M451 Takehome Final

1. Give the meanings of the following mathematical terms. Where you feel it is appropriate, give examples or counter-examples.

   (i) Compact Support

   (ii) Distribution

   (iii) Distributional Derivative

   (iv) Distributional Solution

   (v) Dirac Delta

   (vi) Fundamental Solution, or Free-Space Green's Function

   (vii) Green's Function

   (viii) Characteristic Curves

   (iv) Shock

2. Compute the Green's function for the following 2nd order ODE BVP operator.

$$\mathcal{A}u = \frac{d^2u}{dx^2} + 4u(x), \quad 0 < x < 1,$$
$$u(0) = u(1) = 0.$$

3. Outline a general scheme for computing Green's functions of linear nth order ODE BVP operators.

4. Convert Laplace's equation in 2-D, $u_{xx} + u_{yy} = 0$, to first order system form

$$A\mathbf{w}_x + B\mathbf{w}_y = \mathbf{0}.$$

   Compute the generalized eigenvalues of the matrix pair (A, B) and use them to classify the first order system.

5. Convert the scalar wave equation in one space dimension with wave speed that depends on position, $u_{xx} - \frac{1}{c(x)^2}u_{tt} = 0$, to first order system form

$$A(x)\mathbf{w}_x + I\mathbf{w}_t = \mathbf{0}.$$

Next, use the eigendecomposition of $A(x)$ to transform this system to diagonal form (Note: the transformed system is **not** homogeneous). Finally, apply the method of characteristics to the transformed system. Give the ODEs that determine the characteristics and the components of the transformed solution.

6. Work exercise 3.13 on p. 264 of the text.

7. Denote the 3-D Helmholtz operator by $Lu = \Delta u + k^2 u$, where $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$. Let $B_\epsilon = \{\mathbf{x} \in R^3 \mid |\mathbf{x}| < \epsilon\}$, and take $h(\mathbf{x}) = A\exp(ikr)/r$, where $r = |\mathbf{x}|$ and A is constant. Show that for any test function $\phi \in C_0^\infty(R^3)$,

$$\begin{aligned} \int_{B_\epsilon} h(\mathbf{x})\, L\phi \, d\mathbf{x} &\rightarrow 0 \\ \int_{\partial B_\epsilon} h(\mathbf{x})\, \nabla\phi \cdot \hat{n}\, dS &\rightarrow 0 \\ \int_{\partial B_\epsilon} \phi(\mathbf{x})\, \nabla h \cdot \hat{n}\, dS &\rightarrow -4\pi\, A\, \phi(0) \end{aligned}$$

as $\epsilon \rightarrow 0$.

8. Plot radial cross-sections of the real and imaginary parts of the free-space Green's function for the 3-D Helmholtz equation, $h(r) = -\frac{1}{4\pi}\frac{e^{ikr}}{r}$, vs. radius $r \geq 1$ for a few different values of the wave number k. What is the effect of varying k?

9. Fill in the missing details in the derivation of the first Rayleigh-Sommerfeld diffraction formula, given in the class notes of 4/28/04,

$$U(x_1', x_2', x_3') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_{\mathrm{IN}}(x_1, x_2) \left[\frac{-x_3'}{2\pi}\frac{e^{ikr'}}{(r')^2}(ik \;-\; 1/r')\right] dx_1\, dx_2,$$

where $r' = |\mathbf{x} - \mathbf{x}'| = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3')^2}$.

10. Derive the second Rayleigh-Sommerfeld diffraction formula,

$$U(x_1', x_2', x_3') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_{\mathrm{IN}}(x_1, x_2) \left[\frac{-1}{2\pi}\frac{e^{ikr'}}{r'}\right] dx_1\, dx_2$$

This solves Helmholtz half-space boundary value problem

$$\Delta U + k^2 U(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega,$$
$$\nabla U \cdot \hat{n} = V_{\mathrm{IN}}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega.$$

Here $\Omega = \{\mathbf{x} = (x_1, x_2, x_3) \in R^3 \mid x_3 > 0\}$ and $\partial\Omega$ is the x_1-x_2 plane.

11. Compute a free-space Green's function for the 2-D Helmholtz equation, $\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + k^2 u = 0$.