M273Q Multivariable Calculus
Exam 3

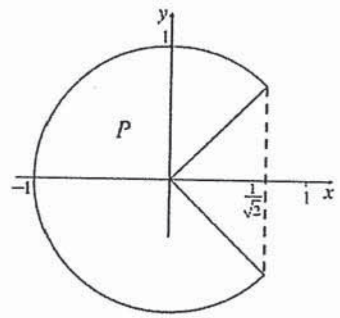
Name and section: ______________________

Instructor's name: Solutions.

Instructions: Closed book. No calculator allowed. Double-sided exam. NO CELL PHONES. **Show all work and use correct notation to receive full credit!** Write legibly.

$dA = r\,dr\,d\theta$	$dV = r\,dz\,dr\,d\theta$	$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

1. Consider the region $\mathcal{P}$ given below.



(a) (1 credit ____) Describe the region P in polar coordinates using mathematically correct notation.

$$P = \left\{ (r,\theta) \mid 0 \le r \le 1,\ \frac{\pi}{4} \le \theta \le \frac{7\pi}{4} \right\}$$

(b) (1 credit ____) Calculate $\iint_{P} y\, dA$

$$\iint_D y\, dA = \int_{\pi/4}^{7\pi/4} \int_0^1 (r \sin\theta)\, r\, dr\, d\theta$$

$$= \int_{\pi/4}^{7\pi/4} \sin\theta\, d\theta \int_0^1 r^2\, dr$$

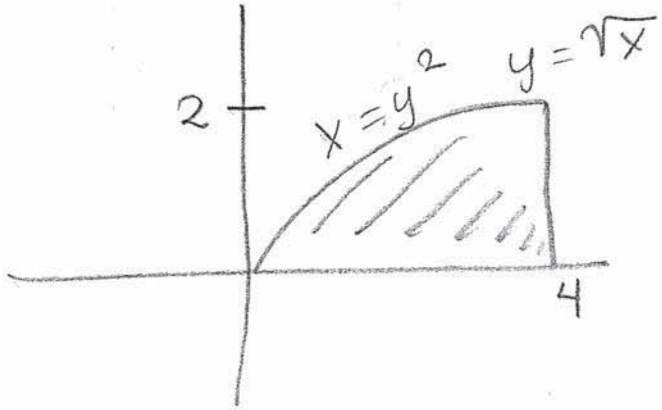
$$= -\cos\theta \Big|_{\pi/4}^{7\pi/4} \frac{r^3}{3}\Big|_0^1$$

$$= -\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) \cdot \frac{1}{3} = 0.$$

Question:	1	Total
Credit	2	2
GPA Credit Points Earned		

2. Consider the integral $\int_0^2 \int_{y^2}^4 \sqrt{1+x^{3/2}}\, dx\, dy$.

(a) (1 credit ____) Sketch the region of integration.



(b) (2 credit ____) Reverse the order of integration and compute the integral.

$$\int_0^4 \int_0^{\sqrt{x}} \sqrt{1+x^{3/2}}\, dy\, dx$$

$$= \int_0^4 \sqrt{x}\sqrt{1+x^{3/2}}\, dx$$

$u = x^{3/2} + 1$ $\quad x = 0 \; u = 1$

$du = \frac{3}{2} x^{1/2} dx$ $\quad x = 2 \; u = 9$

$\frac{2}{3} du = x^{1/2} dx$

$$= \int_1^9 \frac{2}{3} u^{1/2} du = \frac{4}{9} u^{3/2} \Big|_1^9$$

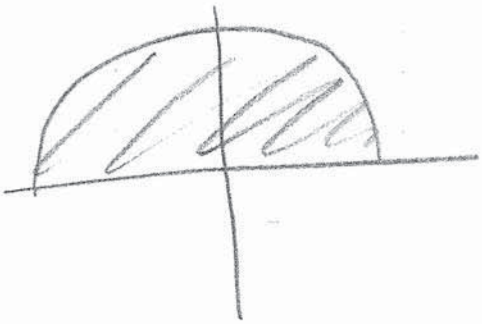
$$= \frac{4}{9}(27 - 1) = \frac{104}{9}$$

