Exam I Review Problems:

1)

a) False. Let $\vec{a} = \langle -1, 0, 1\rangle$, $\vec{b} = \langle 0, 1, 0\rangle$, and $\vec{c} = \langle 2, 0, 1\rangle$.

Then, $\vec{a}\cdot\vec{b} = 0$ and $\vec{b}\cdot\vec{c} = 0$ but $\vec{a}\cdot\vec{c} = -1 \neq 0$

b) True. If $\vec{a}\cdot\vec{b} = 0$ then they are orthogonal to one another, that is, the angle between them is $\pi/2$. Thus, $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin(\frac{\pi}{2})$ ^{1}

c) True. $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$ and $\|\vec{v}\times\vec{u}\| = \|\vec{v}\|\|\vec{u}\|\sin\theta = \|\vec{u}\|\|\vec{v}\|\sin\theta$

d) False. If $\langle 3, -1, 2\rangle$ is parallel to the given plane, then it would also necessarily be orthogonal to the normal vector of the plane $\vec{n} = \langle 6, -2, 4\rangle$ but since $\langle 6, -2, 4\rangle = 2\langle 3, -1, 2\rangle$ that means the vectors are parallel, hence the plane is orthogonal to the given vector, not parallel.

e) False. $\hat{\imath}\cdot\hat{\jmath} = \vec{0}$ but neither $\hat{\imath} = \vec{0}$ nor $\hat{\jmath} = \vec{0}$.

f) False. If $\vec{v} = \lambda\vec{u}$ (i.e. the vectors are parallel) then $\vec{u}\times\vec{v} = \vec{0}$. Since, $\|\vec{u}\times\vec{v}\| = \|\vec{u}\times(\lambda\vec{u})\| = \lambda\|\vec{u}\times\vec{u}\| = \|\vec{u}\|^2\sin(0) = 0$. ^{0}
The only vector of length zero is $\vec{0}$ hence $\|\vec{u}\times\vec{v}\| = \vec{0}$.

g) True. If $\vec{u}\cdot\vec{v} = 0$ then $\vec{u}\perp\vec{v}$. Then, if $\vec{u}\times\vec{v} = \vec{0}$ then $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta = 0$ but since $\theta = \frac{\pi}{2}$ then $0 = \|\vec{u}\|\|\vec{v}\|$ which implies either $\|\vec{u}\| = 0$ or $\|\vec{v}\| = 0$. In either case, the only vector of length zero is $\vec{0}$, so either $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

h) True. Since $x = 0$ the curve lives in the yz-plane. $y = t^2$ and $z = 4t$, so $t = \frac{z}{4}$ and therefore $y = \frac{1}{16}z^2$ which is a parabola.

i) True. Assume t is an arc length parameter (we can do this since, arc length parameterizations exist). Then $\kappa(t) = \|\vec{T}'(t)\| = 0$, which implies $\vec{T}'(t) = \vec{0}$ which further implies that $\vec{T}(t) = \vec{c}$ where $\vec{c}$ is some constant vector. The only curve that has constant unit tangent vector is a straight line.

j) False. Consider two different parameterizations of the line $y=x$: $\vec{r}_1(t) = \langle t, t \rangle$ and $\vec{r}_2(t) = \langle 2t, 2t \rangle$. Then their tangents are $\vec{r}_1'(t) = \langle 1, 1 \rangle$ and $\vec{r}_2'(t) = \langle 2, 2 \rangle$, respectively. Note that these vectors are parallel (point in the same direction) but are not of equal length.

2)

a) Vector. $\vec{a}\cdot\vec{b}$ is a scalar, let's say λ. Then, $(\lambda\vec{c})\times\vec{a} = \lambda(\vec{c}\times\vec{a})$ and $\vec{c}\times\vec{a}$ is a vector, and so is a scalar multiple of a vector.

b) Nonsense. The issue is with "$(\vec{a}\cdot\vec{b})\times\vec{c}$". $\vec{a}\cdot\vec{b}$ produces a scalar, which is then to be crossed with $\vec{c}$ which isn't possible since the cross product is only defined on vectors.

c) Vector. As stated above in (2a) $(\vec{a}\cdot\vec{b})\vec{c}$ is a vector. So this vector crossed with $\vec{c}$ is well-defined and the result is a vector.

d) Scalar. $\vec{a}\times\vec{b}$ produces a vector which is then dotted with $\vec{c}$ which produces a scalar.

