M273Q Multivariable Calculus **Old Exam 1**

Name and section: Key

Instructor's name: ______

Instructions: Closed book. No calculator allowed. Double-sided exam. NO CELL PHONES. **Show all work and use correct notation to receive full credit!** Write legibly.

$$\kappa(s) = \left\|\frac{d\mathbf{T}}{ds}\right\| \qquad \kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$$

$$\kappa(t) = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||} \qquad \kappa(t) = \frac{||\mathbf{r}'(t)\times\mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}$$

1. (1 credit ____) Decide whether the following quantities are vectors, scalars, or are nonsensical (that is, the statement is not defined or does not make sense)

   - ○ Vector ○ Scalar ● Nonsense — $(\mathbf{u}\cdot\mathbf{v})\times\mathbf{w}$
   - ○ Vector ● Scalar ○ Nonsense — $||\mathbf{u}\times\mathbf{v}||$
   - ● Vector ○ Scalar ○ Nonsense — $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$
   - ○ Vector ● Scalar ○ Nonsense — $\kappa(t) = \frac{||\mathbf{r}'(t)\times\mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}$

2. (1 credit ____) Determine whether the following equations describe a plane, a line, or neither in $\mathbf{R}^3$:

   - ○ Plane ● Line ○ Neither — $\mathbf{r}(t) =< 1,-1,5 > +t < 0,2,3 >$
   - ● Plane ○ Line ○ Neither — $x+y+z=1$
   - ● Plane ○ Line ○ Neither — $y = 5+z$
   - ○ Plane ● Line ○ Neither — $x(t) = 7+2t,\ y(t) = 11-5t,\ z(t) = \frac{t}{\pi}$

3. (1 credit ____) Find a **unit** vector parallel to the line $\mathbf{r}(t) =< t+4, -2+2t, -5-2t >$.

$$\vec{v} = \langle 1, 2, -2\rangle$$

$$\hat{e}_{\vec{v}} = \frac{\vec{v}}{||\vec{v}||} = \frac{1}{3}\langle 1, 2, -2\rangle$$

Problem	1	2	3	Total
Credit	1	1	1	3
GPA Credit Points Earned				

4. Let $\mathbf{a} = < 1, 2, -1 >$ and $\mathbf{b} = < 2, -1, 3 >$.

(a) (1 credit ____) Find $\|\mathbf{a}\|$.

$$\|\vec{a}\| = \sqrt{6}$$

(b) (1 credit ____) Find $\mathbf{b} \times \mathbf{a}$.

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 3 \\ 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \hat{k}$$

$$= (1-6)\hat{\imath} - (-2-3)\hat{\jmath} + (4+1)\hat{k}$$

$$= \langle -5, 5, 5 \rangle$$

(c) (1 credit ____) Find $\mathbf{a} \cdot \mathbf{b}$.

$$\vec{a} \cdot \vec{b} = \langle 1, 2, -1 \rangle \cdot \langle 2, -1, 3 \rangle$$

$$= 2 - 2 - 3$$

$$= -3$$

(d) (1 credit ____) Find $3\mathbf{a} - 2\mathbf{b}$.

$$3\vec{a} - 2\vec{b} = 3\langle 1, 2, -1 \rangle - 2\langle 2, -1, 3 \rangle$$

$$= \langle 3, 6, -3 \rangle + \langle -4, 2, -6 \rangle$$

$$= \langle -1, 8, -9 \rangle$$

(e) (1 credit ____) Is the angle between $\mathbf{a}$ and $\mathbf{b}$ acute (less that $\pi/2$), obtuse (greater than $\pi/2$), or neither?

Since $\vec{a} \cdot \vec{b} < 0$, the angle between them is obtuse.

Problem	4	Total
Credit	5	5
GPA Credit Points Earned		

5. (1 credit ____) Find a vector of length 4 that is orthogonal to both $\mathbf{a} = < 1, 2, -1 >$ and $\mathbf{b} = < 2, -1, 3 >$.

Note: $\vec{a} \times \vec{b} = \langle 5, -5, -5 \rangle$ (from #4)

So, $\vec{v} = 4\left(\dfrac{\vec{a} \times \vec{b}}{\|\vec{a} \times \vec{b}\|}\right)$

$= \dfrac{4}{5\sqrt{3}} \langle 5, -5, -5 \rangle = \dfrac{4}{\sqrt{3}} \langle 1, -1, -1 \rangle$

6. This question has two parts.

(a) (1 credit ____) Find the equation of the line through $P(3, 1, 0)$ and $Q(1, 4, -3)$.

