Math 225-01 Quiz 10 Solutions Spring 2008

1. Suppose a given matrix A, with real entries, has the following eigenvalue-eigenvector pair.

$$\lambda_1 = -1 + i, \quad \vec{v}_1 = \begin{bmatrix} 2+3i \\ 1 \end{bmatrix}$$

Give the general real-valued solution to the ODE $\vec{x}' = A\vec{x}$.

A complex-valued solution is $\vec{x}(t) = e^{\lambda_1 t}\vec{v}_1$. Using Euler's formula,

$$e^{\lambda_1 t} = e^{(-1+i)t} = e^{-t}e^{it} = e^{-t}(\cos t + i \sin t) = e^{-t}\cos t + ie^{-t}\sin t,$$

so

$$\vec{x}(t) = \begin{bmatrix} (e^{-t}\cos t + ie^{-t}\sin t)(2 + i3) \\ (e^{-t}\cos t + ie^{-t}\sin t)(1) \end{bmatrix}$$

Recall complex multiplication $(\alpha + i\beta)(\gamma + i\delta) = \alpha\gamma - \beta\delta + i(\alpha\delta + \beta\gamma)$. Applying this gives the first term,

$$(e^{-t}\cos t + ie^{-t}\sin t)(2 + i3) = \underbrace{2e^{-t}\cos t - 3e^{-t}\sin t}_{\text{real part}} + i(\underbrace{3e^{-t}\cos t + 2e^{-t}\sin t}_{\text{imaginary part}}).$$

Thus

$$\begin{aligned} \vec{x}(t) &= \begin{bmatrix} 2e^{-t}\cos t - 3e^{-t}\sin t & + & i(3e^{-t}\cos t + 2e^{-t}\sin t) \\ e^{-t}\cos t & + & ie^{-t}\sin t \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 2e^{-t}\cos t - 3e^{-t}\sin t \\ e^{-t}\cos t \end{bmatrix}}_{\vec{x}_1(t)} + i \underbrace{\begin{bmatrix} 3e^{-t}\cos t + 2e^{-t}\sin t \\ e^{-t}\sin t \end{bmatrix}}_{\vec{x}_2(t)} \end{aligned}$$

The general real-valued solution is $c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$. Note that the term $i = \sqrt{-1}$ is **not** part of $\vec{x}_2(t)$.

2. Let

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}.$$

One (normal) eigenvalue-eigenvector pair for A is

$$\lambda_1 = -2, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

a. Compute a generalized eigenvector for A.

b. Give the general solution to the ODE $\vec{x}' = A\vec{x}$.

Note that $\det(A-\lambda I) = (0-\lambda)(-4-\lambda)-(1)(-4) = \lambda^2+4\lambda+4 = (\lambda+2)^2$, so $\lambda_1 = -2$ is a repeated eigenvalue for A.

2a. If $\vec{w}$ is a generalized eigenvector, then it solves

$$(A-\lambda_1 I)\vec{w} = \vec{v}_1.$$

In component form,

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The second equation is redundant. (If we augment and row-reduce, then the second row can be reduced to all 0s via the elementary row operation Row 2 ← Row 2 + 2*Row 1.) Thus the components of $\vec{w}$ must statisfy $2\xi_1+\xi_2 = 1$, or $\xi_2 = 1-2\xi_1$. Setting $\xi_1 = 0$ gives $\xi_2 = 1$, and

$$\vec{w} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note that $\vec{w}$ is not unique. Setting $\xi_1 = 1$ would have given $\xi_2 = -1$.

2b. The general solution to the ODE is

$$\begin{aligned} \vec{x}(t) &= c_1 e^{\lambda_1 t}\vec{v}_1 + c_2\left(te^{\lambda_1 t}\vec{v}_1 + e^{\lambda_1 t}\vec{w}\right) \\ &= c_1 e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2\left(te^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{-2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= \begin{bmatrix} c_1 e^{-2t} + c_2 te^{-2t} \\ -2c_1 e^{-2t} + c_2(-2te^{-2t} + e^{-2t}) \end{bmatrix}. \end{aligned}$$