3)

a) Meaningful. $\|\vec{w}\|$ is a scalar and $\vec{u}\times\vec{v}$ is a vector, so the result is a vector.

b) Nonsense. $\vec{u}\cdot\vec{v}$ is a scalar and scalars cannot be crossed.

c) Meaningful. $\vec{v}\times\vec{w}$ is a vector which is then being dotted with $\vec{u}$ to produce a scalar.

4)
$$\langle 3,2,x\rangle\cdot\langle 2x,4,x\rangle = 0$$
$$6x+8+x^2 = 0$$
$$(x+4)(x+2)=0$$
$$x=-4 \text{ or } x=-2.$$

5) $\vec{a}=\langle 1,1,1\rangle$ and $\vec{b}=\langle 2,-1,-3\rangle$ $\quad \vec{a}=\vec{a}_{\parallel\vec{b}}+\vec{a}_{\perp\vec{b}}$

$$\vec{a}_{\parallel\vec{b}}=\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b}=\frac{2-1-3}{4+1+9}\langle 2,-1,-3\rangle=-\frac{2}{14}\langle 2,-1,-3\rangle$$

$$\vec{a}_{\perp\vec{b}}=\vec{a}-\vec{a}_{\parallel\vec{b}}=\langle 1,1,1\rangle-\langle -\tfrac{2}{7},\tfrac{1}{7},\tfrac{3}{7}\rangle=\langle \tfrac{9}{7},\tfrac{6}{7},\tfrac{4}{7}\rangle$$

6) $\vec{a}=\langle \frac{1}{\sqrt2},\frac{1}{\sqrt2}\rangle$ and $\vec{b}=\langle \frac{1}{\sqrt2},-\frac{1}{\sqrt2}\rangle$ and $\vec{u}=\langle 3,0\rangle$.

a) $\sqrt{\left(\frac{1}{\sqrt2}\right)^2+\left(\frac{1}{\sqrt2}\right)^2}=\sqrt{\left(\frac{1}{\sqrt2}\right)^2+\left(-\frac{1}{\sqrt2}\right)^2}=1$ so, $\|\vec{a}\|=\|\vec{b}\|=1$ that is, both $\vec{a}$ and $\vec{b}$ are unit vectors. They are also orthogonal since $\vec{a}\cdot\vec{b}=\frac{1}{2}-\frac{1}{2}=0$

bc) Since $\vec{a}$ and $\vec{b}$ differ only in the sign of the y-component, this can be accomplished quickly by doing two calculations at once using $\pm$.

Let $\vec{v}_\pm = \left\langle \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}} \right\rangle$. Note $\vec{a} = \vec{v}_+$ and $\vec{b} = \vec{v}_-$. Then,

$$\vec{u}_{\parallel \vec{v}_\pm} = \frac{\vec{u}\cdot\vec{v}_\pm}{\|\vec{v}_\pm\|^2}\,\vec{v}_\pm = \frac{\frac{3}{\sqrt{2}}}{\frac{1}{2}+\frac{1}{2}}\left\langle \frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right\rangle = \left\langle \frac{3}{2}, \pm\frac{3}{2}\right\rangle$$

$$\vec{u}_{\perp \vec{v}_\pm} = \vec{u} - \vec{u}_{\parallel \vec{v}_\pm} = \langle 3, 0\rangle - \left\langle \frac{3}{2}, \pm\frac{3}{2}\right\rangle = \left\langle \frac{3}{2}, \mp\frac{3}{2}\right\rangle$$

Thus, $\vec{u}_{\parallel \vec{a}} = \vec{u}_{\parallel \vec{v}_+} = \left\langle \frac{3}{2}, \frac{3}{2}\right\rangle$ and $\vec{u}_{\perp \vec{a}} = \vec{u}_{\perp \vec{v}_+} = \left\langle \frac{3}{2}, -\frac{3}{2}\right\rangle$

and $\vec{u}_{\parallel \vec{b}} = \vec{u}_{\parallel \vec{v}_-} = \left\langle \frac{3}{2}, -\frac{3}{2}\right\rangle$ and $\vec{u}_{\perp \vec{b}} = \vec{u}_{\perp \vec{v}_-} = \left\langle \frac{3}{2}, \frac{3}{2}\right\rangle$

