M551 Midterm Takehome Exam

Due Friday, March 7, 2003

Several of the following problems are classical results from Complex Analysis. Please try to work these problems "from scratch" before looking them up in a textbook. Try to construct proofs from basic principles. Please demonstrate that you understand basic ideas, rather than that you can memorize and apply "big theorems".

1. Let U be an open set in $\mathbf{C}$ and let f be continuous on U. Assume $\int_\gamma f(z)\, dz = 0$ for any closed path γ in U. Prove that f is holomorphic in U.

2. Let f be holomorphic in an open set containing the disk $\Delta = \{z \mid |z - z_0| < r\}$, and let C denote the boundary of this disk. Prove for any positive integer n that

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \sup_{z \in C} |f(z)|.$$

3. Prove that f is holomorphic in a neighborhood if and only if it is analytic in that neighborhood (recall f analytic if it has a power series representation with a positive radius of convergence).

4. Compute $\int_\gamma e^z z^{-n}\, dz$, where γ denotes the unit circle with counterclockwise orientation. Justify steps in your compuation. Your answer may depend on n. Hint: Take the power series for e^z about 0.

5. Assuming the open mapping principle, prove the following result: If f is analytic in an open set U, then the maximum of $|f(z)|$ is not attained in U. Then explain why this means that either (i) $|f(z)|$ is unbounded on U, or (ii) the maximum of $|f(z)|$ is attained on the boundary of U. There are no other possibilities.

6. Prove that a function which is holomorphic in $\mathbf{C}$ and satisfies an inequality $|f(z)| \leq |z|^n$ for some positive integer n and all sufficiently large $|z|$ must be a polynomial.