Multivariable Calculus M273
Fall 2018
December 12, 2018
Final Exam

Name: Solutions

Section: ______

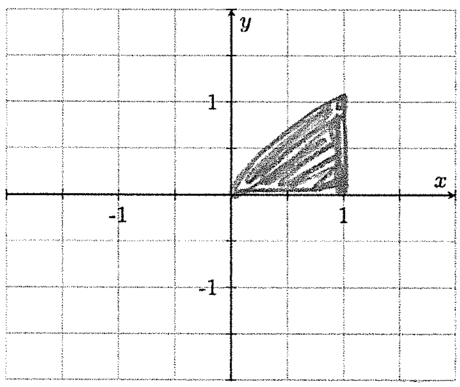
Instructor Name: ______________________

Instructions: Closed book. No calculator allowed. Double-sided exam. NO CELL PHONES. **Show all work and use correct notation to receive full credit!** Write legibly.

$dA = r\,dr\,d\theta$	$dV = r\,dz\,dr\,d\theta$	$dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

1. Consider the integral $\iint_{\mathcal{D}} \frac{1}{1-y^4}\,dA = \int_0^1 \int_0^{\sqrt[4]{x}} \frac{1}{1-y^4}\,dy\,dx$.

(a) (1 Credit) Sketch $\mathcal{D}$, the region of integration.



(b) (2 Credits) Calculate $\int_0^1 \int_0^{\sqrt[4]{x}} \frac{1}{1-y^4}\,dy\,dx$ by switching the order of integration.

$$\int_0^1 \int_0^{x^{1/4}} \frac{1}{1-y^4}\,dy\,dx = \int_0^1 \int_{y^4}^1 \frac{1}{1-y^4}\,dx\,dy$$

$$= \int_0^1 \frac{1-y^4}{1-y^4}\,dy = \int_0^1 dy = 1$$

$\boxed{1}$

2. (1 Credit) True/False.

(a) ● TRUE ○ FALSE The two lines $\mathcal{L}_1 : \mathbf{r}_1(t) = \langle 10-4t, 3+t, 4+t\rangle$ and $\mathcal{L}_2 : \mathbf{r}_2(t) = \langle 6, mt, 9-t\rangle$ intersect if $m = 1$.

$10-4t = 6 \quad t = 1 \qquad 3+1 = S \quad S = 4$

$3+t = 1S$

$4+t = 9-S \qquad 4+1 = 9-4$

(b) ● TRUE ○ FALSE The vectors $\mathbf{v}_1 = \langle 1, \lambda, -2\rangle$ and $\mathbf{v}_2 = \langle 4\lambda, 3, 7\rangle$ are orthogonal when $\lambda = 2$.

(c) ● TRUE ○ FALSE Consider the sphere $x^2+y^2+z^2 = 9$. The plane $x = 3$ is the tangent plane to the sphere at the point $(3, 0, 0)$.

(d) ● TRUE ○ FALSE $\int_1^2 \int_3^4 x^2 e^y \, dy \, dx = \left(\int_1^2 x^2 \, dx\right)\left(\int_3^4 e^y \, dy\right)$

3. (1 Credit) The following equations are all results from theorems we learned this semester.

I: $$\oint_{\partial D} \mathbf{F}\cdot d\mathbf{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

II: $$\int_a^b \int_c^d f \, dy dx = \int_c^d \int_a^b f \, dx dy$$

III: $$\oint_{\partial S} \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}(\mathbf{F})\cdot d\mathbf{S}$$

IV: $$\iint_{\partial W} \mathbf{F}\cdot d\mathbf{S} = \iiint_W \text{div}(\mathbf{F})\, dV$$

V: $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

VI: $$\int_C \nabla f \cdot d\mathbf{r} = f(Q) - f(P)$$

Match each of the following theorems with its corresponding equation above.

