Uniform Space-Filling Designs and Their Applications in Designed Experiments

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Abstract

Experimental design is a widely implemented branch of statistics today, especially in the fields of agriculture, industry, and natural sciences. However, experimenters often find themselves designing poor experiments that result in the waste of precious time and resources. While there are ways to remedy this issue to some extent by using fractional factorial designs that look at minimizing the necessary number of experimental runs to estimate all factors, other issues can arise. Uniform space-filling designs can be thought of as a type of fractional factorial design in that they significantly reduce the necessary number of experimental runs. The UD's have their greatest use when we see issues like an unknown relationship between the predictors and the mean response of interest or possible having a large numbers of factors and factor levels leading to insufficient information when a model is unknown. There are five methods (Good Lattice Point Set, Good Point Sets, and Halton' H-Set) that will be discussed in this paper that can be used to generate uniformly spaced points, and simple examples are provided for each. The basic methodology is similar to that of Monte Carlo approximation, but does differ slightly. While it is nearly impossible to always choose a best candidate design by looking at simple plots, there are measures of uniformity that help us determine mathematically, which design results in the most uniformly spaced points throughout the design region. These methods, while not always being intuitive and computationally accessible, provide experimenters with optimal experimental designs to estimate effects when normal underlying model assumptions cannot hold.

Introduction

Experimental design is one of the most widely used applications of statistical methodology currently in use. Having a response of interest and studying how that response can be affected by certain factors helps experimenters analyze many things. There are many types of experimental designs including but not limited to full factorial designs, fractional factorial designs, and response surface designs. Experimental designs have been extensively and widely applied in many different scientific areas to draw inferences. In general, the goal is for the design to be experimentally efficient. In other words, one would attempt to minimize the number of experimental runs needed to provide sufficient and accurate information. To optimize this experimental process, it is important to analyze the relationship between the response of interest ($\mathbf{Y}$) and the vector of experimental inputs ($\mathbf{X}$). The issue at hand deals with not being able to model this underlying relationship as that is one of the

most common issues statisticians face with experimental design. A commonly assumed relationship between a response variable and certain predictors is that of linearity. This means that we can write our response as a linear combination of the predictor variables where,

$$Y = \beta X + \epsilon$$

After assuming the model form, we then adopt other common assumptions as well. It is common to assume the independence of errors, homoscedasticity (constant variance) and the normal distribution of errors. Even when these assumption are met, issues may still arise if the experiment deals with a large number of factors giving many possible combinations of factor levels. These issues can be classified generally with

- an unknown relationship between the predictors and the mean response of interest,
- having a large numbers of factors and factor levels leading to insufficient information when a model in unknown, and
- imposing assumptions about the error distribution even for an unknown model

This is where we will examine some nice properties of certain space-filling designs, such as Uniform Designs, and how they can help remedy these issues. These designs have been used successfully many times to draw statistical inferences for experimental designs when the relationship between the response and factors is unknown or not justifiably linear. Uniform Designs differ from other experimental designs in that they do not use the common combinatorial principles but instead, look st the spread of the design points throughout the possible domain of the experiment to help with estimation for many possible models.

In classical experimental design methods, statisticians often assume the form of an underlying model, and then use this assumption to estimate the parameters. The design that results will be used to estimate these model parameters in a way that results in high precision (Fang and Lin, 2003). Are these assumptions guaranteed to be correct in every scenario? Intuitively it makes sense that these underlying relationships are not always specified correctly, and this is where space-filling designs have their most use. When using Uniform Designs, the designer is no longer forced to specify the form of the underlying model as that is not how space-filling designs operate. Instead, the goal is to collect data points that will accurately represent the entire design space. This allows an individual

to fit different models from the same data points without restricting researchers to a single option. Another very nice property of Uniform Designs becomes apparent when dealing with a high number of factors and factor levels. Commonly, an experimental design is used with the hope of maximizing the number of factors to be analyzed while still keeping the number of required experimental runs as few as possible. One can think of these space-filling designs as a type of fractional factorial design as it reduces the number of runs significantly while still providing useful information about the relationship between the input variables and the measured output response (Fang et al., 2000). As researchers aren't forced to specify the form of the model for the design, this is a desired situation to be in. When dealing with a design having n points, Uniform Designs will allow one to include the largest possible number of factor levels. These are just a few of the desired properties of space-filling designs.

These space-filling Uniform Designs are based off the theoretical ideas of number-theoretic methods (combinations of both numerical analysis and basic number theory) which, while not exactly identical to many Monte Carlo methods, are actually quite similar. The main difference between these two methods is the scatter of design points throughout the experiment's domain. Monte Carlo methods choose design points scattered at random,while in comparison, Uniform Designs spread design points that are more uniformly (as the name would imply) scattered. Visually, Talke (2012) does a great job showing this difference for a very simple case below.