$\vec{PQ} = \langle -2, 3, -3 \rangle$

$\vec{L}(t) = \langle 3, 1, 0 \rangle + t \langle -2, 3, -3 \rangle$

(b) (1 credit ____) Show that the line you found in part (a) is orthogonal to $x(t) = 3t$, $y(t) = 3 + 8t$, $z(t) = -7 + 6t$.

$\langle -2, 3, -3 \rangle \cdot \langle 3, 8, 6 \rangle = -6 + 24 - 18$

$= 0 \Rightarrow$ they're orthogonal

7. (2 credit ____) Find an equation for the tangent line to the curve $\mathbf{r}(t) = < t, t^2, t^3 >$ at the point $(1, 1, 1)$.

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$\vec{r}\,'(1) = \langle 1, 2, 3 \rangle$

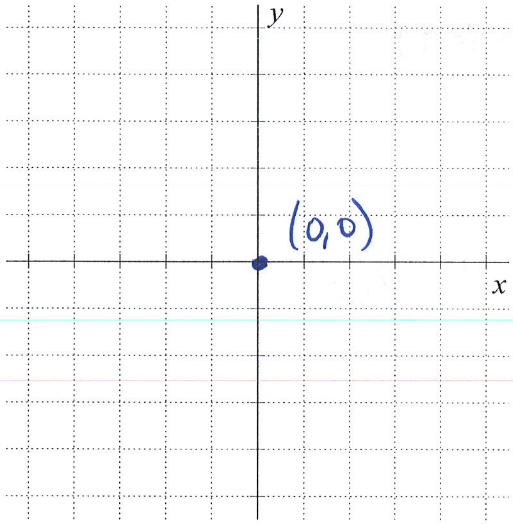
$\vec{L}(t) = \langle 1, 1, 1 \rangle + t \langle 1, 2, 3 \rangle$

Problem	5	6	7	Total
Credit	1	2	2	5
GPA Credit Points Earned				

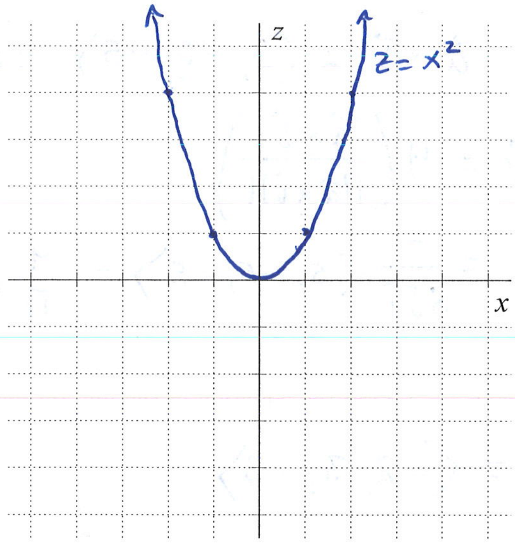
8. (4 credit ____) Provide a clear sketch of the following traces for the quadratic surface $z - x^2 - y^2 = 0$ in the given planes.