(a) Divergence Theorem
○ I ○ II ○ III ● IV ○ V ○ VI

(b) Green's Theorem
● I ○ II ○ III ○ IV ○ V ○ VI

(c) Fundamental Theorem for Conservative Vector Fields
○ I ○ II ○ III ○ IV ○ V ● VI

(d) Stokes' Theorem
○ I ○ II ● III ○ IV ○ V ○ VI

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4. A radioactive substance with strength

$$P(x,y,z) = e^{-x^2-y^2-(z+100)^2}$$

is suddenly discharged. A person standing at the point $(1,1,-100)$ must move away, in the direction of maximum decrease of radiation. Note that the gradient of P is given by

$$\nabla P(x,y,z) = \langle -2xe^{-x^2-y^2-(z+100)^2}, -2ye^{-x^2-y^2-(z+100)^2}, -2(z+100)e^{-x^2-y^2-(z+100)^2} \rangle.$$

(a) (1 Credit) A person standing at the point $(1,1,-100)$ must move away, in the direction of maximum decrease of radiation. What direction should he/she choose to move?

● $\langle 1,1,0\rangle$ ○ $\langle 2,3,0\rangle$ ○ $\langle -1,-1,0\rangle$ ○ $\langle -2,-3,0\rangle$ ○ $\langle 1,-1\rangle$

$\nabla P(1,1,-100) = -2e^{-2}\langle 1,1,0\rangle$

decrease : $-\nabla P(1,1,-100) = 2e^{-2}\langle 1,1,0\rangle$

(b) (1 Credit) A person standing at the point $(1,1,-100)$ must move away, in the direction of maximum decrease of radiation. What is the maximum rate of decrease in radiation?

○ $-\sqrt{2}$ ○ $-e^{-3}\sqrt{13}$ ● $-2\sqrt{2}e^{-2}$ ○ $\sqrt{2}$ ○ 0

$\|-\nabla P(1,1,-100)\| = 2e^{-2}\sqrt{2}$

decrease : $-\|\nabla P(1,1,-100)\| = -2\sqrt{2}e^{-2}$

(c) (1 Credit) The person standing at the point $(1,1,-100)$ decided to move in the direction $\langle 0,1,0\rangle$. What is the rate of change in radiation in this direction?

○ $2e^3$ ○ $-2e^{-3}$ ○ 0 ● $-2e^{-2}$

$D_{\langle 0,1,0\rangle} P(1,1,-100) = \nabla P(1,1,-100)\cdot\langle 0,1,0\rangle$

$= -2e^{-2}$

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5. (2 Credits) Find the equation of the plane containing the points $P = (1, 2, 1)$, $Q = (3, 2, -1)$, $R = (1, 1, 1)$.

$\vec{PQ} = \langle 2, 0, -2 \rangle$

$\vec{PR} = \langle 0, -1, 0 \rangle$

$\vec{n} = \vec{PQ} \times \vec{PR} = \langle -2, 0, -2 \rangle$

use $\vec{n} = \langle 1, 0, 1 \rangle$

$$\boxed{\begin{array}{c} x + z = 2 \\ (x-1) + 0 + (z-1) = 0 \end{array}}$$

6. (2 Credits) Find the tangent line to the curve parametrized by

$$\mathbf{r}(t) = \langle t + \cos(t), te^t, \ln(1+t) \rangle \quad , \quad t > -1$$

at the point where $t = 0$.

$\vec{r}\,'(t) = \langle 1 - \sin t, (t+1)e^t, \frac{1}{1+t} \rangle$

$\vec{r}\,'(0) = \langle 1, 1, 1 \rangle$

$\vec{\ell}(t) = \langle 1, 0, 0 \rangle + t \langle 1, 1, 1 \rangle$

$$\boxed{\vec{\ell}(t) = \langle 1, 0, 0 \rangle + t \langle 1, 1, 1 \rangle}$$

7. (2 Credits) Using the Divergence Theorem, find the flux of the vector field $\mathbf{F}(x,y,z) = \langle 3x, 4y, -2z\rangle$ outwards across the surface of the box $\mathcal{W} = [0,1] \times [0,2] \times [-1,1]$.