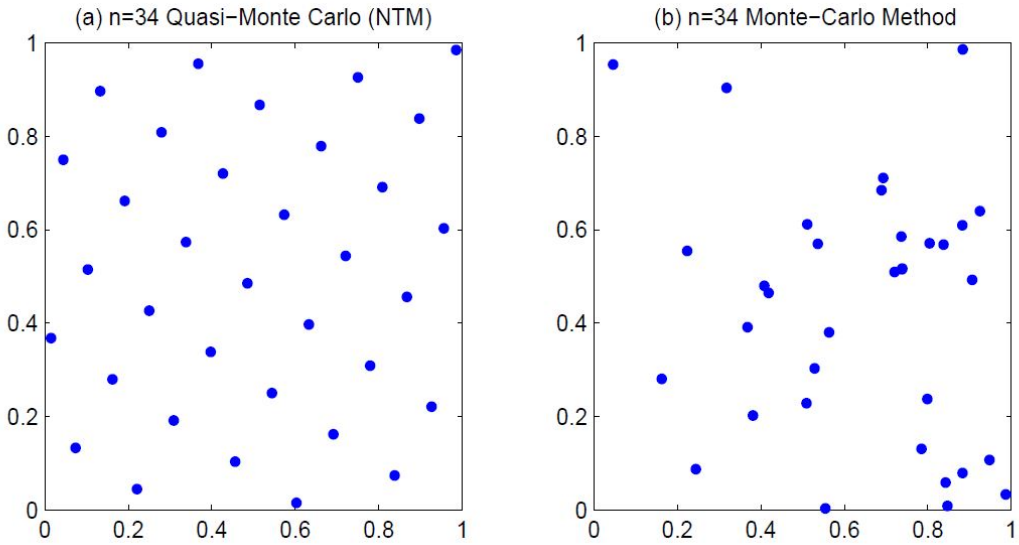

Figure 1: Design points using a uniform design with good lattice point generator of (34;1,13) versus a Monte Carlo approach for $N = 34$ points

What is the benefit of uniformly scattering points versus a random approach? If we define $E(h(x))$

to be the expected value of the mean response for a given input and the sample mean $\overline{h}_N = \frac{1}{N}\sum Y(x)$ for the case when we do not have random error present. This can be viewed in the rate-of-convergence of our mean response where we see the Monte Carlo approach yields a rate mathematically equivalent to

$$\left| E(h(x) - \frac{\sum Y(x)}{N} \right| \leq O[N^{-1/2}]$$

while the uniform take on spreading points would give us

$$\left| E(h(x) - \frac{\sum Y(x)}{N} \right| \leq O[N^{-1}(log(N))^s]$$

From the results above, we see that uniformly spreading design points across a space results in a better rate-of-convergence giving higher coverage for an experiment with n factors being generated in the n-dimensional unit cube $C^n = [0;1]^n$. If we are working with the experimental domain C^n with n experimental runs in our design, then the goal when assessed by a discrepancy measure is to find a design $P_N = (X_1, ..., X_N)$ where $X_i \in C^n$ such that the deviations between the true and estimated approximation model is as "small" as possible for all $X \in C^n$ (Talke, 2012). The easiest way to do this is to scatter the design points uniformly across the design space (Fang and Wang, 1994). When using these space-filling designs regardless of methods used, generating vectors need to be chosen carefully as choosing a poor generator can have an effect on how the points fill the space. Once gain, Talke (2012) shows this scenario in the following plot:

It is important to measure this discrepancy because sometimes it is not possible to visually choose the better design just based on the uniform scattering provided by our generating vector. In Figure 2, four different generators were used. Designs (a) and (b) clearly look better at first glance than designs (c) and (d) because the points appear more uniformly scattered without any large open spaces. One can look at the four designs above and easily see the poor results obtained from using the design generators in (c) and (d), but when looking at designs (a) and (b) which are far more uniform in nature, it is hard to choose one as being a "superior" option. This is where discrepancy measures have their usefulness. There are many possible measures to assess the lack of complete uniformity, and those will be discussed later in further detail.

Many uniform designs exist as possible options to be used in place of classical experimental designs (e.g., factorial designs, fractional factorial designs, etc...). Commonly, the good lattice point set is used but is not the only option. In the next section, we will see examples of this method as well

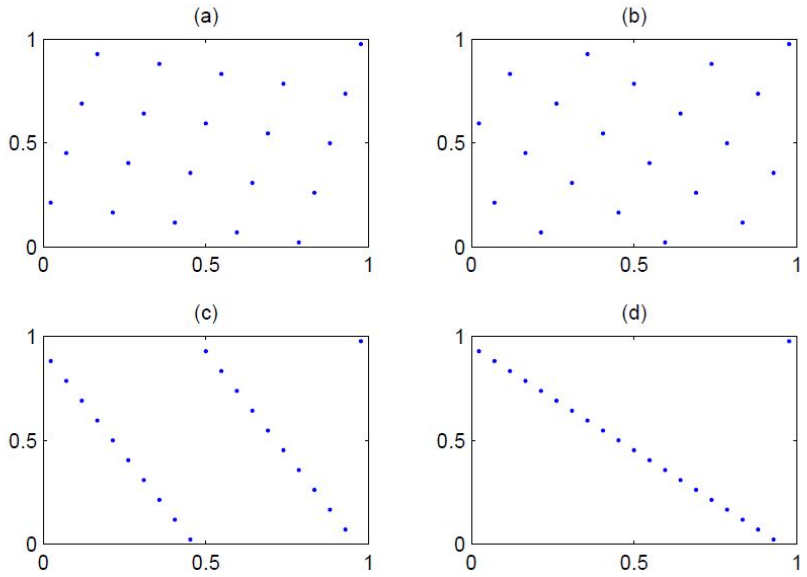

Figure 2: 21 design points for a uniform design with four different generators

as others such as good point methods (the square root sequence, power of p, and cyclotomic field set) and the H-set by Halton.