7)

a) $(x+1)^2 + (y-2)^2 + (z-1)^2 = R^2$

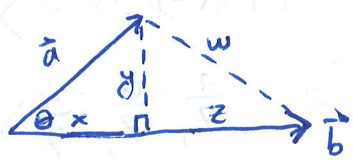
$(6+1)^2 + (-2-2)^2 + (3-1)^2 = R^2$

$49 + 16 + 4 = R^2$

$69 = R^2$ so, the equation is $(x+1)^2 + (y-2)^2 + (z-1)^2 = 69$

b) The yz-plane is given by $x = 0$. So the curve of intersection (trace) is given by $(0+1)^2 + (y-2)^2 + (z-1)^2 = 69$ or $(y-2)^2 + (z-1)^2 = 68$

8)



$\cos\theta = \underline{\quad 5 \quad}$ Since $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$ and therefore $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$

$\sin\theta = \underline{\quad 4 \quad}$ Since $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$ and so $\sin\theta = \frac{\|\vec{a}\times\vec{b}\|}{\|\vec{a}\|\,\|\vec{b}\|}$

$x = \underline{\quad 2 \quad}$ Since x is the scalar component of $\vec{a}$ along $\vec{b}$. This is the amount $\vec{b}$ is scaled in the projection of $\vec{a}$ along $\vec{b}$: $\vec{a}_{\parallel \vec{b}} = \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}\right)\hat{e}_{\vec{b}}$ the scalar component of $\vec{a}$ along $\vec{b}$. $x = \|\vec{a}_{\parallel \vec{b}}\| = \frac{|\vec{a}\cdot\vec{b}|}{\|\vec{b}\|}$

$y = \underline{\quad 3 \quad}$ Since $\sin\theta = \frac{y}{\|\vec{a}\|}$ and $\sin\theta = \frac{\|\vec{a}\times\vec{b}\|}{\|\vec{a}\|\,\|\vec{b}\|}$ we have

$$y = \|\vec{a}\|\sin\theta = \cancel{\|\vec{a}\|}\left(\frac{\|\vec{a}\times\vec{b}\|}{\cancel{\|\vec{a}\|}\,\|\vec{b}\|}\right) = \frac{\|\vec{a}\times\vec{b}\|}{\|\vec{b}\|}$$

$z = \underline{\ \ 7\ \ }$ Since $z = \|\vec{b}\| - x = \|\vec{b}\| - \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|} = \dfrac{\|\vec{b}\|}{\|\vec{b}\|}\left(\|\vec{b}\| - \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}\right)$

$$z = \frac{\|\vec{b}\|^2 - \vec{a}\cdot\vec{b}}{\|\vec{b}\|} = \frac{\vec{b}\cdot\vec{b} - \vec{a}\cdot\vec{b}}{\|\vec{b}\|} = \frac{(\vec{b}-\vec{a})\cdot\vec{b}}{\|\vec{b}\|}$$

$w = \underline{\ \ 6\ \ }$ Process of elimination. But an explanation: If we suppose that the line segment w is a vector $\vec{w}$ beginning at the tip of $\vec{b}$ and pointing to the tip of $\vec{a}$, then it is the case that $\vec{b} + \vec{w} = \vec{a}$ and so $\vec{w} = \vec{a} - \vec{b}$ and its length is given by $\|\vec{w}\| = \|\vec{a} - \vec{b}\|$ but notice that $\|\vec{a} - \vec{b}\| = \|\vec{b} - \vec{a}\|$.

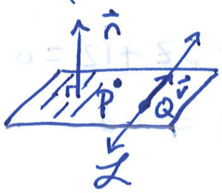
9) $\vec{v} = \langle -3, 2, 3\rangle$ and $P_0 = (4, -1, 2)$. So, $x = 4 - 3t$, $y = -1 + 2t$, and $z = 2 + 3t$

10) Since the line is perpendicular to the plane, that is equivalent to the line being parallel to the plane's normal vector, $\vec{n} = \langle 2, -1, 5\rangle$. So, $\vec{r}(t) = \langle -2, 2, 4\rangle + t\langle 2, -1, 5\rangle$

11) The planes are parallel so their normal vectors are parallel (or equal):

$$(x-2) + 4(y-1) - 3z = 0$$

12) $P = (-1, -3, 2)$ $\quad \vec{\mathcal{L}}(t) = \langle -1, 0, 2\rangle + t\langle -2, 4, 1\rangle$. Let $\vec{v} = \langle -2, 4, 1\rangle$.