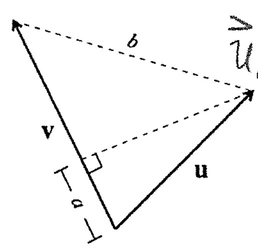
$$\operatorname{div} \vec{F} = 3+4-2=5$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_W \operatorname{div} \vec{F}\, dV = \int_{-1}^{1}\int_0^2\int_0^1 5\, dx\, dy\, dz$$

$$= 5\int_{-1}^{1} dz \int_0^2 dy \int_0^1 dx = 5\cdot 2\cdot 2 = 20$$

$$\iint_{\partial\mathcal{W}} \mathbf{F}\cdot d\mathbf{S} = 20$$

8. (2 Credits) Given $\mathbf{u} = 5\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$, find the **lengths** a and b pictured below.



$$\vec{u}_{\parallel\vec{v}} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v} = \frac{18}{36}\langle 4,4,2\rangle = \langle 2,2,1\rangle$$

$$a = \|\vec{u}_{\parallel\vec{v}}\| = \|\langle 2,2,1\rangle\| = 3$$

$$b = \|\vec{u}-\vec{v}\| = \|\langle 1,-3,-5\rangle\|$$

$a = 3$

$b = \sqrt{35}$

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9. (2 Credits) Let $\mathcal{S}$ be the cone given by $\{(x, y, z) \in \mathbb{R}^3 \mid z = \sqrt{x^2+y^2} \text{ and } 0 \le z \le 1\}$ and let the vector field $\mathbf{F}$ be given by $\mathbf{F}(x, y, z) = \langle -yz, xz, z^2 \rangle$.
Given that $\partial\mathcal{S}$ is parametrized by $\mathbf{r}(t) = \langle \cos(t), \sin(t), 1\rangle$, $0 \le t \le 2\pi$, use Stokes' Theorem to find the flux of $\operatorname{curl}(\mathbf{F})$ upwards across $\mathcal{S}$, that is, find $\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi$$

$$\vec{F}(\vec{r}(t)) = \langle -\sin t, \cos t, 1\rangle$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0\rangle$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \sin^2 t + \cos^2 t = 1$$

$$\boxed{\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = 2\pi}$$

10. (2 Credits) Consider the surface $\mathcal{S}$ parametrized by $\mathbf{r}(u, v) = \langle u+v, u-v, 2u+3v\rangle$ where $0 \le u \le 1$ and $0 \le v \le 1$. Find the surface area of $\mathcal{S}$.

$$\iint_S 1\,dS = \iint_D \|\vec{r}_u \times \vec{r}_v\|\,du\,dv = \int_0^1\int_0^1 \sqrt{30}\,du\,dv = \sqrt{30}$$

$$\vec{r}_u = \langle 1, 1, 2\rangle$$

$$\vec{r}_v = \langle 1, -1, 3\rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \|\langle 5, -1, -2\rangle\| = \sqrt{30}.$$

$$\boxed{\sqrt{30}}$$

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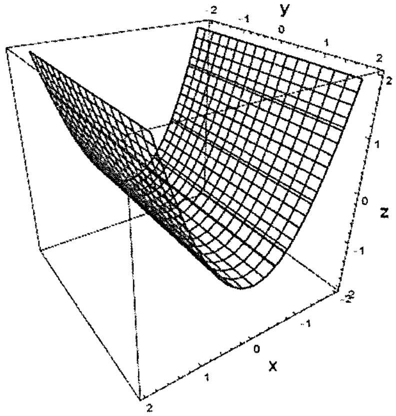
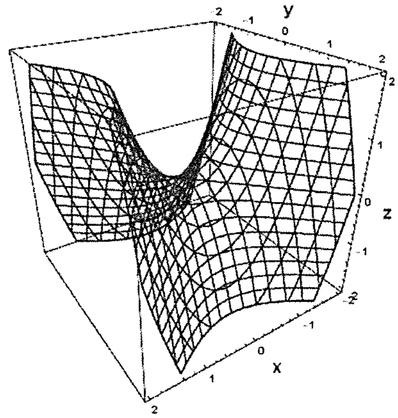
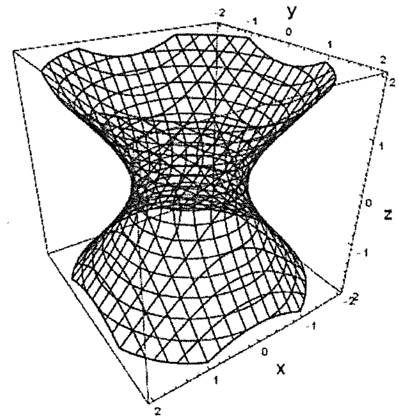
11. (1 Credit) Consider the following quadratic surfaces in $\mathbb{R}^3$.