Methods

1 Measures of Uniformity

As the name "Uniform Space-Filling Design" implies, the goal with these methods is to produce a set of design points that are uniformly spaced in our experiment's design region. Five different methods of filling designs with points in a uniform manner will be reviewed. These methods, however, do not all produce results of equal uniformity. Just because points are spread uniformly throughout a space, there is still a need to assess the uniformity of spreading. This is where the measures of uniformity have their use. Many different ways to measure this strength exist including different versions of the L_2 and L_2 star discrepancy measures, as well as methods based on metric distances between points in the design space. Here in this paper, we will be analyzing the following few different measures of uniformity: the discrepancy measures briefly introduced earlier that are described by Fang and Wang (1994), methods using the metric distance between points discussed by Borkowski and Piepel (2009), and MSE measurements. These methods will be discussed for designs

in the unit cube C^n and how they can be generalized to design regions all rectangles of the form $[a, b)$.

Discrepancy measures allow us to to analyze how well a set of N points, $S = (x_1, ..., x_N)$ represent some multivariate distribution $F(x)$ in R^n. We can define the empirical distribution of these points in the following manner:

$$F_N(x) = \frac{1}{N} \sum_{i=1}^{N} I_i(x_i \leq x)$$

when we define our indicator functions to be

$$I_i(x_i \leq x) = \begin{cases} 1 & : x_i \leq x \\ 0 & : x_i > x \end{cases}$$

When we have a multivariate distribution $F(x)$ and a set of points in R^n, Fang and Wang (1994) describe the F-discrepancy of these points to be

$$D_F(N) = sup|F_N(x) - F(x)|$$

What exactly is this F-discrepancy, and how can it be used? It is seen that this measure is analyzing how well our set of N points represent the distribution $F(x)$. This discrepancy value acts as the Kolmogorov-Smirnov test statistic when testing goodness of fit for our distribution. When looking at a more specific use of this discrepancy (in the case of uniform space-filling methods), the discrepancy of these N points $S = (x_1, ..., x_N)$ is

$$D(N, S) = sup \left| \frac{M(\gamma, S)}{N} - v([0, \gamma]) \right|$$

The above relationship will hold for $\gamma \in C^n$, and $M(\gamma, S)$ is the number of points that will satisfy $x_k \leq \gamma$ for $k = 1, 2, ...$ Also, $v[0, \gamma])$ is the volume of the hyper-rectangle defined by $[0, \gamma]$. For simplicity, this discrepancy will be denoted as $D(N)$. Often, it is difficult to find a set with the smallest discrepancy when $n \geq 2$ since the distribution of points tends to be very complicated. Therefore, the goal is to find sets with asymptotically small discrepancies (Fang and Wang, 1994).

The star discrepancy D^* is the measure that expands the $D(N)$ discrepancy results to more

general rectangular regions. Here, for a set of S points satisfying the requirement that $x_k \in [a, b)$,

$$D^*(N, S) = sup \left| \frac{M([a, b), S)}{N} - v([a, b)) \right|$$

for the set of rectangles $[a, b)$ that satisfy $0 \leq a \leq b \leq 1$. The star discrepancy and the original discrepancy described above are very similar and satisfy the relationship $D(N) \leq D^*(N) \leq 2^n D(N)$.

Next, we take a look at the approach using metric distances proposed by Borkowski and Piepel (2009). Assume we have the following design matrix for N points in the region C^n:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_N \end{pmatrix} \tag{1}$$

We can define the distance between this design matrix and any point in our region C^n to be

$$d(x, X) = min \sqrt{(x - x_j)(x - x_j)^T}$$

When dealing with a design of N points, we calculate a distance for each of the nearest matrix points to the N points in our design region. Out of all these calculations, the smallest is kept. There are different criteria that use $d(x, X)$ to measure space-filling properties of a design. More specifically, Borkowski and Piepel (2009) define the following three measures:

- The root mean square distance is defined as $RMSD(X) = \sqrt{E[d(x, X)^2]}$
- The average distance is defined as $AD(X) = E[d(x, X)]$
- The maximum distance is defined as $MD(X) = max(d(x, X))$

These methods follow just as their names imply. For instance, the root mean square distance looks at taking the square root of the square mean distance between any point in the design region (x) and the design matrix ($\mathbf{X}$). The average distance looks at taking the average distance between

any point in the design region (x) and the design matrix ($\mathbf{X}$). Lastly, the maximum distance method looks at the maximum distance between any point in the design region (x) and the design matrix ($\mathbf{X}$).