$x^2+y^2=0$

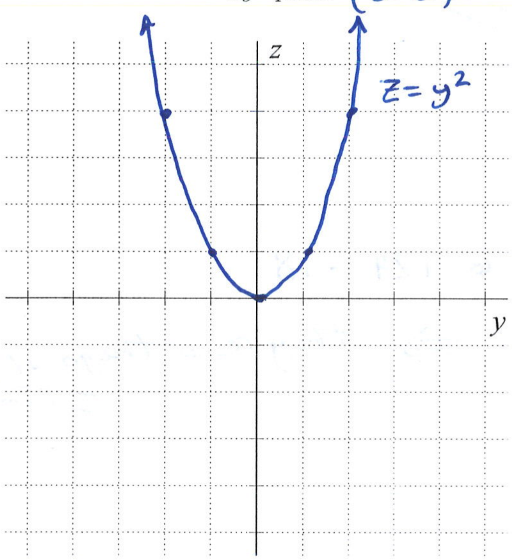
$\Rightarrow (0,0)$



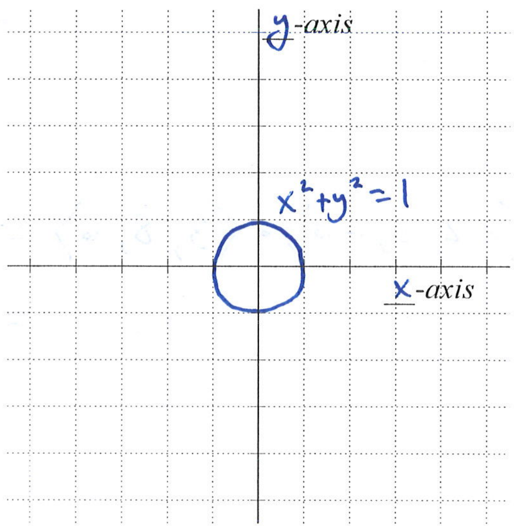
xy–plane $(z=0)$



xz–plane $(y=0)$

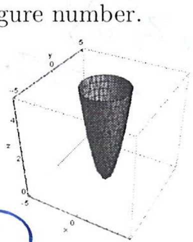


yz–plane $(x=0)$

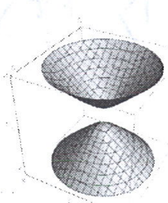


$z = 1$ label the appropriate axes.

9. (1 credit ____) Based on the traces you found above, identify the graph of $z - x^2 - y^2 = 0$ by circling the figure number.



I.



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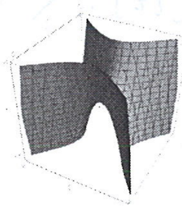
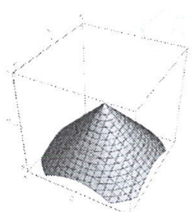
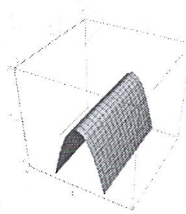


III.



IV.



V.

Problem	8	9	Total
Credit	4	1	5
GPA Credit Points Earned			

10. Given position $\mathbf{r}(t) = \langle 6\sin t, 6\cos t, 8t\rangle$, $a > 0$ at time t, find the following:

(a) (1 credit ____) The unit tangent vector $\mathbf{T}(t) =$

$\vec{r}\,'(t) = \langle 6\cos t, -6\sin t, 8\rangle$

$\|\vec{r}\,'(t)\| = \sqrt{36\cos^2 t + 36\sin^2 t + 64} = 10$

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{1}{10}\langle 6\cos t, -6\sin t, 8\rangle$

(b) (1 credit ____) The unit normal vector $\mathbf{N}(t) =$

$\vec{T}\,'(t) = \frac{1}{10}\langle -6\sin t, -6\cos t, 0\rangle$

$\|\vec{T}\,'(t)\| = \frac{1}{10}\sqrt{36\sin^2 t + 36\cos^2 t} = \frac{3}{5}$

$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = \frac{5}{3}\left(\frac{1}{10}\right)\langle -6\sin t, -6\cos t, 0\rangle$

(c) (1 credit ____) The curvature of the graph of $\mathbf{r}(t)$ at $t = 0$, $\kappa(0) =$ ______________.

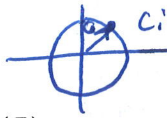
$K(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}$

$= \frac{3}{5} \cdot \frac{1}{10}$

$= \frac{3}{50}$

Problem	10	Total
Credit	3	3
GPA Credit Points Earned		

11. Let $\mathbf{c}(t) = \langle 6\sin 2t, 6\cos 2t \rangle$.



circle of radius 6.

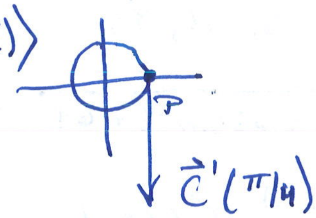
(a) (1 credit ____) Sketch $\mathbf{c}(t)$ for $0 \le t \le \pi$.

(b) (1 credit ____) Label the point corresponding to $\mathbf{c}\left(\frac{\pi}{4}\right)$ on your graph.

$\vec{c}\left(\frac{\pi}{4}\right) = \langle 6\sin(\pi/2), 6\cos(\frac{\pi}{2})\rangle$

$= \langle 6, 0 \rangle$

$P = (6, 0)$

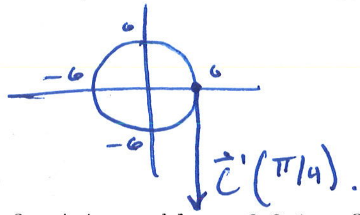


(c) (1 credit ____) Calculate $\mathbf{c}'\left(\frac{\pi}{4}\right)$.

