Math 225-01 Supplemental Notes Spring 2008

Stability Analysis for Autonomous ODE Systems

Autonomous ODE systems take the form

$$\vec{x}' = \vec{F}(\vec{x}). \tag{1}$$

A vector $\vec{p}$ is an **equilibrium solution**, or a **fixed point**, for the ODE system (1) provided the components of $\vec{p}$ are constants and $\vec{x} = \vec{p}$ solves the ODE system. Since derivatives of constants are zero, the equation

$$\vec{F}(\vec{p}) = \vec{0} \tag{2}$$

characterizes fixed points.

Example 1. The linear case is

$$\vec{x}' = \underbrace{A\vec{x}}_{\vec{F}(\vec{x})},$$

where A is an $n \times n$ matrix with coefficients that are independent of t and $\vec{x}$. Since $A\vec{0} = \vec{0}$, by equation (2), $\vec{p} = \vec{0}$ is always a fixed point for any linear system. If A is invertible, the $\vec{p} = \vec{0}$ is the only fixed point. This is the case for all the ODEs for mechanical systems that we have seen this semester.

Example 2. A simplified damped pendulum model is

$$\vec{x}' = \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix}.$$

This comes from the scalar pendulum model in the text by setting $g = L = m = \gamma = 1$ and converting to system form with $x_1 = \theta$, the angular displacement of the pendulum, and $x_1 = \theta'$, the angular velocity. To compute the fixed points, we apply characterization (2) and set both components of the right-hand-side vector equal to zero. This gives

$$p_2 = 0, \qquad -\sin(p_1) - 0 = 0,$$

so $\sin(p_1) = 0$ and $p_2 = 0$. Consequently, the fixed points are

$$(0,0),\ (0,\pi),\ (0,-\pi),\ (0,2\pi),\ (0,-2\pi),\ (0,3\pi),\ldots.$$

Definition of Stability. A mathematical definition of stability is given on p. 496 of the text. In words, a critical point $\vec{p}$ for system (1) is **stable** if, given any trajectory $\vec{x}(t)$ that starts close enough to $\vec{p}$, the trajectory stays close to $\vec{p}$ for all time t. (Recall that a trajectory for (1) is a solution to (1) that satisfies some specified initial condition.) If a critical point is not stable, then it is called **unstable**. In words, this means that we can find trajectories that may start arbitrarily close to $\vec{p}$ (but not exactly at $\vec{p}$) but they eventually move away from $\vec{p}$.

A critical point $\vec{p}$ for system (1) is called **asymptotically stable** if, given any trajectory $\vec{x}(t)$ that starts close enough to $\vec{p}$,

$$\lim_{t\to\infty} \vec{x}(t) = \vec{p}.$$

Stability for Linear Systems. Linear systems have a simple characterization of stability in terms of their eigenvalues. Denote the eigenvalues of A by $\lambda_1, \dots, \lambda_n$, and recall that if $\lambda = \alpha + i\beta$ is complex-valued, then $Real(\lambda) = \alpha$ is the real part of λ. If λ is real-valued, then $Real(\lambda) = \lambda$, and the condition $Real(\lambda) < 0$ just means that λ is negative. Then

$$\vec{x}' = A\vec{x} \text{ is stable at } \vec{p} = \vec{0} \quad \Longleftrightarrow \quad Real(\lambda_k) \le 0 \text{ for all k=1,...,n.}$$

Equivalently,

$$\vec{x}' = A\vec{x} \text{ is unstable at } \vec{p} = \vec{0} \quad \Longleftrightarrow \quad Real(\lambda_k) > 0 \text{ for some k.}$$

There is a corresponding simple characterization of asymptotic stability.

$$\vec{x}' = A\vec{x} \text{ is asymptotically stable at } \vec{p} = \vec{0} \quad \Longleftrightarrow \quad Real(\lambda_k) < 0 \text{ for all k=1,...,n.}$$

The double-headed arrow above means “if and only if”, or “is equivalent to”.

Linearized Perturbation Analysis.

In the 1-d case, the autonomous system (1) reduces to a scalar ODE

$$y' = f(y). \tag{3}$$

Suppose that p is a fixed point for this ODE, so that $f(p) = 0$. The Taylor approximation for f about p gives

$$f(p + \Delta y) = \underbrace{f(p)}_{=0} + f'(p)\Delta y + e(\Delta y),$$

where the error term $e(\Delta y)$ vanishes in the limit as $\Delta y \to 0$. But since p is constant,

$$\frac{d}{dt}(p + \Delta y(t)) = 0 + \frac{d}{dt}\Delta y(t) = \Delta y'(t).$$

Thus if $y(t)$ solves the ODE (3) and is close to p, then $\Delta y(t) = y(t) - p$ satisfies

$$\Delta y' \approx \lambda \Delta y, \qquad \lambda = f'(p) \tag{4}$$

Replacing $\approx$ by $=$, we obtain a scalar linear ODE

$$\Delta y' = \lambda\, \Delta y, \quad \text{where } \lambda = f'(p),$$

which is called the **linearized perturbation equation**. Since $\lambda = f'(p)$ is constant, this equation has solutions $\Delta y(t) = Ce^{\lambda t}$ which decay to zero exponentially if $\lambda = f'(p) < 0$ and grow exponentially if $f'(p) > 0$. Because this linearized perturbation analysis is valid near p, we obtain the following result.