$Q = (-1, 0, 2) \Rightarrow \overrightarrow{PQ} = \langle 0, 3, 0\rangle$

$$\begin{aligned}
\vec{n} &= \overrightarrow{PQ} \times \vec{v} \\
&= 3\hat{j} \times (-2\hat{i} + 4\hat{j} + \hat{k}) \\
&= -6\hat{j}\times\hat{i} + 12\hat{j}\times\hat{j} + 3\hat{j}\times\hat{k} \\
&= 6\hat{k} + 3\hat{i} \\
&= \langle 3, 0, 6\rangle \Rightarrow 3(x+1) + 6(z-2) = 0
\end{aligned}$$

13) $x = 1 - t$ $\quad z = 1 - 2x + y$

$y = t \quad 1 + t = 1 - 2 + 2t + t$

$z = 1 + t \quad 2 = 2t$

$t = 1 \Rightarrow (0, 1, 2)$

14) $A = (2,1,1)$, $B = (-1,-1,10)$, and $C = (1,3,-4)$

a) $\vec{AB} = \langle -3, -2, 9 \rangle \quad \vec{AC} = \langle -1, 2, -5 \rangle$

$\vec{n} = \vec{AB} \times \vec{AC}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -2 & 9 \\ -1 & 2 & -5 \end{vmatrix} = \begin{vmatrix} -2 & 9 \\ 2 & -5 \end{vmatrix} \hat{i} - \begin{vmatrix} -3 & 9 \\ -1 & -5 \end{vmatrix} \hat{j} + \begin{vmatrix} -3 & -2 \\ -1 & 2 \end{vmatrix} \hat{k}$$

$$= (10 - 18)\hat{i} - (15 + 9)\hat{j} + (-6 - 2)\hat{k}$$

$\vec{n} = \langle -8, -24, -8 \rangle \Rightarrow -8(x-2) - 24(y-1) - 8(z-1) = 0$

$\Rightarrow P_1: (x-2) + 3(y-1) + (z-1) = 0$

$\Rightarrow \vec{n}_1 = \langle 1, 3, 1 \rangle$

b) $P_2: 2(x-2) - 4y - 3(z-4) = 0 \Rightarrow \vec{n}_2 = \langle 2, -4, -3 \rangle$

The line of intersection has direction vector given by: $\vec{v} = \vec{n}_1 \times \vec{n}_2$

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 1 \\ 2 & -4 & -3 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ -4 & -3 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} \hat{k}$$

$$= (-9 + 4)\hat{i} - (-3 - 2)\hat{j} + (-4 - 6)\hat{k}$$

$$= \langle -5, 5, -10 \rangle$$

$P_1: x - 2 + 3y - 3 + z - 1 = 0$

$x + 3y + z = 6$

$\underline{x=0}$: $z = 6 - 3y$

$P_2: 2x - 4 - 4y - 3z + 12 = 0$

$2x - 4y - 3z = -8$

$\underline{x=0}$: $z = \frac{8}{3} - \frac{4}{3}y$

$\vec{L}(t) = \langle 0, 2, 0 \rangle + t\langle -5, 5, -10 \rangle$

$\vec{L}(t) = \langle 0, 2, 0 \rangle + t\langle -5, 5, -10 \rangle$

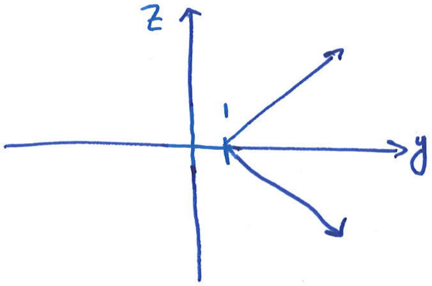
Set Equal: $6 - 3y = \frac{8}{3} - \frac{4}{3}y$