The last measure of uniformity we will discuss looks at the mean square error (MSE) of our set of design points. Suppose we have our set of design points, S, such that

$$d(x, x_{k(x)}) = minimum\ \ d(x, x_j) \qquad 1 \leq j \leq N$$

We then can analyze the integral of $d(x, x_{k(x)})^2$ over the n-dimension cube design region (C^n) to measure how uniform the design points are being spread. The resulting measure stems from

$$MSE(S) = \int_{C^n} min\ d(x, x_j)^2 dx$$

In the past the estimation of discrepancy was obtained usually by neglecting some terms of lower orders, this may lead to a large error when N is small (Fang and Wang, 1994). Often, the glp method is the best as determined by the measures of uniformity whch is consistent with the idea that this method is good for small n. The reason for this is because the glp method has more than one candidate of design generators for each design size. At least one of these generators usually leads to good coverage rates of the region of interest. For large n, the glp method produces excellent designs, but it is computationally expensive and often not feasible to implement (Talke, 2012). For a table analyzing a few measures of uniformity for simple designs of 10 runs to estimate 2 factors, see the Appendix below.

2 Good Lattice Point Sets

A nice property of uniform space-filling designs is that they can be applied to different experimental regions such as cubes, spheres, etc... As stated earlier, the idea of uniform space-filling designs does not boil down to a single concept. Rather, multiple different types of space-filling designs exist in this family. One of the most efficient and commonly-used space-filling methods is called the good lattice point (glp) method. Computation of this method is fairly straight forward. Begin with a generating vector say $h = (h_1, ...h_n)$ where n represents the number of experimental factors in a design with N runs. This method uses the concept of modular arithmetic such that the design rows can be

calculated by $jh = (jh_1, ...jh_n) \bmod N$, for $j = 1, 2, ..., N$. A good lattice point set will result in a final row of $(N, ..., N)$, and when h_j and N are coprime meaning the $\gcd(h_j, N) = 1$, we see the resulting jth column of the design is merely a permutation of $1, ..., N$ (Zhou and Xu, 2015). To determine the number of generators to use, we introduce Euler's function. This function's values equal the maximum possible number of h_i components for a given number of N runs. Mathematically, Euler's function is

$$p(N) = N\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)...\left(1 - \frac{1}{p_r}\right)$$

where $p_1 \times ... \times p_r$ help form the prime decomposition of N (Fang et al., 2006). Normally, in a regular design, we would see elements in a design space take on values in $0, 1, ..., N-1$, but according to Zhou and Xu, if we replace 0 elements in our set with N, we see that glp sets contain elements taking on values in $1, ..., N$. While changing $0's$ to $N's$ will indeed change the structure and properties of our design, we will see later that it does not have an effect on some of our discrepancy measures of uniformity.

Once we have defined our generating vector h, we can create an $n \times k$ matrix U using modular arithmetic operations. The elements in this matrix can be defined as $u_{ij} = ih_j \bmod(N)$. From this generated matrix U, we form our design matrix X. Using generators, we can form the jth row in our design matrix $\mathbf{X} = (x_{i1}, ..., x_{ik})$ where

$$x_{ij} = \frac{2u_{ij} - 1}{2N} \quad i = 1, ..., N \quad j = 1, ..., k$$

There may be many possible designs that result from this process even for the same exact generators. For instance, if we take a subset of columns from our design matrix X, we create a $k-$factor uniform design with N points. Intuitively, we see then that different subsets of columns of X will result in different uniform designs. The good lattice point method will look all the possible designs resulting from these different subsets. What design then do we choose? Fang and Wang (1994) state that taking the the first column of our generating matrix U can be selected because this will make the number of possible designs smaller and easier to work with. Once we have selected this first column, we need only select the remaining columns of our subset from the $k-1$ remaining columns from our design matrix.

Recall from Figure 2 that some uniform designs will be better than others (and largely so at that). To determine which of the uniform designs to use, we will address this analyzing the different

measures of uniformity in a later section, but for the time being, we can define discrepancy of a glp set to be

$$D(N) < c(n)p^{-1}(logp)^n$$

where $c(n)$ is a constant dependent on the number of our factors, and p is our chosen generating prime number(s). Now that we have discussed the theoretical approaches to constructing a space-filling design using the glp method, let's consider an example. Say we have an experiment with $N = 10$ runs and $n = 3$ factors in the design region in $C^3 = [0, 1]^3$. The prime decomposition of N, $10 = 2^1 \times 5^1$, will result in primes of $p_1 = 2$ and $p_2 = 5$. Euler's result from these primes give us $p(10) = 4$. Thus, a good generating vector for this analysis could possibly be $h = (1, 3, 7, 9)$. This would give us

$$U_4 = \begin{pmatrix} 1 & 3 & 7 & 9 \\ 2 & 6 & 4 & 8 \\ 3 & 9 & 1 & 7 \\ 4 & 2 & 8 & 6 \\ 5 & 5 & 5 & 5 \\ 6 & 8 & 2 & 4 \\ 7 & 1 & 9 & 3 \\ 8 & 4 & 6 & 2 \\ 9 & 7 & 3 & 1 \\ 10 & 10 & 10 & 10 \end{pmatrix} \tag{2}$$

