M442 TAKEHOME FINAL EXAM

Spring Semester 2009

This exam is due at 4:00 PM Wednesday, May 6.

1. Take equispaced grid points $x_i = ih$, $i = 1, \ldots, N$, $h = 1/(N+1)$, on the interior of the unit interval $[0,1]$, and define the continuous, piecewise linear "hat" basis functions

$$\phi_i(x) = \begin{cases} 0, & \text{if } x \le x_{i-1} \\ (x - x_{i-1})/h, & \text{if } x_{i-1} \le x \le x_i \\ (x_{i+1} - x)/h, & \text{if } x_i \le x \le x_{i+1} \\ 0, & \text{if } x \ge x_{i+1} \end{cases}$$

Compute the entries of the corresponding N-by-N Galerkin representation matrix for the first derivative operator

$$\mathcal{L}u = u'(x), \quad 0 < x < 1,$$

with homogeneous Dirichlet boundary conditions.

2. You are to apply the Galerkin finite element method to solve the ODE BVP

$$-\kappa u''(x) - u'(x) = 0, \quad 0 < x < 1$$
$$u(0) = 0, u(1) = 1,$$

where κ is a positive parameter, and compare your approximate solution to the true solution. This is a 1-dimensional advection-diffusion problem. When κ becomes small, then advection (active flow) dominates diffusion and a "boundary layer" forms at the left boundary and numerical solution becomes problematic.

a. Compute the true solution to this BVP and generate plots for κ equal to 1, .1, and .01. Be sure to take enough grid points in your plots to resolve the boundary layer.

b. Derive explicit formulas for the components of the (tridiagonal) FEM stiffness matrix A and load vector $\mathbf{b}$ for this BVP. These components will depend on the diffusion coefficient κ and the grid spacing $h = 1/(N+1)$.

c. Compute and plot Galerkin solutions to the BVP for a "representative sample" of values of κ and h to illustrate what happens when h is small enough and not small enough to resolve the boundary layer. This will require you to first validate your solution scheme on "easy" problems (large κ) and then experiment with various values of κ and h to understand the behavior of numerical solutions.

Comment: The numerical solution to advection-diffusion equations in more than one space dimension has important applications (e.g., in fluid mechanics) and is an active area of research.

3. You are to apply the shooting method to solve the ODE BVP from problem 2. Feel free to modify the code provided in `Shooting.m`. Hand in (a) a listing of your code; and (b) a

representative sample of plots showing approximate solutions to demonstrate how your code performs as you vary the diffusion coefficient κ. Also include plots of the step size vs. x with each of your solution plots.

Some things to note: Shooting relies on an IVP solver, so the independent variable x plays the role of time t in our earlier IVP development. The ability of the IVP solver to adjust the step size helps to resolve the boundary layer.

4. Replace the backward difference IVP solver in your shooting method for Problem 3 with an explicit 4th order Runge-Kutta IVP solver (you may use MATLAB's ode45, if you don't want to write your own). Is the numerical approach as effective as before when κ becomes very small? Provide numerical/graphical evidence to justify your answer. Then explain why or why not, based on numerical ODE IVP theory that we have covered in class.