$18 - 9y = 8 - 4y$

$10 = 5y \Rightarrow y = ~~5~~\ 2 \Rightarrow z = 6 - 15 = -9$

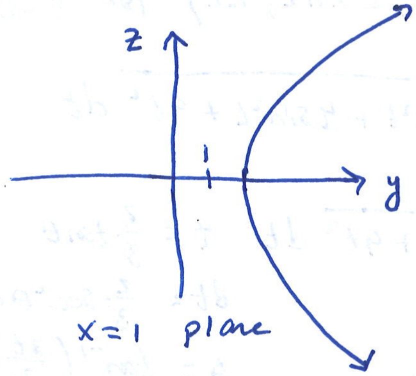
15) $y = \sqrt{x^2+z^2} + 1$

$x = 0 \Rightarrow y = |z| + 1$



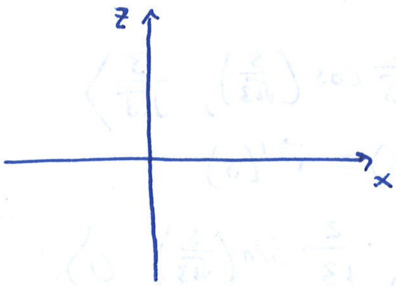
$x=0$ (yz-plane)

$x = 1 \Rightarrow y = \sqrt{1+z^2} + 1$



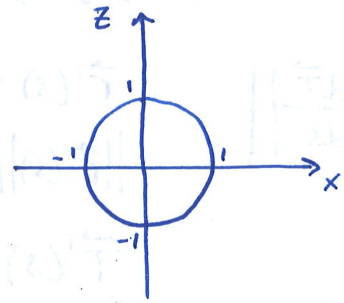
$x=1$ plane

$y=0 \Rightarrow -1 = \sqrt{x^2+z^2}$
(No Solution)



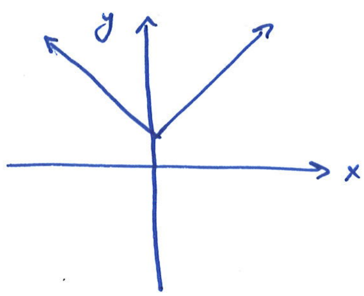
$y=0$ (xz-plane)

$y = 2 \Rightarrow 1 = x^2 + z^2$



$y=2$ plane

$z=0 \Rightarrow y = |x| + 1$



$z=0$ (xy-plane)

16)

a) $8x + 2y + 3z = 0$ is a plane (II)

b) $z = \sin x + \cos y$

$x = 0 \Rightarrow z = \cos y$ in yz-plane

$y = 0 \Rightarrow z = \sin x + 1$ in xz-plane (I)

c) $z = \sin\left(\frac{\pi}{2+x^2+y^2}\right)$ is radially symmetric about the z axis (IV)

d) $z = e^y$ all traces parallel to the yz-plane are the same $\Rightarrow$ (III)

17) $x^2 + y^2 = 16$ intersected with $x + z = 5$

Let $x = 4\cos t$ and $y = 4\sin t$ then: $z = 5 - 4\cos t$

So, $\vec{r}(t) = \langle 4\cos t, 4\sin t, 5 - 4\cos t \rangle$ for $t \in [0, 2\pi]$.

18) $x = 2\sin t \Rightarrow x' = 2\cos t \Rightarrow x'(\pi/6) = \sqrt{3}$

$y = 2\sin(2t) \Rightarrow y' = 4\cos(2t) \Rightarrow y'(\pi/6) = 2$

$z = 2\sin(3t) \Rightarrow z' = 6\cos(3t) \Rightarrow z'(\pi/6) = 0$

Note: $x\left(\frac{\pi}{6}\right) = 2\sin\left(\frac{\pi}{6}\right) = 2\left(\frac{1}{2}\right) = 1$

$y\left(\frac{\pi}{6}\right) = 2\sin\left(\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

$z\left(\frac{\pi}{6}\right) = 2\sin\left(\frac{\pi}{2}\right) = 2(1) = 2$

$\Bigg\}$ so $t = \frac{\pi}{6}$

$\vec{r}(t) = \langle 1, \sqrt{3}, 2 \rangle + t\langle \sqrt{3}, 2, 0 \rangle$

19) a) $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t \rangle$ for $t \in [0, 2\pi]$

b) $s = \int_0^{2\pi} \sqrt{4\sin^2 t + 4\cos^2 t + 9}\, dt$

$= \int_0^{2\pi} \sqrt{13}\, dt$

$= 2\pi\sqrt{13}$

c) $s = \int_0^t \sqrt{13}\, dt \Rightarrow s = t\sqrt{13} \Rightarrow t = \frac{s}{\sqrt{13}} \Rightarrow \vec{r}(s) = \left\langle 2\cos\left(\frac{s}{\sqrt{13}}\right), 2\sin\left(\frac{s}{\sqrt{13}}\right), \frac{3s}{\sqrt{13}} \right\rangle$

