Math 225-01 Final Exam Spring 1997

Please show your work when appropriate! **Name:**

1. Consider the ODE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sin(t)$$

   a. Find the general solution to this ODE. Use whatever method you feel is appropriate, but **state which method you use** and show important details.

   b. Describe the behavior of the solution for large t, i.e., if it "blows up" or decays to zero, say so; if it goes to a constant value, give the constant value; if it oscillates without decaying to zero, give the frequency of oscillation.

2. Solve the ODE initial value problem

$$\begin{aligned} \frac{d^2y}{dt^2} + 9y &= \delta_5(t), \\ y(0) = -1, \qquad & y'(0) = 0. \end{aligned}$$

a. Find the general solution to this ODE. Use whatever method you feel is appropriate, but **state which method you use** and show important details.

b. Describe the behavior of the solution for large t.

3. Consider a full tank of salty water with a 10 liter capacity. Water is pumped into the tank at a rate of 1 liter per minute. Salty water is pumped out at the same rate. Suppose at time $t = 0$ minutes the tank contains 1 gram of salt. Suppose also that from time $t = 0$ until just before time $t = 3$ minutes, the water being pumped into the tank is pure (no salt). Beginning at time $t = 3$ minutes (and continuing indefinitely thereafter), salty water with a fixed salt concentration of 2 grams per liter is pumped into the tank.

a. Write down a first order, linear, constant coefficient, nonhomogeneous ODE whose solution describes the amount of salt (measured in grams) in the tank at time t minutes. Also give the initial condition for the ODE.

b. Solve the ODE with the initial condition from part a.

c. Describe the behavior of the solution for large time. Then use this to determine the salt concentration in the tank at large time t. Your answer should make physical sense.

4. Consider the model for a nonlinear "hard" spring,

$$\frac{d^2y}{dt^2} + y(1 - y^2) = 0.$$

a. Put this equation into system form, and then find all the equilibrium points for the system (there are 3).

b. For each equilibrium point, classify the linearized system about that equilibrium point (for example, stable with a spiral sink at the origin).

c. Verify that the system is Hamiltonian, and then using information from part b, provide a rough sketch of the contours of the Hamiltonian function (you shouldn't have to actually compute the Hamiltonian).

d. Describe the behavior of a solution that starts out at $y = 1/2$, $\frac{dy}{dt} = 0$ at time $t = 0$. Is this physically reasonable for a spring?

e. Describe the behavior of a solution that starts out at $y = 2$, $\frac{dy}{dt} = 0$ at time $t = 0$. Is this physically reasonable for a spring?