$\vec{c}\,'(t) = \langle 12\cos(2t), -12\sin(2t)\rangle$

$\vec{c}\,'\left(\frac{\pi}{4}\right) = \langle 12\cos(\frac{\pi}{2}), -12\sin(\pi/2)\rangle = \langle 0, -12 \rangle$

(d) (1 credit ____) Sketch the vector $\mathbf{c}'\left(\frac{\pi}{4}\right)$ at the appropriate point on your graph.



12. Given $\mathbf{a} = \langle 3, -4, 4 \rangle$ and $\mathbf{b} = \langle 2, 2, 1 \rangle$, find vectors $\mathbf{a}_{||\mathbf{b}}$ and $\mathbf{a}_{\perp\mathbf{b}}$:

(a) (1 credit ____) $\mathbf{a}_{||\mathbf{b}} = \langle \frac{4}{9}, \frac{4}{9}, \frac{2}{9} \rangle$

$\vec{a}_{||\vec{b}} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b}$

$= \dfrac{6-8+4}{9}\vec{b} = \frac{2}{9}\langle 2,2,1\rangle = \langle \frac{4}{9}, \frac{4}{9}, \frac{2}{9} \rangle$

(b) (1 credit ____) $\mathbf{a}_{\perp\mathbf{b}} =$ ____________

$\vec{a}_{\perp\vec{b}} = \vec{a} - \vec{a}_{||\vec{b}}$

$= \langle 3, -4, 4\rangle - \langle \frac{4}{9}, \frac{4}{9}, \frac{2}{9}\rangle = \langle \frac{23}{9}, -\frac{40}{9}, \frac{34}{9} \rangle$

(c) (1 credit ____) Show that $\mathbf{a}_{\perp\mathbf{b}}$ is orthogonal to $\mathbf{b}$.

$\langle \frac{23}{9}, -\frac{40}{9}, \frac{34}{9} \rangle \cdot \langle 2, 2, 1 \rangle$

$= \frac{1}{9}(46 - 80 + 34)$

$= \frac{1}{9}(80 - 80)$

$= 0 \Rightarrow \vec{a}_{\perp\vec{b}} \perp \vec{b}$

Problem	11	12	Total
Credit	4	3	7
GPA Credit Points Earned			

13. A curve is parameterized by $\mathbf{r}(t) = < 3 + \cos 3t, 3 - \sin 3t, 4t >$.

(a) (1 credit ____) Find the arc length of the piece of the curve $0 \le t \le \frac{2\pi}{3}$.

$\vec{r}\,'(t) = \langle -3\sin(3t), -3\cos(3t), 4 \rangle$ $\quad \|\vec{r}\,'(t)\| = \sqrt{9\sin^2(3t) + 9\cos^2(3t) + 16}$

$= 5$

$$S = \int_0^{2\pi/3} 5\,dt = \frac{10\pi}{3}$$

(b) (1 credit ____) Re-parameterize the curve with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

$$S = \int_0^t 5\,du$$

$$S = 5t$$

$$t = \frac{S}{5} \Rightarrow \vec{r}(s) = \left\langle 3 + \cos\left(\frac{3s}{5}\right), 3 - \sin\left(\frac{3s}{5}\right), \frac{4s}{5} \right\rangle$$

Problem	13	Total
Credit	2	2
GPA Credit Points Earned		

14. (2 credit ____) Find an equation for the plane that contains the points $A(1,2,3)$, $B(2,4,2)$ that is parallel to $\mathbf{v} =< -3, -1, -2 >$.

$$\vec{AB} = \langle 1, 2, -1 \rangle$$

$$\vec{v} \times \vec{AB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -1 & -2 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} \hat{i} - \begin{vmatrix} -3 & -2 \\ 1 & -1 \end{vmatrix} \hat{j} + \begin{vmatrix} -3 & -1 \\ 1 & 2 \end{vmatrix} \hat{k}$$

$$= (1+4)\hat{i} - (3+2)\hat{j} + (-6+1)\hat{k}$$

$$= \langle 5, -5, -5 \rangle$$

$$5(x-1) - 5(y-2) - 5(z-3) = 0$$

Question	Points	Score
14	2	
Total:	2	

Page:	1	2	3	4	5	6	7	8	Total
Credit	3	5	5	5	3	7	2	2	32
GPA Credit Points Earned									

Name and section: ____________________