Keeping the fractional component for our X design matrix using $x_{ij} = \frac{2u_{ij}-1}{2N}$ would result in the following design matrix for this example:

$$X_4 = \begin{pmatrix} 0.05 & 0.25 & 0.65 & 0.85 \\ 0.15 & 0.55 & 0.35 & 0.75 \\ 0.25 & 0.85 & 0.05 & 0.65 \\ 0.35 & 0.15 & 0.75 & 0.55 \\ 0.45 & 0.45 & 0.45 & 0.45 \\ 0.55 & 0.75 & 0.15 & 0.35 \\ 0.65 & 0.05 & 0.85 & 0.25 \\ 0.75 & 0.35 & 0.55 & 0.15 \\ 0.85 & 0.65 & 0.25 & 0.05 \\ 0.95 & 0.95 & 0.95 & 0.95 \end{pmatrix} \tag{3}$$

Now, as we have 3 factors of interest, we can see that there are three possible designs in this glp example. Three factor uniform designs are any of the 10×3 matrices created by taking a subset of 3 columns from our X design matrix above. Remember that the first column is always chosen as specified by Fang and Wang. Thus, we merely need to choose two remaining columns from the three left giving us a total of three possible designs. To choose the best of these three designs, plots of all candidate designs can be used along with measures of uniformity to choose the most optimal design. A plot of one candidate design using glp methods for estimating two factors with 10 runs is shown in the Appendix below.

3 Good Point Sets

While computationally convenient and common to implement, glp methods aren't always the most desirable choice of a space-filling uniform design. This leads us to the following methodology to discuss called the good point methods. Three good point (gp) methods will be considered: cycolotomic fields, square root sequence, and root of a prime. The main difference between these good point methods and the glp method previously described shows itself in the mathematical calculations in obtaining the generators and design points. Here, define $\gamma = (\gamma_1, ...\gamma_n)$ where the resulting set equals

$(\gamma_1 j, ..., \gamma_n j)$. The discrepancy for the good point set is such that

$$D(N) \leq c(\gamma, \epsilon) N^{-1+\epsilon}$$

where γ refers to a single good point (Fang and Wang, 1994). First, the cyclotomic field method will be analyzed.

For cyclotomic fields, we will define the following set of points:

$$\gamma = k\left(\left[2cos\frac{2\pi}{p}\right], \left[2cos\frac{4\pi}{p}\right], ..., \left[2cos\frac{2n\pi}{p}\right]\right)$$

where p is some prime number $\geq 2n+3$ and $k = 1, 2, ...10$. This method is straight forward. To analyze this good point method with an example, consider the earlier design in which we have $N = 10$ runs and $n = 3$ factors. Here, we need a prime number that is $\geq 2n+3$. For this example, this means the first candidate prime we could consider is $p = 11$. If we use $p = 11$, we will obtain the following set of design points where we keep only the fractional component of our cosine evaluations:

$$\begin{pmatrix} .6825 & .8308 & .7154 \\ .3650 & .6617 & .4308 \\ .0475 & .4925 & .1462 \\ .7300 & .3233 & .8616 \\ .4125 & .1542 & .5770 \\ .0950 & .9850 & .2924 \\ .7775 & .8158 & .0078 \\ .4601 & .6466 & .7232 \\ .1426 & .4775 & .4386 \\ .8251 & .3083 & .1540 \end{pmatrix} \tag{4}$$

A plot of one candidate design using CF methods for estimating two factors with 10 runs is shown in the Appendix below.

Next, we will look at the square root sequence method of producing uniform space-filling design points. A very easy method computationally, this sequence is formed by taking

$$\gamma = k(\sqrt{p_1}, \sqrt{p_2}, ..., \sqrt{p_n})$$

where all p_j are different primes (usually the first n primes). Keeping with our originally chosen design with $N = 10$ runs and $n = 3$ factors, we can build the following design matrix using the fractional part of the products with the first 3 primes (2, 3, and 5) to find our design points with $k = 1, 2, ..., 10$.

$$\begin{pmatrix} .4142 & .7321 & .2361 \\ .8284 & .4641 & .4721 \\ .2426 & .1962 & .7082 \\ .6569 & .9282 & .9443 \\ .0711 & .6603 & .1803 \\ .5853 & .3923 & .4164 \\ .8995 & .1244 & .6525 \\ .3137 & .8564 & .8885 \\ .7279 & .5885 & .1246 \\ .1421 & .3205 & .3607 \end{pmatrix} \tag{5}$$

A plot of one candidate design using SRS methods for estimating two factors with 10 runs is shown in the Appendix below.