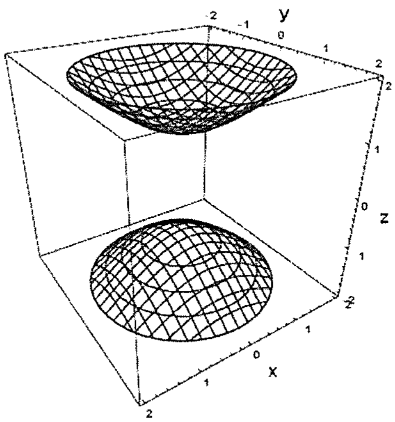
I



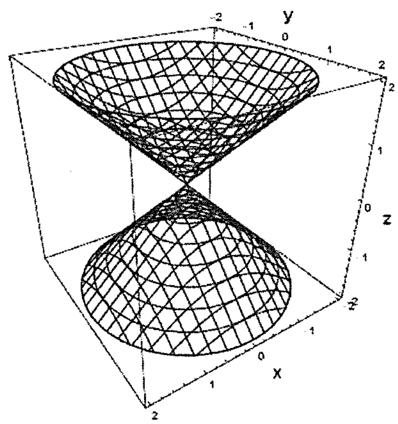
II



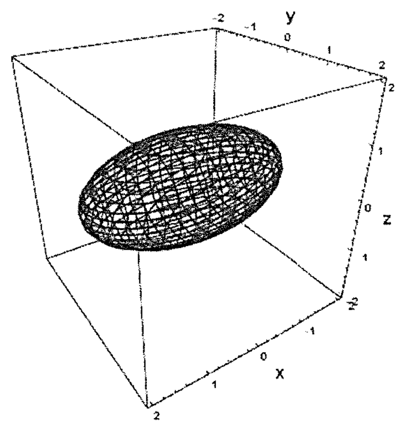
III



IV



V



VI

Match each of the following equations with its corresponding graph above.

(a) $z = x^2 - y^2$

○ I ● II ○ III ○ IV ○ V ○ VI

(b) $x^2 + y^2 = z^2 - 1$

○ I ○ II ○ III ● IV ○ V ○ VI

(c) $z^2 = x^2 + y^2$

○ I ○ II ○ III ○ IV ● V ○ VI

(d) $z = x^2 - 1$

● I ○ II ○ III ○ IV ○ V ○ VI

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12. (2 Credits) Let $\mathcal{E}$ be the region in $\mathbb{R}^3$ where $1 \leq x^2 + y^2 + z^2 \leq 4$ and $x, y, z \geq 0$ (i.e., in the first octant). Evaluate the integral

$$\iiint_{\mathcal{E}} (x^2 + y^2 + z^2)^{3/2}\, dV.$$

$$\mathcal{E} = \{(\rho, \phi, \theta) \mid 1 \leq \rho \leq 2, 0 \leq \phi \leq \pi/2, 0 \leq \theta \leq \pi/2\}$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} (\rho^2)^{3/2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\phi\, d\phi \int_1^2 \rho^5\, d\rho = \frac{\pi}{2}\left(\cos 0 - \cos \pi/2\right) \frac{\rho^6}{6}\Big|_1^2$$

$$= \frac{\pi}{2}(1)\left(\frac{63}{6}\right) = \frac{21\pi}{4}$$

$$\boxed{\frac{21\pi}{4}}$$

Problem	1	2	3	4	5	6	7	8	9	10	11	12	Total
Credit	3	1	1	3	2	2	2	2	2	2	1	2	23
Credit Points													

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