First Test Solutions, MATH 224, Spring 2007

1. (5 pts) Find the distance from $(3, -4, -2)$ to the xy-plane, the xz-plane, the yz-plane, the z-axis, and the origin.

The distances are $2, 4, 3, \sqrt{3^2 + 4^2} = 5$, and $\sqrt{3^2 + 4^2 + 2^2} = \sqrt{29} \approx 5.39$, respectively.

2. (5 pts) Find a vector of length 6 which has the opposite direction as $\langle 1, 2, -2\rangle$.

$\mathbf{v} = \frac{-6}{\sqrt{1^2+2^2+2^2}}\langle 1, 2, -2\rangle = -2\langle 1, 2, -2\rangle = \langle -2, -4, 2\rangle$.

3. (20 pts) Which of the following statements are true, which are false? Justify your answers.

- The dot product of two unit vectors is a number between -1 and 1. — True, because $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta = \cos\theta$, and cos has values between -1 and 1.
- The cross product of two unit vectors is a unit vector. — False, e.g. $\langle 1, 0, 0\rangle \times \langle 1, 0, 0\rangle = \langle 0, 0, 0\rangle$.
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$ for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. — True, because $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$.
- If a force of magnitude 5 Newton moves an object along a vector of length 2 meters, the work done is 10 Joules. — False, the work done is $10 \cos\theta$ Joules, it depends on the angle between the force and displacement vectors.
- If the derivative of a space curve is a constant vector, then the curve is a line. — True, if $\mathbf{r}'(t) = \langle a, b, c\rangle$, then integration yields $\mathbf{r}(t) = \langle at + C_1, bt + C_2, ct + C_3\rangle$, which is the standard parameterization of a line.

4. (15 pts) Reduce the equation $y^2 + 2z^2 - 2x - 6y + 4z = 0$ to standard form, classify the surface, and sketch it.

The left side is equal to $(y - 3)^2 - 9 + 2(z + 1)^2 - 2 - 2x$, so the equation becomes $2(x + 11/2) = (y - 3)^2 + 2(z + 1)^2$, which is an elliptic paraboloid along the x-axis with tip at $(-11/2, 3, -1)$.

5. (10 pts) Find the area of the triangle with vertices $(-2, 1, 0)$, $(0, 1, 2)$, and $(1, 0, 1)$.

If we call the three points P, Q, and R, respectively, then $\overrightarrow{PQ} = \langle 2, 0, 2\rangle$ and $\overrightarrow{PR} = \langle 3, -1, 1\rangle$, so $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2, 4, -2\rangle$, which has length $\sqrt{2^2 + 4^2 + 2^2} = \sqrt{24}$. The area of the triangle is half of that, i.e $\sqrt{24}/2 = \sqrt{6} \approx 2.45$.

6. (10 pts) (a) Where does the line through $(2, 1, 0)$ and $(3, 0, 2)$ intersect the plane $x + y + 2z = 6$?

The direction vector for the line is $\mathbf{v} = \langle 1, -1, 2\rangle$, so the equation of the line is $x = 2+t$, $y = 1-t$, $z = 2t$. Plugging these into the equation for the plane gives $2+t+1-t+4t = 4t + 3 = 6$, which has the solution $t = 3/4$. Thus $x = 2 + 3/4 = 11/4 = 2.75$, $y = 1 - 3/4 = 1/4 = 0.25$, $z = 2 \cdot 3/4 = 3/2 = 1.5$ is the point of intersection.

(10 pts) (b) Find the angle of intersection. (You do not need to calculate the numerical value of square roots and inverse trigonometric functions.)

The angle of intersection of a plane with normal vector $\mathbf{n}$ and a line with direction vector $\mathbf{v}$ is $\pi/2 - \theta$, where θ is the angle between $\mathbf{n}$ and $\mathbf{v}$. (This should become clear from a sketch.) In our case, $\mathbf{n} = \langle 1, 1, 2\rangle$, and $\mathbf{v} = \langle 1, -1, 2\rangle$, so $\mathbf{n} \cdot \mathbf{v} = 1 - 1 + 4 = 4$, and $|\mathbf{n}| = |\mathbf{v}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$, so $\theta = \cos^{-1} \frac{4}{\sqrt{6}\cdot\sqrt{6}} = \cos^{-1} \frac{2}{3} \approx 0.84$ and the angle we were looking for is $\pi/2 - \cos^{-1}\left(\frac{2}{3}\right) \approx 0.73 \approx 42°$.

7. (10 pts) Find parametric equations for the tangent line to the curve $x = \cos t$, $y = 1 + 2t^{3/2}$, $z = \sin t$, at the point $(1, 1, 0)$.

We have $\mathbf{r}(t) = \langle \cos t, 1 + 2t^{3/2}, \sin t$ and $\mathbf{r}'(t) = \langle -\sin t, 3t^{1/2}, \cos t\rangle$. The parameter corresponding to the point $(1, 1, 0)$ is $t = 0$, and $\mathbf{r}'(0) = \langle 0, 0, 1\rangle$, so the equations for the line are $x = 1$, $y = 1$, $z = t$.

8. (15 pts) Find the length of the curve $x = \cos t$, $y = 1 + 2t^{3/2}$, $z = \sin t$, $0 \le t \le 2\pi$.

The length is $L = \int_0^{2\pi} |\mathbf{r}'(t)|\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + (3t^{1/2})^2 + \cos^2 t}\, dt = \int_0^{2\pi} \sqrt{1 + 9t}\, dt =$ $\left[\frac{2}{27}(1 + 9t)^{3/2}\right]_0^{2\pi} = \frac{2}{27}\left[(1 + 18\pi)^{3/2} - 1\right] \approx 32.3.$