivalently, whenever $\mathcal{L}(f_1(t)) = F_1(s)$ and $\mathcal{L}(f_2(t)) = F_2(s)$. This gives us

$$\begin{aligned}y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} &= \mathcal{L}^{-1}\left\{\frac{11}{12}\frac{1}{s-2} + \frac{1}{12}\frac{1}{s} + \frac{1}{6}\frac{1}{s^2}\right\} \\ &= \frac{11}{12}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{12}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \\ &= \frac{11}{12}e^{2t} + \frac{1}{12} + \frac{1}{6}t.\end{aligned}$$

The last step follows by applying Table 6.21 entries no. 2, no. 1, and no. 3 with $n = 1$. Note that this solution is the same as that obtained for the exam problem using either variation of parameters or integrating factors.

A Note on Partial Fraction Expansions. It was suggested in class that in place of equation (2) we try the expansion

$$\frac{1}{(s-2)s^2} = \frac{A}{s-2} + \frac{Bs + C}{s^2} \tag{3}$$

This expansion is equivalent to the expansion in equation (2) because

$$\frac{Bs + C}{s^2} = \frac{Bs}{s^2} + \frac{C}{s^2} = \frac{B}{s} + \frac{C}{s^2}$$

The expansion in equation (2) is preferable here because it typically yields terms that are elementary Laplace transforms. For more on partial fraction expansions, check the web at

`http://en.wikibooks.org/wiki/Calculus/Integration_techniques`

Quiz 6, Problem 2. Solve the ODE initial value problem

$$\begin{aligned} y'' + y' + 9y &= 4 \\ y(0) &= y_0 \\ y'(0) &= v_0. \end{aligned}$$

Apply Laplace transforms to both sides of the ODE. On the LHS we get

$$\begin{aligned} \mathcal{L}\{y'' + y' + 9y\} &= \mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 9\mathcal{L}\{y\}, \quad \text{from linearity} \\ &= (s^2\mathcal{L}\{y\} - sy(0) - y'(0)) + (s\mathcal{L}\{y\} - y(0)) + 9\mathcal{L}\{y\}, \quad \text{Table no. 18, n=2, n=1} \\ &= (s^2 + s + 9)\mathcal{L}\{y\} - sy_0 - v_0 - y_0, \quad \text{from initial conditions} \end{aligned}$$

On the RHS we get

$$\begin{aligned} \mathcal{L}\{4\} &= 4\mathcal{L}\{1\}, \quad \text{from linearity} \\ &= 4\frac{1}{s}, \quad \text{from Table 6.2.1 no. 1.} \end{aligned}$$

Set LHS = RHS and isolate terms involving $\mathcal{L}\{y\}$ to get

$$(s^2 + s + 9)\mathcal{L}\{y\} = (v_0 + y_0) + y_0 s + 4\frac{1}{s}$$

so

$$\mathcal{L}\{y\} = (v_0 + y_0)\frac{1}{s^2 + s + 9} + y_0\frac{s}{s^2 + s + 9} + 4\frac{1}{s(s^2 + s + 9)}. \tag{4}$$

Next, we need to rewrite the RHS of equation (4) as the superposition of elementary Laplace transforms. We first need to know whether or not we can decompose $s^2 + s + 9$ into linear factors, in which case we could apply Table 6.2.1 entry no. 2 to each of the factors. If $s^2 + s + 9 = (s - a_1)(s - a_2)$, then a_1 and a_2 would be roots of the polynomial equation

$$s^2 + s + 9 = 0.$$

But this is just the characteristic equation for the ODE with the Laplace transform variable s substituted for the roots r. We know from Quiz 6 that the roots are complex. This means we can factor $s^2 + s + 9 = (s - a_1)(s - a_2)$ where a_1 and a_2 are the complex roots of the characteristic equation. Complex factors are a major pain, so lets drop this approach.

Lets try to express

$$s^2 + s + 9 = (s - a)^2 + b^2. \tag{5}$$

This looks like the denominator in Table 6.2.1 entries no. 9 and 10, and is what you would expect, given that complex exponential roots give exponentially modulated sine and cosine solutions. Expanding the RHS of equation (5) gives

$$s^2 + s + 9 = s^2 - 2as + a^2 + b^2.$$

Equating coefficients of like powers of s gives

$$\begin{aligned} 1 &= -2a \quad \text{(from coefficients of } s\text{)} \\ 9 &= a^2 + b^2 \quad \text{(from coefficients of } s^0 = 1\text{)} \end{aligned}$$

Thus $a = -1/2$ and $b = \sqrt{9 - a^2} = \sqrt{9 - 1/4} = \sqrt{35}/2$. An alternative approach is to apply the quadratic formula to the characteristic equation, generating roots whose real and imaginary parts give a and b:

$$r_\pm = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 9}}{2 \cdot 1} = \underbrace{-1/2}_{a} \pm i \underbrace{\sqrt{35}/2}_{b}, \quad i = \sqrt{-1}.$$

We can conclude that we the first and second terms on the RHS of equation (4) look like the superposition of elementary Laplace transforms. To deal with the 3rd term, we need to apply partial fractions.