The last good point method to introduce looks at taking the $(n + 1)$th root of a prime number to determine the space-filling design points. In other words, if we are working with some prime p, and we define $q = p^{1/n+1}$, then we obtain

$$\gamma = (q, q^2, ..., q^n),$$

and this will lead us to the following results when using $n = 3$ factors and $N = 10$ runs in our experiment with chosen prime of $p = 2$:

$$\begin{pmatrix} .1892 & .4142 & .6818 \\ .3784 & .8284 & .3636 \\ .5676 & .2426 & .0454 \\ .7568 & .6569 & .7272 \\ .9460 & .0711 & .4090 \\ .1352 & .5853 & .0908 \\ .3244 & .8995 & .7725 \\ .5137 & .3137 & .4543 \\ .7029 & .7279 & .1361 \\ .8921 & .1421 & .8179 \end{pmatrix} \tag{6}$$

A plot of one candidate design using power prime methods for estimating two factors with 10 runs is shown in the Appendix below.

4 Halton's H-set

We have currently looked at the good lattice point method as well as three "good point" methods, but it doesn't stop there. Around 1960, Halton proposed another space-filling method that focuses on the p-adic representation of natural numbers (Fang and Wang 1994). The mathematics behind this method are a bit more complex than the methods we have already discussed, therefore, further detail should be analyzed. If we have a prime $p \geq 2$, we can write out representations of p digits for any other natural number k. Thus, it can be displayed that

$$k = b_0 + b_1 p + b_2 p^2 + .. + b_r p^r$$

where $0 \leq b_i \leq p$. Say we are dealing with a small space such that we have a rational number $c \in (0, 1)$. The unique representation for c is as follows:

$$c = c_0 p^{-1} + c_1 p^{-2} + ..., \quad 0 \leq c_i \leq p - 1$$

There is a $1-1$ correspondence between the rational numbers in the space (0, 1) and the positive integers. This relationship allows us to establish that for any integer k, we can write

$$y_p(k) = b_0 p^{-1} + b_1 p^{-2} + ... + b_r p^{-(r+1)}$$

These points help build the H-set defined by Halton where $x_k = (y_{p_1}(k), ..., y_{p_n}(k))$ for $k = 1, 2, 3, ..., 10$ Halton showed that this set formed by the first m points has the following discrepancy:

$$D(m) \leq O(m^{-1}(logm)^n)$$

which shows that the H-set is in fact uniformly scattered in the space C^n (Fang and Wang, 1994). We can look at a simple example for when $p = 2$ and $k = 1, 2, ..., 8$. The results are as follows:

	1	2	4	8	
	$\frac{1}{2^1}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	
k	b_0	b_1	b_2	b_3	$y_p(k)$
1	1				0.5
2	0	1			0.25
3	1	1			0.75
4	0	0	1		0.125
5	1	0	1		0.625
6	0	1	1		0.375
7	1	1	1		0.875
8	0	0	0	1	0.0625
9	1	0	0	1	0.5625
10	0	1	0	1	0.3125

Now, we can take a look at the set of 10 points that would form the H-set for when we have $n = 3$ factors and $N = 10$ runs. Here, we will use the first three prime numbers to generate our points. Thus, $p_1 = 2$, $p_2 = 3$, and $p_3 = 5$.

$$\begin{pmatrix} .5000 & .3333 & .2000 \\ .2500 & .6667 & .4000 \\ .7500 & .1111 & .6000 \\ .1250 & .4444 & .8000 \\ .6250 & .7778 & .0400 \\ .3750 & .2222 & .2400 \\ .8750 & .5556 & .4400 \\ .0625 & .8889 & .6400 \\ .5625 & .0370 & .8400 \\ .3125 & .3704 & .0800 \end{pmatrix} \tag{7}$$

A plot of one candidate design using H-set methods for estimating two factors with 10 runs is shown in the Appendix below.

A few Details and Examples

While being computationally intensive/expensive, the glp methodology discussed in this paper often yields the most optimal uniform design. Talke (2012) generates a good example to see how uniform data can be used. For instance, if we are interested in estimating two factors with $N = 9$ runs, the best resulting design using the glp methodology yields a generating vector of $(9; 1, 4)$. This specific design can be seen in the table below. It is also important to consider randomizing the order of the 9 experimental runs when implementing.

Uniform Design				
No.	$U_9(9^2)$		x_1	x_2
1	1	4	-1.00	-0.25
2	2	8	-0.75	0.75
3	3	3	-0.50	-0.5
4	4	4	-0.25	0.5
5	5	2	0.00	-0.75
6	6	6	0.25	0.25
7	7	1	0.50	-1.00
8	8	5	0.75	0.00
9	9	9	1.00	1.00

Table 7.1: UD data layout for $U_9(9^2)$ design

Fields like agriculture, industry and natural sciences are areas of science that tend to rely on the important of statistical experimental design. However, bad experimental designs are frequent, and they usually result in wasting good resources due to their improper implementation. While uniform designs do help alleviate much of this issue as they allow for the maximum number of possible factors to be analyzed with the fewest possible number of runs, they aren't exactly straightforward. The reason is because uniform space-filling designs are not orthogonal. Thus, the linear coefficients in the model are not uniquely estimable. Talke (2012) notes that one way to perform the analysis while still bypassing this issue often requires the use of R packages like the R Development Core Team and MASS packages to perform the stepwise regression analysis to estimate the model effects from a model chosen using the AIC criteria.

