Math 225-01 Supplemental Notes Spring 2008
Topics from Linear Algebra

Fundamental Theorem of $n \times n$ Linear Systems

Consider the linear system in matrix-vector form, $A\vec{x} = \vec{b}$, where the matrix A is $n \times n$. The following statements are equivalent, i.e., either all are true or none are true.

S1. The inverse matrix A^{-1} exists.

S2. $\det(A) \neq 0$.

S3. Given **any** $n \times 1$ vector $\vec{b}$, the system $A\vec{x} = \vec{b}$ has a solution.

S4. The homogeneous system $A\vec{x} = \vec{o}$ has $\vec{x} = \vec{o}$ as its **only** solution.

S5. We can apply Gauss-Jordan reduction to convert the matrix A to the $n \times n$ identity matrix I.

S6. The column vectors of A are linearly independent.

If these conditions hold, the matrix A is called **nonsingular**, or **invertible**. Otherwise, the matrix A is called **singular**.

Linear Independence

Vectors $\vec{v}_1, \ldots, \vec{v}_n$ are called **linearly independent** if the equation (with coefficients $c_1, \ldots, c_n$ as the variables)

$$c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = \vec{0} \tag{1}$$

has $c_1 = 0$, ..., $c_n = 0$ as its **only** solution. Note that this equation can be rewritten in matrix-vector form

$$\underbrace{[\vec{v}_1 \ldots \vec{v}_n]}_{A}\,\vec{x} = \vec{0},$$

where the coefficients $c_1, \ldots, c_n$ form the entries of $\vec{x}$. The columns of the matrix A are just the vectors $\vec{v}_i$, $i = 1, \ldots, n$ in the superposition (1). If each $\vec{v}_i$ is an $n \times 1$ vector, then A is an $n \times n$ matrix.

From statements S2, S4, and S6 of the Fundamental Theorem, we see that $n \times 1$ vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent if and only if $\det(A) \neq 0$, where the columns of A are the vectors $\vec{v}_i$, $i = 1, ..., n$.

Cramer's Rule

This is a determinant-based method for solving $n \times n$ linear systems $A\vec{x} = \vec{b}$. Let $\vec{v}_i$ denote the ith column of A. If A is invertible, then the ith component of the solution vector is given by

$$x_i = \frac{\det([\vec{v}_1 \cdots \vec{v}_{i-1}\vec{b}\vec{v}_{i+1} \cdots \vec{v}_n])}{\det(A)}.$$

The matrix in the numerator is obtained by replacing the ith column of A by the vector $\vec{b}$.

An ODE Application of Cramer's Rule. This example is motivated by the method of variation of parameters to solve nonhomogeneous linear systems of ODEs, which take the form

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{g}(t).$$

In the 2×2 case, one assumes a solution of the form

$$\vec{x}(t) = k_1(t)\vec{v}_1(t) + k_2(t)\vec{v}_2(t),$$

where $\vec{v}_1(t)$ and $\vec{v}_2(t)$ are linearly independent solutions to the homogeneous equation $\vec{x}' = A\vec{x}$. By substituting this form into the ODE and doing some algebra, one obtains the to equation

$$k_1'(t)\vec{v}_1(t) + k_2'(t)\vec{v}_2(t) = \vec{g}(t),$$

Cramer's rule gives the solution as

$$k_1'(t) = \frac{\det([\vec{g}(t)\ \vec{v}_2(t)])}{\det([\vec{v}_1(t)\ \vec{v}_2(t)])}$$

$$k_2'(t) = \frac{\det([\vec{v}_1(t)\ \vec{g}(t)])}{\det([\vec{v}_1(t)\ \vec{v}_2(t)])}$$

The textbook calls the determinant in the denominator the **Wronskian**.

To be more specific, suppose

$$\vec{v}_1 = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}, \quad \vec{g} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Then the Wronskian is

$$\det([\vec{v}_1(t)\ \vec{v}_2(t)]) = \cos^2(t) - (-\sin^2(t)) = 1$$

This confirms that $\vec{v}_1$ and $\vec{v}_2$ really are independent. Also,

$$\det([\vec{g}(t)\ \vec{v}_2(t)]) = +0 \cdot (\cos(t)) - t \cdot (\sin(t)) = -t\sin(t)$$

and

$$\det([\vec{v}_1(t)\ \vec{g}(t)]) = +t \cdot (\cos(t)) - 0 \cdot (-\sin(t)) = t\cos(t)$$

Consequently,

$$k_1'(t) = -t\sin(t), \quad k_2'(t) = t\cos(t).$$