$\kappa(s) = \left\| \frac{d\vec{T}}{ds} \right\|$

$\vec{r}'(s) = \left\langle -\frac{2}{\sqrt{13}}\sin\left(\frac{s}{\sqrt{13}}\right), \frac{2}{\sqrt{13}}\cos\left(\frac{s}{\sqrt{13}}\right), \frac{3}{\sqrt{13}} \right\rangle$

$\|\vec{r}'(s)\| = 1 \Rightarrow \vec{T}(s) = \vec{r}'(s)$

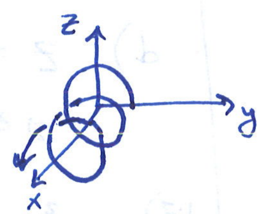
$\vec{T}'(s) = \left\langle -\frac{2}{13}\cos\left(\frac{s}{\sqrt{13}}\right), -\frac{2}{13}\sin\left(\frac{s}{\sqrt{13}}\right), 0 \right\rangle$

$\|T'(s)\| = \sqrt{\frac{4}{169}\cos^2\left(\frac{s}{\sqrt{13}}\right) + \frac{4}{169}\sin^2\left(\frac{s}{\sqrt{13}}\right)} = \frac{2}{13}$

$\Rightarrow \kappa(s) = \frac{2}{13}$

20) $\vec{r}(t) = \langle t, \cos(\pi t), \sin(\pi t) \rangle$

a) Helix extending in the $+x$ direction



b) $\vec{r}'(t) = \langle 1, -\pi\sin(\pi t), \pi\cos(\pi t) \rangle$

$\vec{r}''(t) = \langle 0, -\pi^2\cos(\pi t), -\pi^2\sin(\pi t) \rangle$

21) $\vec{r}(t) = \langle \sin(\pi t), \cos(\pi t), \frac{1}{4}t^2 \rangle$ for $t \in [0, 2]$. The curve begins @ $(0, 1, 0)$ and z is always increasing.

22) $\vec{r}(t) = \langle t+1, 2t^2 - 2, 5 \rangle$

$\vec{r}'(t) = \langle 1, 4t, 0 \rangle$

$\langle 1, 4t, 0 \rangle \cdot \langle 1, 2, 4 \rangle = 0$

$1 + 8t = 0 \Rightarrow t = -\frac{1}{8}$

$\vec{r}\left(-\frac{1}{8}\right) = \left\langle \frac{7}{8}, -\frac{63}{42}, 5 \right\rangle \Rightarrow \left(\frac{7}{8}, -\frac{63}{42}, 5\right)$

23) $\vec{r}(t) = \left\langle \sqrt{2-t},\ \frac{e^t-1}{t},\ \ln(t+1) \right\rangle$

a) $2-t \geq 0$, $t \neq 0$, $t+1>0$

$2 \geq t$, $t \neq 0$, $t > -1$

$t \in (-1,0) \cup (0,2]$

b) $\lim_{t\to 0} \vec{r}(t) = \left\langle \sqrt{2},\ \lim_{t\to 0} \frac{e^t-1}{t},\ \ln(1) \right\rangle$ "$\frac{0}{0}$" $\Rightarrow$ use L'Hopitals.

$\overset{L'H}{=} \left\langle \sqrt{2},\ \lim_{t\to 0} e^t,\ 0 \right\rangle$

$= \langle \sqrt{2}, 1, 0 \rangle$

c) $\vec{r}\,'(t) = \left\langle -\frac{1}{2\sqrt{2-t}},\ \frac{e^t t - (e^t-1)}{t^2},\ \frac{1}{t+1} \right\rangle$

24) $\vec{r}(t) = \int \vec{v}(t)\,dt$

$= \left\langle 3\left(\frac{2}{3}\right)(1+t)^{3/2},\ -\cos(2t),\ 2e^{3t} \right\rangle + \vec{C}$

$\vec{r}(t) = \left\langle 2(1+t)^{3/2},\ -\cos(2t),\ 2e^{3t} \right\rangle + \vec{C}$

