Math 225-01 Supplemental Notes Spring 2008

Variation of Parameters—System Version

This method applies to nonhomogeneous linear systems

$$\vec{x}'(t) = A(t)\vec{x}(t) + \vec{g}(t). \tag{1}$$

Let $A(t)$ be $n \times n$ and assume we can find n linearly independent solutions $\vec{x}_1(t)$, $\vec{x}_2(t)$, ..., $\vec{x}_n(t)$ to the homogeneous system

$$\vec{x}'(t) = A(t)\vec{x}(t). \tag{2}$$

In the constant coefficient case, $A(t) = A$ and techniques from Sections 7.5, 7.6, and 7.8 can be applied to compute $\vec{x}_i(t),\ i = 1, \ldots, n$.

Let

$$X(t) \ = \ [\vec{x}_1(t)\vec{x}_2(t) \cdots \vec{x}_n(t)]$$

denote the $n \times n$ matrix whose ith column is the vector $\vec{x}_i(t)$. Since the $\vec{x}_i(t)$s are linearly independent, the determinant

$$W(t) = \det(X(t)), \tag{3}$$

which is called the **Wronskian** determinant, is nonzero.

Generalizing the variation of parameters approach we took for scalar equations (see web posting from 2/25/08), we assume a solution to (1) of the form

$$\vec{x}(t) = k_1(t)\vec{x}_1(t) + k_2(t)\vec{x}_2(t) + \ldots + k_n(t)\vec{x}_n(t). \tag{4}$$

Using a straightforward generalization of the product rule,

$$\begin{aligned} \vec{x}' &= k_1'\vec{x}_1 + k_2'\vec{x}_2 + \ldots + k_n'\vec{x}_n + k_1\vec{x}_1' + k_2\vec{x}_2' + \ldots + k_n\vec{x}_n' \\ &= k_1'\vec{x}_1 + k_2'\vec{x}_2 + \ldots + k_n'\vec{x}_n + k_1A\vec{x}_1 + k_2A\vec{x}_2 + \ldots + k_nA\vec{x}_n. \end{aligned} \tag{5}$$

The second equality follows because each $\vec{x}_i$ solves the homogeneous equation (2). Note that we have suppressed dependence on t for notational simplicity, e.g., k_1 means $k_1(t)$. But since matrix-vector multiplication is a linear operation,

$$A\vec{x} = A(k_1\vec{x}_1 + k_2\vec{x}_2 + \ldots + k_n\vec{x}_n) = k_1A\vec{x}_1 + k_2A\vec{x}_2 + \ldots + k_nA\vec{x}_n. \tag{6}$$

By substituting the form (4) into the nonhomogeneous equation (1), and then using equations (5) and (6), we obtain the linear system

$$k_1'\vec{x}_1 + k_2'\vec{x}_2 + \ldots + k_n'\vec{x}_n = \vec{g}.$$

This can be rewritten in matrix-vector form as

$$X(t)\vec{k}'(t) = \vec{g}(t). \tag{7}$$

To this system we can apply Cramer's rule (see web posting from 4/4/08) to obtain

$$k_1'(t) = \frac{\det[\vec{g}\,\vec{x}_2 \cdots \vec{x}_n]}{\det[\vec{x}_1 \vec{x}_2 \cdots \vec{x}_n]} = \frac{\det[\vec{g}\,\vec{x}_2 \cdots \vec{x}_n]}{W(t)},$$

where the $W(t)$ in the denominator is the Wronskian determinant from equation (3). Similarly,

$$\begin{aligned} k_2'(t) &= \frac{\det[\vec{x}_1\,\vec{g} \cdots \vec{x}_n]}{W(t)} \\ &\vdots \\ k_n'(t) &= \frac{\det[\vec{x}_1 \cdots \vec{x}_{n-1}\,\vec{g}]}{W(t)} \end{aligned}$$

We next integrate to get

$$k_i(t) = \int k_i'(t)\,dt + c_i, \quad i = 1, \ldots, n,$$

and substitute back into the form (4) to obtain the general solution

$$\vec{x}(t) = \underbrace{\int k_1'(t)\,dt\,\vec{x}_1(t) + \ldots + \int k_n'(t)\,dt\,\vec{x}_n(t)}_{particular} + \underbrace{c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t)}_{homogeneous}.$$

Note that the first n terms on the RHS give a particular solution to the nonhomogeneous ODE (1), while the last n terms give the general solution to the homogeneous ODE (2).

For an example, see the web posting from 4/4/08.