Question:	2	Total
Credit	3	3
GPA Credit Points Earned		

$$\begin{array}{r} 26 \\ \times\ 4 \\ \hline 104 \end{array}$$
$/9$

3. (1 credit ____) Convert to polar coordinates to evaluate

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy. \quad = V$$



$$V = \int_0^{\pi} \int_0^3 \frac{1}{\sqrt{r^2}}\, r\, dr\, d\theta = \int_0^{\pi} d\theta \int_0^3 \frac{r}{r}\, dr$$

$$= \pi \cdot r \Big|_0^3$$

$$= 3\pi$$

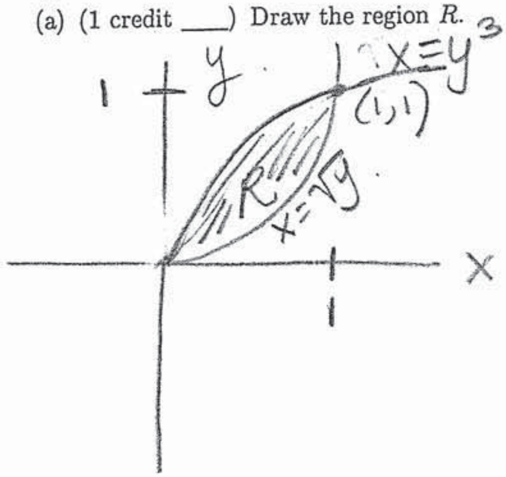
Question:	3	Total
Credit	1	1
GPA Credit Points Earned		

4. Consider the triple integral

$$\int_0^1 \int_{y^3}^{\sqrt{y}} \int_0^{xy} dz\, dx\, dy$$

representing a solid S. Let R be the projection of S onto the plane $z = 0$.

(a) (1 credit ____) Draw the region R.



$\sqrt{y} = x$

$\Rightarrow y = x^2$

$y^3 = x$

$y = x^{1/3}$

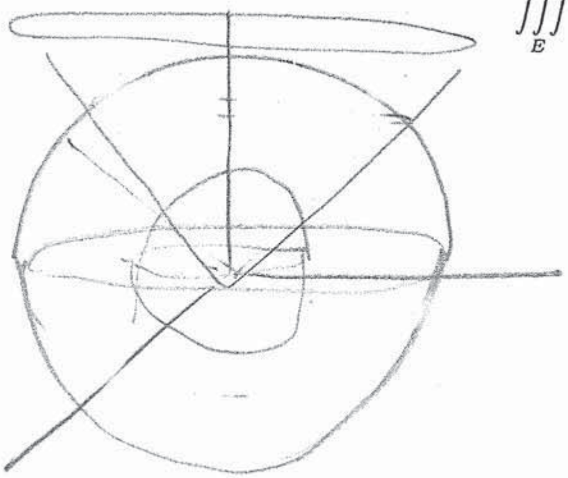
(b) (1 credit ____) Rewrite this integral as a triple integral in the order $dz\, dy\, dx$. Do not compute the resulting integral.

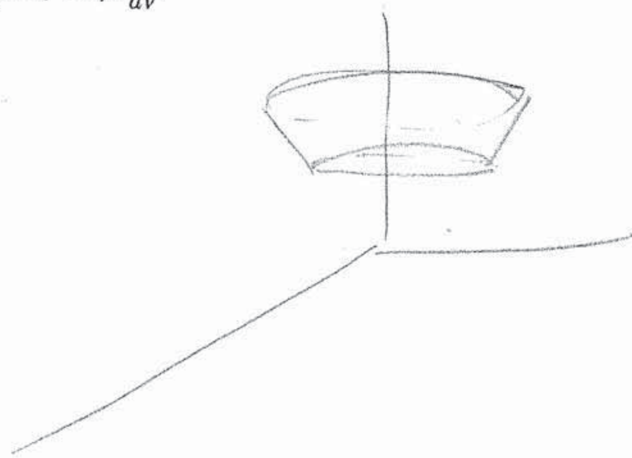
$$\int_0^1 \int_{x^2}^{x^{1/3}} \int_0^{xy} dz\, dx\, dy$$

