Third Practice Test Solutions
MATH 224, Fall 2007

1. Calculate $\displaystyle\int_0^1\int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\,dx\,dy$ by reversing the order of integration.

$$\begin{aligned}\int_0^1\int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\,dx\,dy &= \int_0^1\int_0^{x^2} \frac{ye^{x^2}}{x^3}\,dy\,dx = \int_0^1 \left[\frac{y^2e^{x^2}}{2x^3}\right]_{y=0}^{y=x^2} dx \\ &= \int_0^1 \frac{x^4e^{x^2}}{2x^3}\,dx = \int_0^1 \frac{x}{2}e^{x^2}\,dx = \left[\frac{1}{4}e^{x^2}\right]_{x=0}^{x=1} = \frac{e-1}{4}\end{aligned}$$

2. Consider a spherical shell E between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ with mass density equal to the distance to $(0,0,0)$. Find the total mass.

In spherical coordinates this shell is described by $1 \le \rho \le 2$, $0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$. The density is given by ρ, so

$$m = \int_1^2\int_0^\pi\int_0^{2\pi} \rho\cdot\rho^2\sin\phi\,d\theta\,d\phi\,d\rho = 2\pi\int_1^2 \rho^3\,d\rho\int_0^\pi \sin\phi\,d\phi = 15\pi.$$

3. Calculate $\displaystyle\int_C y^3\,ds$, where C is the part of the graph $y=2x^3$ from $(0,0)$ to $(1,2)$.

Standard parameterization is $\mathbf{r}(t) = \langle t, 2t^3\rangle$, $0 \le t \le 1$, with $\mathbf{r}'(t) = \langle 1, 6t^2\rangle$, and thus

$$\int_C y^3\,ds = \int_0^1 8t^9\sqrt{1+36t^4}dt.$$

Due to a typo (it was supposed to be $\int_C y\,ds$), this integral turned out quite a bit harder to solve than anticipated. Don't worry if you gave up at this point. For completeness' sake, the rest of the solution follows. (Feel free to ignore it, it is indeed scary.)

Substitute $u = 1 + 36t^4$ to get $du = 144t^3 dt$, and since $t^6 = \frac{(u-1)^{3/2}}{216}$ we get

$$\begin{aligned}
\int_1^{37} \frac{8}{144 \cdot 216}(u-1)^{3/2}u^{1/2}du &= \frac{1}{3888}\int_1^{37}(u-1)\sqrt{u(u-1)}du \\
&= \frac{1}{3888}\int_1^{37}(u-1)\sqrt{\left(u-\frac{1}{2}\right)^2 - \frac{1}{4}}\,du.
\end{aligned}$$

Now we substitute $r = u - \frac{1}{2}$ and get

$$\begin{aligned}
&\frac{1}{3888}\int_{1/2}^{73/2}\left(r-\frac{1}{2}\right)\sqrt{r^2-\frac{1}{4}}\,dr \\
&\quad = \frac{1}{3888}\int_{1/2}^{73/2} r\sqrt{r^2-\frac{1}{4}}\,dr - \frac{1}{7776}\int_{1/2}^{73/2}\sqrt{r^2-\frac{1}{4}}\,dr.
\end{aligned}$$

The first integral is easy to solve directly (or by substituting $v = r^2 - 1/4$), and the second one can be solved mysteriously by substituting $r = (e^w + e^{-w})/4$. Fortunately it is also one of the standard integrals in the back of the textbook (table of integrals, number 39, with $a = 1/2$), so we get

$$\begin{aligned}
&\frac{1}{3888}\left[\frac{1}{3}\left(r^2-\frac{1}{4}\right)^{3/2}\right]_{1/2}^{73/2} - \frac{1}{7776}\left[\frac{r}{2}\sqrt{r^2-\frac{1}{4}} - \frac{1}{8}\ln\left|r+\sqrt{r^2-\frac{1}{4}}\right|\right]_{1/2}^{73/2} \\
&\quad = \frac{37\sqrt{37}}{54} - \frac{73\sqrt{37}}{5184} + \frac{1}{62208}\ln(73+12\sqrt{37}) \approx 4.082
\end{aligned}$$

Certainly no integral like this will be on the test.

4. Which of the following vector fields are conservative? Find a potential for one of them and use it to calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the arc of the unit circle from $(1,0)$ to $(0,1)$ in counterclockwise direction.

$$\mathbf{F}_1(x,y) = \langle x^2, x^2\rangle \qquad \mathbf{F}_2(x,y) = \langle 2xy, x^2\rangle$$
$$\mathbf{F}_3(x,y) = \langle e^y, e^x\rangle \qquad \mathbf{F}_4(x,y) = \langle e^x, e^y\rangle$$

We have to test whether $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in all four cases. This test shows that $\mathbf{F}_2$ and $\mathbf{F}_4$ are conservative, whereas $\mathbf{F}_1$ and $\mathbf{F}_3$ are not. Potentials for the conservative fields are $f_2(x,y) = x^2y$ and $f_4(x,y) = e^x + e^y$. From the fundamental theorem for line integrals we get that

$$\begin{aligned}
\int_C \mathbf{F}_2\cdot d\mathbf{r} &= f_2(0,1) - f_2(1,0) = 0^2\cdot 1 - 1^2\cdot 0 = 0\\
\int_C \mathbf{F}_4\cdot d\mathbf{r} &= f_4(0,1) - f_4(1,0) = e^0 + e^1 - (e^1 + e^0) = 0
\end{aligned}$$

5. Use Green's Theorem to evaluate $\int_C \sqrt{1+x^2}dx + x(1+\sin y)\,dy$, where C is the unit circle, parameterized in counterclockwise direction. (Don't even try to solve this integral directly.)

With $P(x,y) = \sqrt{(1+x^2)}$ and $Q(x,y) = x(1+\sin y)$, we have $\frac{\partial P}{\partial y}(x,y) = 0$ and $\frac{\partial Q}{\partial x} = 1 + \sin y$. Applying Green's Theorem gives

$$\begin{aligned}
\int_C \sin(1+x^2)dx + x(1+y)\,dy &= \iint_D (1+\sin y)\,dA\\
&= \iint_D 1\,dA + \iint_D \sin y\,dA = \pi + 0 = \pi,
\end{aligned}$$

where the first integral is just the area of the unit disk D, and the second one is zero by symmetry. If this is not immediately clear, here is one example where it might be easier not to use polar coordinates. The second integral is

$$\begin{aligned}
\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin y\,dy\,dx &= \int_{-1}^{1} [-\cos y]_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\,dx\\
&= \int_{-1}^{1}\left(-\cos(-\sqrt{1-x^2}) + \cos\sqrt{1-x^2}\right)\,dx = 0,
\end{aligned}$$

since $\cos(-\sqrt{1-x^2}) = \cos\sqrt{1-x^2}$.