Math 450 Midterm Exam

October 29, 2001

1. In his 1940 analysis of turbulent fluid flow, the Russian applied mathematician A. Kolmogorov observed that the strength of the turbulence depends on the density ρ of the fluid, the length scale ℓ of the system, the energy E in the system, and the energy dissipation rate R of the system. This motivated his conjecture that there exists a function f for which $f(E, R, \rho, \ell) = 0$. From this he was able to derive the "power law" dependence of energy on length scale,

$$E = const \times \ell^p.$$

You are to use dimensional analysis to find the power p in this power law. Energy has dimensions $[E] = ML^2T^{-2}$; density ρ has dimensions of mass per unit volume; obviously $[\ell] = L$; and the energy dissipation rate R has dimensions of energy per unit time per unit volume.

(i) Show that only one independent dimensionless variable can be constructed from E, R, ρ, and ℓ, and find one.

(ii) Use part (i) to determine the power p in the power law.

2. Use singular perturbation techniques to find the leading order uniform approximation to the solution to the ODE boundary value problem

$$\epsilon \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y^3 = 0, \quad 0 < t < 1,$$
$$y(0) = 0,$$
$$y(1) = \frac{1}{2}.$$

3. Recall that when we applied the WKB methold to the second order ODE

$$\epsilon^2 \frac{d^2y}{dx^2} + k(x)^2 y = 0,$$

we substituted

$$y(x) = \exp\left(\frac{i}{\epsilon}\phi(x)\right), \quad i = \sqrt{-1},$$

and obtained the nonlinear ODE

$$i\epsilon\frac{d^2\phi}{dx^2} - \left(\frac{d\phi}{dx}\right)^2 + k(x)^2 = 0.$$

You are to

(i) Compute the first 2 terms in the regular perturbation series for the solution $\phi(x)$ of the nonlinear ODE.

(ii) Show that the corresponding approximate solution to the linear ODE can be written in the form

$$y_{approx}(x) = A(x)\exp\left(\frac{i}{\epsilon}\phi_{approx}(x)\right).$$

In particular, determine $\phi_{approx}(x)$ and $A(x)$.