Math 225-01 Supplemental Notes Spring 2008

n^{th} Order Linear Homogeneous ODEs

n^{th} order linear ODEs take the form

$$\underbrace{a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots a_1(t)\frac{dy}{dt} + a_0(t)y(t)}_{L[y](t)} = g(t) \tag{1}$$

Here $a_0(t), \ldots, a_n(t)$ are called the *coefficients* of the ODE. The ODE has *constant coefficients* if the coefficients are all constants, independent of t. (Note that the coefficients cannot depend on y, or else the ODE would become nonlinear.) The ODE is called *homogeneous* if the right-hand-side function $g(t) = 0$; otherwise it is called *nonhomogeneous*. The ODE is in *standard form* if the coefficient of the highest order derivative is one, i.e., if $a_n(t) = 1$. The ODE can always be put in standard form by dividing through by $a_n(t)$, provided $a_n(t)$ is never zero.

Lets assume that the coefficents $a_0(t), \ldots, a_n(t)$ and the forcing function $g(t)$ make sense for $t_0 > 0$. Corresponding to the n^{th} order ODE in equation (1), we have n *initial conditions*

$$\begin{aligned} y(t_0) &= b_0 \\ y'(t_0) &= b_1 \\ \vdots \;\; &\vdots \;\; \vdots \\ \frac{d^{n-1}y}{dt^{n-1}}(t_0) &= b_{n-1}, \end{aligned} \tag{2}$$

where $b_0, b_1, \ldots, b_n$ are all constants. Note that the highest order derivative in the initial conditions (n-1) is one less than the highest order derivative (n) that appears in the ODE.

Linearity. The symbol $L[y]$ on the left-hand-side of equation (1) defines the n^{th} order ordinary differential operator L. The term "operator" means a transformation that maps functions to other functions. The operator L is called *linear* because it has following property: If $y_1(t)$ and $y_2(t)$ are functions and c_1 and c_2 are constants, then

$$L[c_1y_1 + c_2y_2](t) = c_1L[y_1](t) + c_2L[y_2](t) \tag{3}$$

See the top of p. 146 in the text for a proof in the special case of 2^{nd} order linear ODEs in standard form. An important consequence of the linearity of L is the
Principle of Superposition: If L is a linear operator, y_1 and y_2 are both solutions to the homogeneous equation $L[y] = 0$, and c_1 and c_2 are constants, then $c_1y_1 + c_2y_2$ is also a solution to $L[y] = 0$. To verify this,

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= c_1L[y_1] + c_2L[y_2], && \text{from linearity of } L \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0, && \text{since } L[y_1] = 0 \text{ and } L[y_2] = 0. \end{aligned}$$

Some Theory for Linear Homogeneous ODEs. The general solution to an n^{th} order, linear, homogeneous ODE $L[y] = 0$ takes the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \dots c_n y_n(t), \tag{4}$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants and $y_1(t), y_2(t), \dots, y_n(t)$ are solutions to the homogeneous ODE (i.e., $L[y_1] = 0, \dots, L[y_n] = 0$) which are **linearly independent**. This means the *only way* that

$$c_1 y_1(t) + c_2 y_2(t) + \dots + c_n y_n(t) = 0 \quad \text{for all } t > 0$$

is for $c_1 = 0$, $c_2 = 0$, ..., $c_n = 0$.

If the coefficients $a_0, \dots, a_n$ of the ODE in equation (1) are well behaved, then we can always find n linearly independent solutions to the homogeneous ODE. Among the important consequences of the existence of n independent solutions is the following:

- Given any initial conditions (i.e., no matter what values the constants $b_0, b_1, \dots, b_{n-1}$ in equation (2) take on, we can always find a unique solution to the ODE in (1) which also satisfies the initial conditions.

To actually compute this solution, you will need to find the solution to an $n \times n$ linear system of equations whose unknowns are the coefficients $c_1, c_2, \dots, c_n$ in equation (4). The theory guarantees that this linear system always has a unique solution.

Practical Solution of Linear, Homogeneous, Constant Coefficient ODEs. In case the ODE (1) is homogeneous (so $g(t) = 0$) and has constant coefficients (so $a_0, a_1, \dots, a_n$ are all constants), the following approach will generate at least some of the required n linearly independent solutions:

Assume a solution to $L[y] = 0$ of the form $y(t) = e^{rt}$ where r is constant. As a consequence of the linearity of L, the fact that

$$\frac{d}{dt} e^{rt} = r e^{rt}, \ \frac{d^2}{dt^2} e^{rt} = r^2 e^{rt}, \dots, \frac{d^n}{dt^n} e^{rt} = r^n e^{rt},$$

and the fact that e^{rt} can never be zero for all $t > 0$, no matter how we choose r, we obtain the corresponding *characteristic equation*,

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0. \tag{5}$$

If r_k is a solution of the characteristic equation, then $y_k(t) = e^{r_k t}$ satisfies $L[y_k] = 0$, and we get a solution to the linear, homogeneous, constant coefficient ODE.

Note that the left hand side of the characteristic equation (5) is a polynomial of degree n, called the *characteristic polynomial* for the ODE. This polynomial has n roots (equivalently, the characteristic equation has n solutions), which we can label as $r_1, r_2, \dots, r_n$. It can be proved that if these roots are *distinct*, then the required n linearly independent solutions can be taken to be

$$y_1(t) = e^{r_1 t}, \ y_2(t) = e^{r_2 t}, \dots, y_n(t) = e^{r_n t}.$$

However, the following complications can arise:

(i) The roots need not be real-valued. Suppose the the first root is complex-valued, so it can be written as

$$r_1 = \lambda + i\mu, \quad i = \sqrt{-1},$$

where λ and μ are real-valued constants. Then its *complex conjugate* $r_2 = \lambda - i\mu$ is also a root of the characteristic polynomial. It can be shown that a corresponding pair of homogeneous solutions to the (linear, constant coefficient) ODE are

$$y_1(t) = e^{\lambda t}\sin(\mu t), \quad y_2(t) = e^{\lambda t}\cos(\mu t).$$

This situation is discussed in more detail in Section 3.4 of the text.

(ii) The roots need not be distinct. Suppose the r_1 is a double root of the characteristic polynomial, i.e., the characteristic polynomial can be factored as

$$(r - r_1)^2 Q(r),$$

where $Q(r)$ is a polynomial of degree $n - 2$ which does not have r_1 as a root. In this case, we obtain a pair of linearly independent, homogeneous solutions

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = te^{r_1 t}.$$

If $r_1 = \lambda + i\mu$ happens to be complex valued, then its complex conjugate is also a root, and we can combine this with case (i) to get 4 linearly independent solutions,

$$y_1(t) = e^{\lambda t}\sin(\mu t),\ y_2(t) = e^{\lambda t}\cos(\mu t),\ y_3(t) = te^{\lambda t}\sin(\mu t),\ y_4(t) = te^{\lambda t}\cos(\mu t).$$

See section 3.5 of the text for details. If the root r_1 is repeated more times, we just append higher powers of t to get more linearly independent solutions.

The Variable Coefficient Case. If the coefficients of the ODE (1) are not constant with respect to t, but the ODE is first order, we compute a nonzero homogeneous solution $y_1(t)$ by separation of variables, provided we can handle the integration. (Note that linear combinations of a single function $y_1(t)$ are simply constant multiples $c_1 y_1(t)$, which give us the general homogeneous solution in the first order case).

If the ODE is higher order, we can compute linearly independent, homogeneous solutions using series techniques from Chapter 6 of the text.