$\vec{r}(0) = \langle 2, -1, 2 \rangle + \vec{C} = \langle 0, 1, 2 \rangle \Longrightarrow \vec{C} = \langle -2, 2, 0 \rangle$

$\Rightarrow \vec{r}(t) = \left\langle 2(1+t)^{3/2},\ -\cos(2t),\ 2e^{3t} \right\rangle + \langle -2, 2, 0 \rangle$

IBP: $u = t$ $v = \frac{1}{\pi}\sin(\pi t)$
$du = dt$ $dv = \cos(\pi t)\,dt$

25) $\int_0^1 \left\langle t^2,\ t\cos(\pi t),\ \sin(\pi t) \right\rangle dt$

$= \left\langle \frac{1}{3}t^3\Big|_0^1,\ \cancel{\frac{t}{\pi}\sin(\pi t)\Big|_0^1}^{\,0} - \frac{1}{\pi}\int_0^1 \sin(\pi t)\,dt,\ -\frac{1}{\pi}\cos(\pi t)\Big|_0^1 \right\rangle$

$= \left\langle \frac{1}{3},\ \frac{1}{\pi^2}\cos(\pi t)\Big|_0^1,\ \frac{1}{\pi}+\frac{1}{\pi} \right\rangle = \left\langle \frac{1}{3},\ -\frac{1}{\pi^2}-\frac{1}{\pi^2},\ \frac{2}{\pi} \right\rangle = \left\langle \frac{1}{3},\ -\frac{2}{\pi^2},\ \frac{2}{\pi} \right\rangle$

26) $\vec{r}(t) = \left\langle 2\cos(2t),\ 2t^{3/2},\ 2\sin(2t) \right\rangle$ for $t \in [0,1]$.

$\vec{r}\,'(t) = \left\langle -4\sin(2t),\ 3t^{1/2},\ 4\cos(2t) \right\rangle \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{16\sin^2(2t) + 9t + 16\cos^2(2t)}$

$= \sqrt{16+9t}$

$$s = \int_0^1 (16+9t)^{1/2}\,dt$$

$$= \frac{1}{9}\left(\frac{2}{3}\right)(16+9t)^{3/2}\Big|_0^1$$

$$= \frac{2}{27}(25)^{3/2} - \frac{2}{27}(16)^{3/2}$$

$$= \frac{122}{27}$$

27) $\vec{r}(t) = \left\langle e^t,\ e^t\sin t,\ e^t\cos t \right\rangle = e^t\left\langle 1,\ \sin t,\ \cos t \right\rangle$. Note: $\vec{r}(0) = \langle 1, 0, 1 \rangle$.

$\vec{r}\,'(t) = e^t\langle 1, \sin t, \cos t\rangle + e^t\langle 0, \cos t, -\sin t\rangle = e^t\langle 1,\ \sin t + \cos t,\ \cos t - \sin t\rangle$

$\|\vec{r}\,'(t)\| = e^t\sqrt{1 + \sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t + \cos^2 t - \cancel{2\sin t\cos t} + \sin^2 t}$

$= e^t\sqrt{3}$

$$s = \int_0^t \sqrt{3}e^u\,du$$

$$s = \sqrt{3}e^t - \sqrt{3} \Rightarrow t = \ln\left(\frac{s-\sqrt{3}}{\sqrt{3}}\right)$$

$$\vec{r}(s) = e^{\ln\left(\frac{s-\sqrt{3}}{\sqrt{3}}\right)}\left\langle 1,\ \sin\left(\ln\left(\frac{s-\sqrt{3}}{\sqrt{3}}\right)\right),\ \cos\left(\ln\left(\frac{s-\sqrt{3}}{\sqrt{3}}\right)\right)\right\rangle$$

$$= \frac{s-\sqrt{3}}{\sqrt{3}}\left\langle 1,\ \sin\left(\ln\left(\frac{s-\sqrt{3}}{\sqrt{3}}\right)\right),\ \cos\left(\ln\left(\frac{s-\sqrt{3}}{\sqrt{3}}\right)\right)\right\rangle$$

28) $x^2+y^2=25$ and $x=z$. Let $x = 5\cos t$ and $y = 5\sin t$. Then, $z = 5\cos t$ and:

$\vec{r}(t) = \langle 5\cos t,\ 5\sin t,\ 5\cos t\rangle$ and $\vec{r}\,'(t) = \langle -5\sin t,\ 5\cos t,\ -5\sin t\rangle$

at $t=t_0$: $\vec{r}(t_0) = \langle 3,4,3\rangle \Rightarrow 5\cos(t_0) = 3$ and $5\sin(t_0) = 4$, thus:

$\vec{r}\,'(t_0) = \langle -4, 3, -4\rangle$ and so the line of intersection $\vec{L}(t) = \langle 3,4,3\rangle + t\langle -4,3,-4\rangle$

29) $\vec{r}(t) = \langle \frac{1}{3}t^3, t^2, 2t \rangle$

a) $\vec{r}\,'(t) = \langle t^2, 2t, 2 \rangle \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{t^4 + 4t^2 + 4} = t^2 + 2$

$\vec{T}(t) = \dfrac{\langle t^2, 2t, 2 \rangle}{t^2+2}$ $\Rightarrow \vec{T}\,'(t) = \dfrac{\langle 2t, 2, 0 \rangle (t^2+2) - \langle t^2, 2t, 2 \rangle 2t}{t^4+4t^2+4}$

b)

$$= \frac{\langle \cancel{2t^3} + 4t - \cancel{2t^3}, 2t^2+4-4t^2, -4t \rangle}{t^4+4t^2+4}$$

$$= \frac{\langle 4t, 4-2t^2, -4t \rangle}{t^4+4t^2+4}$$

$$\|\vec{T}\,'(t)\| = \frac{\sqrt{16t^2 + 16 \cancel{-16t^2} + 4t^4 \cancel{+16t^2}}}{t^4+4t^2+4}$$

$$= \frac{\sqrt{4(t^4+4t^2+4)}}{t^4+4t^2+4}$$

$$= \frac{2(t^2+2)}{(t^2+2)^2}$$

$$= \frac{2}{t^2+2} \Rightarrow \vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = \frac{\langle 4t, 4-2t^2, -4t \rangle}{(t^2+2)^2} \cdot \frac{t^2+2}{2} = \frac{\langle 4t, 4-2t, -4t \rangle}{2(t^2+2)}$$

$$= \frac{\langle 2t, 2-t, -2t \rangle}{t^2+2}$$

c) $K(t) = \dfrac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \dfrac{2}{(t^2+2)^2}$

30) $\vec{r}(t) = \langle t\ln t, t, e^{-t} \rangle \Rightarrow \vec{v}(t) = \langle \ln t + 1, 1, -e^{-t} \rangle \Rightarrow \vec{a}(t) = \langle \frac{1}{t}, 0, e^{-t} \rangle$

$\|\vec{v}(t)\| = \sqrt{(\ln t + 1)^2 + 1 + e^{-2t}}$

31) $\vec{v}(t) = \int \vec{a}(t)\,dt$

$= \int \langle 6t, 12t^2, -6t \rangle dt$

$= \langle 3t^2, 4t^3, -3t^2 \rangle + \vec{C}_1$

$\vec{v}(0) = \vec{C}_1 = \langle 1, -1, 3 \rangle$.

$\vec{r}(t) = \int \vec{v}(t)\,dt$

$= \langle t^3, t^4, -t^3 \rangle + \langle 1, -1, 3 \rangle t + \vec{C}_2$

$\vec{r}(0) = \vec{C}_2 = \langle 0, 0, 0 \rangle \Rightarrow \vec{r}(t) = \langle t^3 + t, t^4 - t, 3t - t^3 \rangle$

32) $\vec{r}(t) = \langle t + \frac{t^2}{2}, 1 - t, 2 + t^2 \rangle$ $\quad \vec{r}\,'(t) = \vec{v}(t) = \langle 1 + t, -1, 2t \rangle$

a) $\vec{v}(1) = \langle 2, -1, 2 \rangle$

b) $\|\vec{v}(1)\| = \sqrt{4 + 1 + 4} = 3$

33) $B \rightarrow A \rightarrow C$. Osculating Circles won't be tested. It's the circle whose curvature $\left(\frac{1}{R}\right)$ matches the curvature at that point, i.e. a circle of radius $\frac{1}{\kappa(t)}$ where $\kappa(t)$ is the curvature at $\vec{r}(t)$.