Appendix

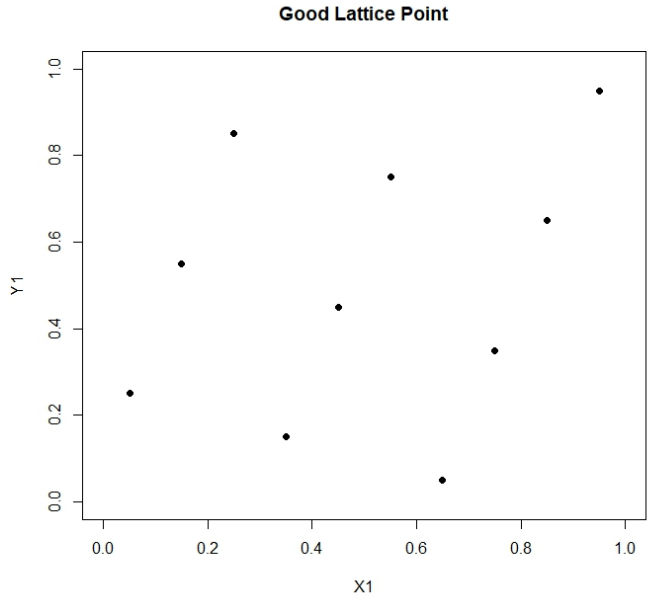

Figure 3: Plot of a Good Lattice Point Design for $N = 10$ runs and $n = 2$ factors with generating primes of (1,3)

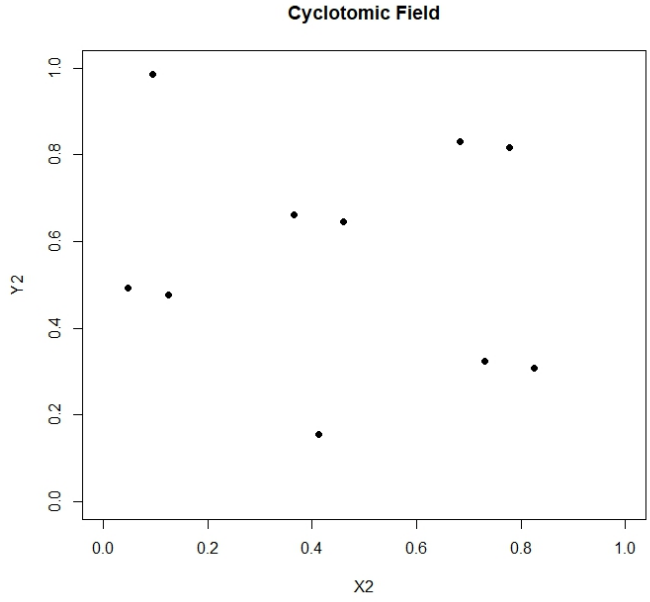

Figure 4: Plot of a Cyclotomic Field for Design $N = 10$ runs and $n = 2$ factors with generating primes of (2,3)

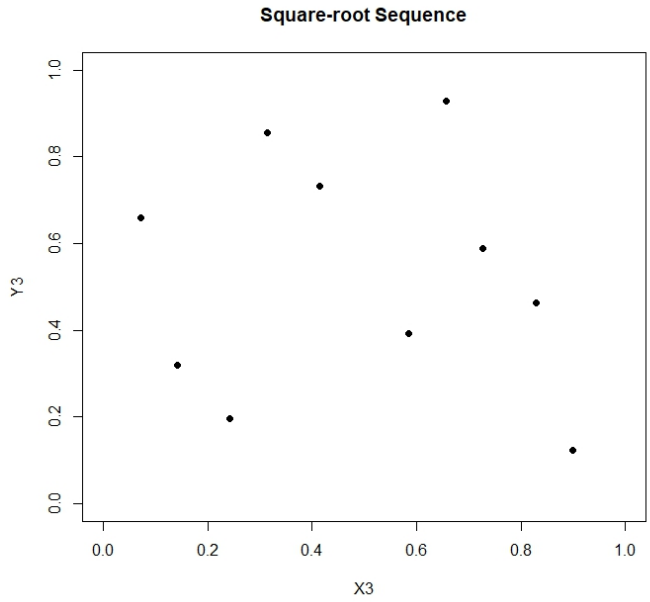

Figure 5: Plot of a Square-Root Sequence Design for $N = 10$ runs and $n = 2$ factors with generating primes of (1,3)

