M551 Homework

Due Friday, May 2, 2003

1. Prove that if $f = u + iv$ is analytic, then v is harmonic.

2. Let $u(x, y) = x - xy$. Find a harmonic conjugate v of u and plot the level curves of both u and v in a neighborhood of the origin.

3. Find an analytic function f whose real part is $u(x, y) = 3x^2y - y^3$.

4. Consider the partial differential equation (PDE)

$$\begin{aligned} \Delta u &= 0, \quad (x, y) \in U \\ u &= g(x, y), \quad (x, y) \in \partial U, \end{aligned}$$

   where U is an open, bounded, simply connected set in $\mathbf{R}^2$. Also assume that the PDE has a solution and that g is continuous on ∂U. Use the maximum principle to show that u is continuous with respect to perturbations in g.

5. Use Poisson's formula combined with the appropriate conformal mapping the solve the following PDE:

$$\begin{aligned} \Delta u &= 0, \quad -\infty < x < \infty, \quad 0 < y < \infty \\ u(x, 0) &= \frac{1}{1 + x^2}, \qquad -\infty < x < \infty. \end{aligned}$$

   Plot the $u = 1/2$ contour, both in original domain and in the transformed domain. What is $\lim_{y \to \infty} u(x, y)$ for any fixed x?

6. Show that if F, G are fractional linear (Mobius) transformations, then so is the composition $F \cdot G$.

7. Find a fractional linear transformation that maps $\{(x, y) | x + y > 1\}$ onto the unit disk.