Theorem. (Linearized stability for scalar autonomous ODEs)

- If $f(p) = 0$ and $f'(p) < 0$, then p is an asymptotically stable fixed point for the ODE $y' = f(y)$.
- If $f(p) = 0$ and $f'(p) > 0$, then p is an unstable fixed point for $y' = f(y)$.
- If $f(p) = 0$ and $f'(p) = 0$, the above analysis gives no indication of the stability of the fixed point p.

Example. Consider the simplified logistic equation

$$y' = y(1 - y) = y - y^2.$$

Setting $f(y) = y(1 - y) = 0$ gives us the fixed points $p = 0$ and $p = 1$. Since $f'(y) = 1 - 2y$, $f'(0) = 1 > 0$, so $p = 0$ is unstable. On the other hand, $f'(1) = 1 - 2 = -1 < 0$, so $p = -1$ is asymptotically stable.

In the vector case, the above linearized perturbation analysis holds, so that if we have a solution $\vec{x}(t) = \vec{p} + \Delta\vec{x}(t)$ which is close to a fixed point $\vec{p}$, then $\Delta\vec{x}(t)$ approximately solves the linearized perturbation equation

$$\Delta\vec{x}' = A\vec{\Delta}x,$$

where the matrix $A = \vec{F}'(\vec{p})$ is called the **Jacobian matrix** for system (1) at $\vec{p}$. This matrix has eigenvalues $\lambda_1, \ldots, \lambda_n$. Since the nonlinear system behaves like its linearized approximation when we are close to $\vec{p}$, we obtain the following result.

Theorem. (Linearized stability for autonomous ODE Systems).

- If $\vec{F}(\vec{p}) = \vec{0}$ and $Real(\lambda_k) < 0$ for **all** the eigenvalues $\lambda_k,\ k = 1, ..., n$ of the Jacobian matrix $A = \vec{F}'(\vec{p})$, then $\vec{p}$ is an asymptotically stable fixed point for the ODE (1).
- If $\vec{F}(\vec{p}) = \vec{0}$ and $Real(\lambda_k) > 0$ for **any** eigenvalue λ_k of the Jacobian matrix, then $\vec{p}$ is an unstable fixed point for (1).

Jacobian matrix computation. If $\vec{F}$ has components $f_k(x_1, \ldots, x_n),\ k = 1, \ldots, n$, then $A = \vec{F}'(\vec{p})$ has components which are the partial derivatives

$$a_{j,k} = \frac{\partial f_j}{\partial x_k}(\vec{p})$$

For example, for the damped pendulum model, the Jacobian matrix is

$$\begin{aligned} F'(\vec{x}) &= \begin{bmatrix} \frac{\partial}{\partial x_1} x_2 & \frac{\partial}{\partial x_2} x_2 \\ \frac{\partial}{\partial x_1}(-\sin x_1 - x_2) & \frac{\partial}{\partial x_2}(-\sin x_1 - x_2) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -\cos(x_1) & -1 \end{bmatrix}. \end{aligned}$$

To determine stability, we evaluate the Jacobian at the fixed points, compute the eigenvalues, and check their sign if they are real, or check the sign of the real parts if they are complex.

At $\vec{p} = (0, 0)$,

$$A = \begin{bmatrix} 0 & 1 \\ -\cos(0) & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Then $\det(A - \lambda I) = -\lambda(-1 - \lambda) + 1 = \lambda^2 + \lambda + 1$. Setting this equal to zero gives the eigenvalues

$$\lambda_{1,2} = -1/2 \pm i\sqrt{3}/2.$$

Since the real part is negative, $\vec{p} = (0, 0)$ is an asymptotically stable fixed point. Since the cosine function is 2π periodic, fixed points $\vec{p} = (2n\pi, 0)$ are also asymptotically stable for $n = \pm 1, \pm 2, \ldots$.

At $\vec{p} = (\pi, 0)$,

$$A = \begin{bmatrix} 0 & 1 \\ -\cos(\pi) & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

Then $\det(A - \lambda I) = -\lambda(-1 - \lambda) - 1 = \lambda^2 + \lambda - 1$. Setting this equal to zero gives the eigenvalues

$$\lambda_{1,2} = -1/2 \pm \sqrt{5}/2 \approx -1.6180,\ .6180$$

These eigenvalues are real with mixed signs, so $\vec{p} = (\pi, 0)$ is a saddle point, which is unstable. The same holds for $\vec{p} = (-\pi, 0),\ (\pm 3\pi, 0), (\pm 5\pi, 0), \ldots.$