Question:	4	Total
Credit	2	2
GPA Credit Points Earned		

5. (2 credit ____) A solid object occupies the region inside the cone $z = \sqrt{x^2+y^2}$ (that is $z \geq \sqrt{x^2+y^2}$) and between the two spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$.
Rewrite *BUT DO NOT EVALUATE* the triple integral in the spherical coordinate system.

$$\iiint_E e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV$$





$2 \leq \rho \leq 3$

$0 \leq \phi \leq \pi/4$

$0 \leq \theta \leq 2\pi$

$\rho\cos\phi = \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2}$

$\rho\cos\phi = \rho\sqrt{\sin^2\phi(\cos^2\theta+\sin^2\theta)}$

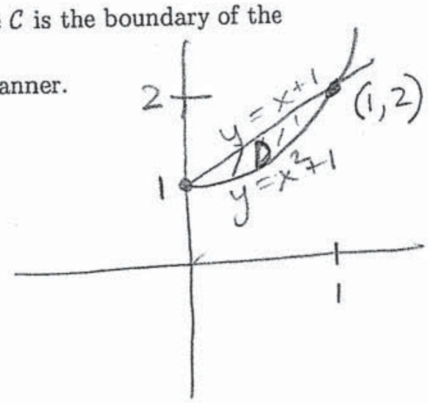
$\cos\phi = \sin\phi$

$\phi = \frac{\pi}{4}.$

$$\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{2\pi}\int_0^{\pi/4}\int_2^3 e^{(\rho^2)^{3/2}}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

Question:	5	Total
Credit	2	2
GPA Credit Points Earned		

6. (2 credit ____) Use Green's Theorem to evaluate $\int_C (2xy - y^2)\,dx + x^2\,dy$ where C is the boundary of the region enclosed by $y = x + 1$ and $y = x^2 + 1$, traversed in a counterclockwise manner.



$P = 2xy - y^2 \qquad P_y = 2x - 2y$

$Q = x^2 \qquad Q_x = 2x$

$Q_x - P_y = 2x - (2x - 2y) = 2y.$

$$\int_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$$

$$= \iint_D 2y \, dA$$

$\frac{1}{5} + \frac{1}{3} = \frac{8}{15}$

$D = \{(x,y) \mid 0 \le x \le 1,\ x^2+1 \le y \le x+1\}$

$$= \int_0^1 \int_{x^2+1}^{x+1} 2y \, dy\, dx$$

$$= \int_0^1 y^2 \Big|_{x^2+1}^{x+1} dx = \int_0^1 (x+1)^2 - (x^2+1)^2 \, dx$$

Question:	6	Total
Credit	2	2
GPA Credit Points Earned		

$$= \int_0^1 x^2 + 2x + 1 - (x^4 + 2x^2 + 1)\, dx = \int_0^1 -x^4 - x^2 + 2x \, dx$$

$$= \left(-\frac{x^5}{5} - \frac{x^3}{3} + x^2\right)\Big|_0^1 = 1 - \frac{1}{5} - \frac{1}{3} = \boxed{\frac{7}{15}}$$

7. Let $\mathbf{F}(x, y, z) = < 2x - y,\ z - x,\ y + 1 >$.

(a) (1 credit ____) Show that $\mathbf{F}$ is conservative. Justify your answer.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & z-x & y+1 \end{vmatrix}$$

The domain of $\vec{F}$, $\mathbb{R}^3$, is simply connected.

$$= < 1 - 1, -(0-0), -1-(-1) >$$

$$= \vec{0}$$

(b) (2 credit ____) Find a function f so that $\nabla f = \mathbf{F}$.