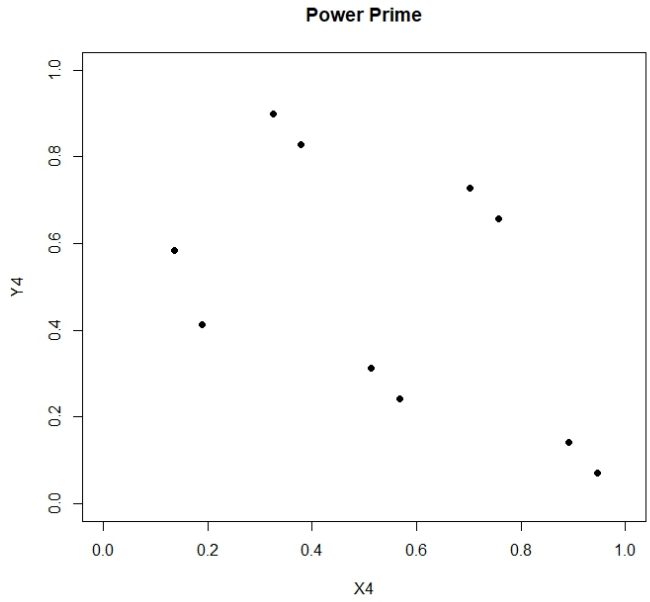

Figure 6: Plot of a Power Prime Design for $N = 10$ runs and $n = 2$ factors with generating primes of (1,3)

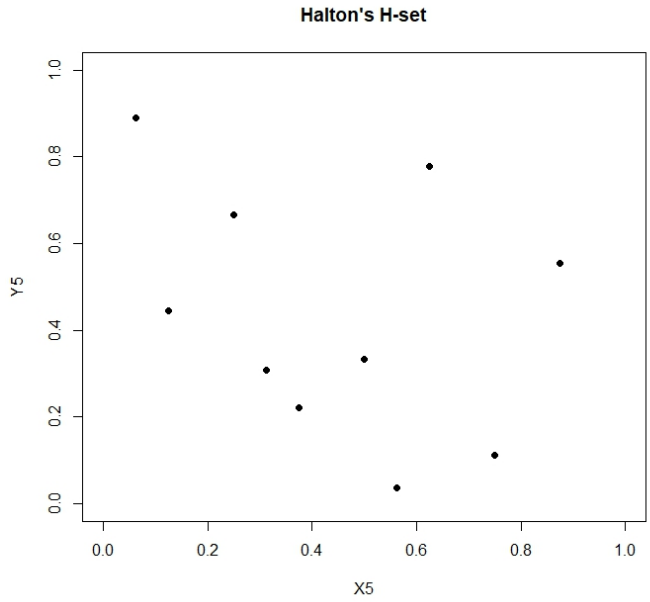

Figure 7: Plot of an H-set Design for $N = 10$ runs and $n = 2$ factors with generating primes of (2,3)

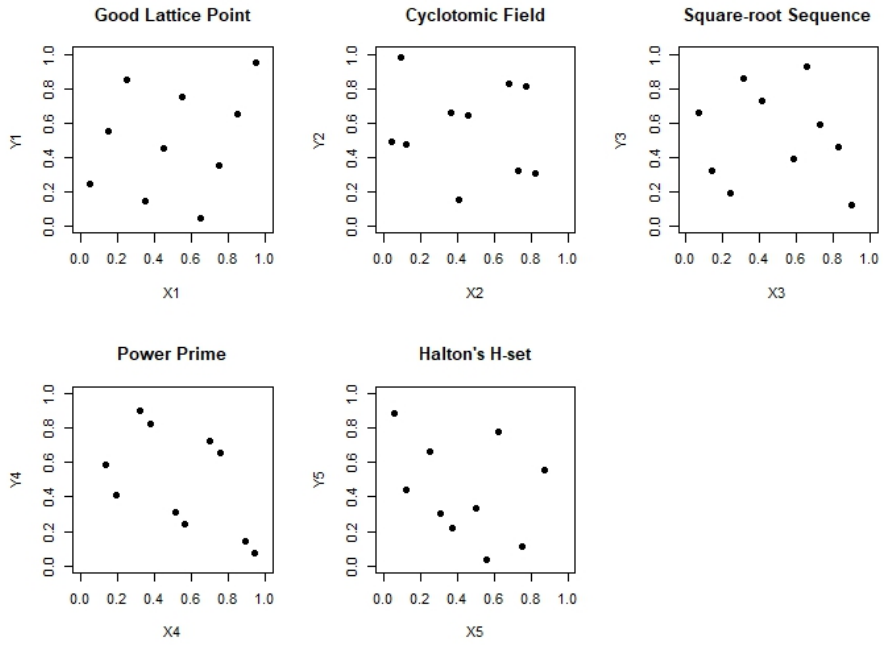

Figure 8: A side-by-side comparison of the five different methods for a design $N = 10$ runs and $n = 2$ factors

Design	**Average Distance**	**Maximum Distance**
Good Lattice Point	0.1333	0.3498
Cyclotomic Field	0.1564	0.4296
Square-root Sequence	0.1411	0.3597
Power Prime	0.1620	0.4403
Halton's H-set	0.1471	0.4299

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