Weekly Write-Up on Sections 4.1 and 4.2

Problem 1. From the suggested homework:

[4.1 #42] The dosage $D$ of diphenhydramine for a dog of body mass $w$ kg is $D = 4.7w^{2/3}$ mg. Estimate the maximum allowable error in $w$ for a cocker spaniel of mass $w = 10$ kg if the percentage error in $D$ must be less than 3%.

\[
D = 4.7w^{2/3} \\
\frac{dD}{dw} = 4.7 \left( \frac{2}{3} \right) w^{-1/3} \\
\frac{dD}{D} = \frac{2}{3w} \frac{dw}{dw} \\
\]

\[
.03 = \frac{2}{3(10)} dw \\
.9 = 2dw \\
dw = \left[ .45 \right] kg
\]

Problem 2. From an old exam:

Suppose you are driving along a highway in a car with a broken speedometer at a nearly constant speed and you record the number of seconds it takes to travel between two consecutive mile markers. If it takes 60 seconds, then your average speed is 1 mi/60 s or 60 mi/hr. Now suppose that you travel one mile in $60 + x$ seconds; for example if it takes 62 seconds then $x = 2$ and if it takes 57 seconds, then $x = -3$. In this case, your speed over one mile is $1 \text{ mi} / (60 + x)$ seconds or 63.16 mi/hr. Because there are 3600 seconds in 1 hour, the function

\[
s(x) = \frac{3600}{60 + x}
\]

gives your average speed in mi/hr if you travel one mile in $x$ seconds more or less than 60 seconds. For example, if $x = 2$ then your average speed is $s(2) \approx 58.06 \text{ mi/hr}$. If you travel one mile in 57 seconds, then $x = -3$ and your average speed is $s(-3) \approx 63.16 \text{ mi/hr}$. Because you don’t want to use your phone or calculator while driving, you need an easy approximation to this function. Use linear approximation to derive such a formula.

\[
s(x) = 3600(60+x)^{-1} \\
s'(x) = -3600(60+x)^{-2} \\
s(6) = 60 \\
s'(6) = 1
\]

\[
L(x) = 60 + 1(x-6) \\
L(x) = 60 + x \text{ mph}
\]
Problem 3. For in-class discussion:

In 1919, physicist Alfred Betz argued that the maximum efficiency of a wind turbine is about 59%. If wind enters a turbine with speed \( v_1 \) and exits with speed \( v_2 \), then the power extracted is the difference in kinetic energy per unit time:

\[
P = \frac{1}{2} m v_1^2 - \frac{1}{2} m v_2^2
\]

where \( m \) is the mass of wind flowing through the rotor. Betz assumed that \( m = \rho A (v_1 + v_2)/2 \), where \( \rho \) is the density of air and \( A \) is the area swept out by the rotor. Wind flowing undisturbed through the same area \( A \) would have mass per unit time \( \rho A v_1 \) and power \( P_0 = \frac{1}{2} \rho A v_1^3 \). The fraction of power extracted by the turbine is \( F = P/P_0 \).

a. Show that \( F \) depends only on the ratio \( r = v_2/v_1 \) and is equal to

\[
F(r) = \frac{1}{2} (1 - r^2) (1 + r)
\]

where \( 0 \leq r \leq 1 \).

\[
F = \frac{P}{P_0} = \frac{\frac{1}{2} m v_1^2 - \frac{1}{2} m v_2^2}{\frac{1}{2} \rho A v_1^3} = \frac{\frac{1}{2} \rho A (v_1 + v_2)}{\frac{1}{2} \rho A v_1^3} \left( \frac{v_1^2 - v_2^2}{v_1^3} \right)
\]

\[
= \frac{1}{2} \left( \frac{v_1 + v_2}{v_1} \right) \left( \frac{v_1^2 - v_2^2}{v_1} \right)
\]

let \( r = v_2/v_1 \)

\[
F(r) = \frac{1}{2} (1+r) (1-r^2) \quad \text{on } [0,1]
\]

b. Show that the maximum value of \( F(r) \), called the Betz Limit, is 16/27.

\[
F'(r) = \frac{1}{2} \left[ (1-r^2) + (1+r)(-2r) \right]
\]

\[
= \frac{1}{2} \left[ 1 - r^2 - 2r - 2r^2 \right]
\]

\[
= \frac{1}{2} \left( 1 - 2r - 3r^2 \right)
\]

\[
0 = \frac{1}{2} \left( 1 - 3r \right) \left( r + 1 \right)
\]

C.P. \( r = \frac{1}{3} \)

\[
\begin{array}{l}
\text{Max Value } \frac{16}{27} \\
\end{array}
\]