$f_x = 2x - y \qquad f = x^2 - yx + g(y,z)$

$f_y = z - x \qquad f = zy - xy + h(x,z)$

$f_z = y + 1 \qquad f = yz + z + i(x,y)$

$x^2 - \cancel{yx} + g(y,z) = zy - \cancel{xy} + h(x,z) = yz + z + i(x,y)$.

$f(x,y,z) = x^2 + z + zy - xy$.

Question	Points	Score
7	3	
Total:	3	

8. For each part of problem a-d below, let C be the straight line segment from $(1,0,1)$ to $(0,3,6)$.

(a) (1 credit ____) Give a parametrization for C, the straight line segment from $(1,0,1)$ to $(0,3,6)$.

$$\vec{r}(t) = (1-t)\langle 1,0,1\rangle + t\langle 0,3,6\rangle$$
$$= \langle 1-t, 3t, 1+5t\rangle \quad 0 \le t \le 1$$

$x(t) = 1-t \qquad x'(t) = -1$

$y(t) = 3t \qquad y'(t) = 3$

$z(t) = 1+5t \qquad z'(t) = 5$

(b) (1 credit ____) Calculate $\int_C y\,dx$.

$$\int_C y\,dx = \int_0^1 3t(-1)\,dt = -\frac{3t^2}{2}\Big|_0^1 = -\frac{3}{2}.$$

(c) (1 credit ____) Let $\mathbf{F}$ be the vector field $\mathbf{F} = \langle xy, y^2, 1\rangle$, calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$$\vec{F}(\vec{r}(t)) = \langle (1-t)(3t), (3t)^2, 1\rangle$$
$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \langle 3t-3t^2, 9t^2, 1\rangle\cdot\langle -1,3,5\rangle$$
$$= 3t^2 - 3t + 27t^2 + 5$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 30t^2 - 3t + 5\,dt$$
$$= \left(10t^3 - \frac{3}{2}t^2 + 5t\right)\Big|_0^1 = 10 - \frac{3}{2} + 5$$
$$= 15 - \frac{3}{2} = \frac{27}{2}.$$

(d) (1 credit ____) Calculate $\int_C xy^2z\,ds$.

$\|\vec{r}\,'(t)\| = \sqrt{(-1)^2 + 3^2 + 5^2}$

$= \sqrt{35}$

$\int_C xy^2z\,ds = \int_0^1 (1-t)(3t)^2(1+5t)\sqrt{35}\,dt$

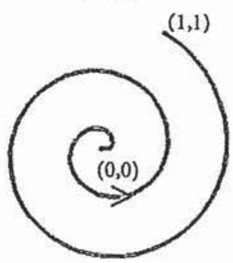
$= \sqrt{35}\int_0^1 9t^2(1+4t-5t^2)\,dt$

$= 9\sqrt{35}\int_0^1 t^2 + 4t^3 - 5t^4\,dt$

$= 9\sqrt{35}\left(\frac{t^3}{3} + t^4 - t^5\right)\Big|_0^1$

$= 9\sqrt{35}\left(\frac{1}{3}\right) = 3\sqrt{35}$

9. (1 credit ____) Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle (1+xy)e^{xy}, e^y + x^2e^{xy}\rangle$ and C is as pictured. Note that if $f(x,y) = e^y + xe^{xy}$, then $\nabla f = \mathbf{F}$.



C

$f(0,0) = e^0 + 0 = 1$

$f(1,1) = e^1 + e^1 = 2e$

$\int_C \vec{F}\cdot d\vec{r} = f(1,1) - f(0,0) = 2e - 1$

Question	Points	Score
8	4	
9	1	
Total:	5	

Page:	1	2	3	4	5	6	7	8	9	Total
Credit	2	3	1	2	2	2	3	3	2	20
GPA Credit Points Earned										

19.