Math 225-01 Quiz 9 Solutions Spring 2008

1. Let

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

One eigenpair for A is

$$\lambda_1 = -1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

You are to (a) use linear algebra techniques to compute the second eigenvalue and a corresponding eigenvector for A; (b) give the general solution to the ODE $\vec{x}' = A\vec{x}$; and (c) solve the ODE, given the initial condition $\vec{x}(0) = [3\ \text{-}4]^T$.

Recall that $(\lambda_i, \vec{v}_i)$ is an eigenpair if and only if $A\vec{v}_i = \lambda_i \vec{v}_i,\ \vec{v}_i \neq \vec{0}$ if and only if $(A - \lambda_i I)\vec{v}_i = \vec{0},\ \vec{v}_i \neq \vec{0}$ if and only if $A - \lambda_i I$ is not invertible if and only if $\det(A - \lambda_i I) = 0$. So ...

Part a1. To get the eigenvalues, solve the characteristic equation, $\det(A - \lambda I) = 0$.

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 0 - \lambda & 1 \\ -2 & -3 - \lambda \end{bmatrix} = (-\lambda)(-3 - \lambda) - (-2)1 = \lambda^2 + 3\lambda + 2.$$

We can factor $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$ Setting this equal to zero gives the eigenvalues $\lambda_1 = -1$, $\lambda_1 = -2$.

Part a2. To get an eigenvector $\vec{v}_2$ corresponding to the eigenvalue $\lambda_1 = -2$, find a nonzero solution to $\vec{0} = (A - (-2)I)\vec{v} = (A + 2I)\vec{v}$. The associated augmented matrix is

$$\left[\begin{array}{cc|c} 0+2 & 1 & 0 \\ -2 & -3+2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The symbol "$\sim$" means "is equivalent to under row reduction", and the elementary row operation is Row 2 $\leftarrow$ Row 2 + Row 1. The right-most augmented matrix is equivalent to the single linear equation $2\xi_1 + \xi_2 = 0$, or $\xi_2 = -2\xi_1$. Setting $\xi_1 = 1$ gives a corresponding eigenvector,

$$\vec{v}_2 = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Part b. The general solution is the superposition

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 = \begin{bmatrix} c_1 e^{-t} & + & c_1 e^{-2t} \\ -c_1 e^{-t} & + & -2c_1 e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Part c Evaluating the right-most solution form in part b at $t = 0$ and setting this equal to the initial vector gives the linear system

$$\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}.$$

Applying the elementary row operations Row 2 $\leftarrow$ Row 2 + Row 1, followed by Row 1 $\leftarrow$ Row 1 + Row 2, followed by Row 2 $\leftarrow$ -Row 2 to the augmented system gives

$$\begin{bmatrix} 1 & 1 & | & 3 \\ -1 & -2 & | & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & | & 3 \\ 0 & -1 & | & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & -1 & | & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 1 \end{bmatrix},$$

so the coefficients in the superposition in part b are $c_1 = 2$, $c_2 = 1$.