Take the partial fraction expansion

$$\frac{1}{s(s^2 + s + 9)} = \frac{A}{s} + \frac{Bs + C}{s^2 + s + 9}. \tag{6}$$

Find a common denominator on the RHS to get

$$\frac{1}{s(s^2 + s + 9)} = \frac{A(s^2 + s + 9) + (Bs + C)s}{s(s^2 + s + 9)} = \frac{(A + B)s^2 + (A + C)s + 9A}{s(s^2 + s + 9)}$$

and equate coefficient of like powers of s on the LHS and RHS to get $9A = 1$, so $A = 1/9$; $A + B = 0$, so $B = -A = -1/9$; and $A + C = 0$, so $C = -A = 1/9$. We plug A, B, C back into the partial fraction expansion (6) to get

$$\begin{aligned} \frac{1}{s(s^2 + s + 9)} &= \frac{1}{9} \cdot \frac{1}{s} + \frac{1}{9} \cdot \frac{-s + 1}{s^2 + s + 9} \\ &= \frac{1}{9} \cdot \frac{1}{s} - \frac{1}{9} \cdot \frac{s}{s^2 + s + 9} + \frac{1}{9} \cdot \frac{1}{s^2 + s + 9} \end{aligned}$$

We then substitute this expansion back into equation (4) to get

$$\begin{aligned} \mathcal{L}\{y\} &= (v_0 + y_0)\frac{1}{s^2 + s + 9} + y_0 \frac{s}{s^2 + s + 9} + \left(\frac{4}{9}\right)\frac{1}{s} - \left(\frac{4}{9}\right)\frac{s}{s^2 + s + 9} + \left(\frac{4}{9}\right)\frac{1}{s^2 + s + 9} \\ &= \left(\frac{4}{9}\right)\frac{1}{s} + (v_0 + y_0 + 4/9)\frac{1}{s^2 + s + 9} + (y_0 - 4/9)\frac{s}{s^2 + s + 9}. \end{aligned}$$

Almost done now. $1/s = \mathcal{L}\{1\}$. The other two terms look almost like multiples of Table 6.2.1 entries 9 and 10. To make them look exactly like these entries, we make use of the representation in equation (5),

$$s^2 + s + 9 = (s - a)^2 + b^2, \quad \text{with } a = -1/2,\ b = \sqrt{35}/2 \tag{7}$$

and we express

$$1 = \frac{1}{b} \cdot b, \quad s = (s - a) + a.$$

This now gives us

$$\begin{aligned} \mathcal{L}\{y\} &= \left(\frac{4}{9}\right)\frac{1}{s} + (v_0/b + y_0/b + 4/9b)\frac{b}{(s - a)^2 + b^2} + (y_0 - 4/9)\frac{(s - a) + a}{(s - a)^2 + b^2} \\ &= \left(\frac{4}{9}\right)\underbrace{\frac{1}{s}}_{\mathcal{L}\{1\}} + c_1 \underbrace{\frac{b}{(s - a)^2 + b^2}}_{\mathcal{L}\{e^{at}\sin(bt)\}} + c_2 \underbrace{\frac{s - a}{(s - a)^2 + b^2}}_{\mathcal{L}\{e^{at}\cos(bt)\}}, \end{aligned}$$

where c_1 and c_2 are constants that depend on the initial conditions and constants $a = -1/2$, $b = \sqrt{35}/2$, are real and imaginary parts of the roots of the characteristic equation. Finally we apply the inverse Laplace transform to both sides to get the solution of the ODE,

$$y(t) = 4/9 + c_1 e^{-t/2} \sin(\sqrt{35}/2 \cdot t) + c_2 e^{-t/2} \cos(\sqrt{35}/2 \cdot t)$$

This the same thing that we got using the Method of Undetermined Coefficients.

An obvious question is the following: What are the advantages and disadvantages of using Laplace transforms instead of the Method of Undetermined Coefficients? Probably the most obvious advantage that we don't need to guess the form of the particular solution. (To be fair to the textbook, they give you a user-vicious 16-part recipe that takes the guesswork out of the method; see details in Section 3.6) The downside of Laplace transform approach is that you need to manipulate the transformed solution so that it looks like the superposition of elementary Laplace transforms. This typically requires partial fraction expansions and some ugly algebraic manipulations, as we have seen above. While ugly, these manipulations are sufficiently straightforward that they can be relatively easily implemented in computer software. They form the basis for automatic circuit design codes. The Laplace transform is also a basic tool for the design of control systems.