16.2 Line Integrals II

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Integrals Over Piecewise Curves

Definition

If $\mathcal{C}$ consists of n smooth curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$, we write

$$\mathcal{C} = \mathcal{C}_1 + \ldots + \mathcal{C}_n,$$

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} + \ldots + \int_{\mathcal{C}_n} \mathbf{F} \cdot d\mathbf{s},$$

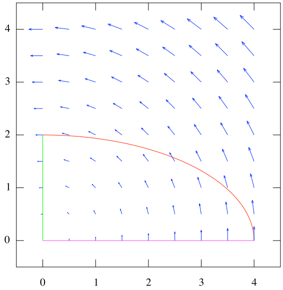
and

$$\int_{\mathcal{C}} f\, ds = \int_{\mathcal{C}_1} f\, ds + \ldots + \int_{\mathcal{C}_n} f\, ds.$$

Piecewise Curve Integration Example I

Example

Calculate $\int_{\mathcal{C}} x\,dy - y\,dx$ where $\mathcal{C}$ is the boundary of the quarter-ellipse $x^2 + 4y^2 = 16$, $x, y \geq 0$, in counterclockwise orientation.



Sketch of the vector field

$$\mathbf{F}(x, y) = \langle -y, x \rangle$$

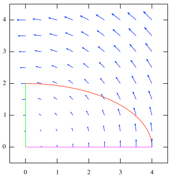
and (starting at $\mathbf{0}$) the quarter ellipse

$$\mathcal{C} = \mathcal{C}_1 \text{ (magenta)} + \mathcal{C}_2 \text{ (red)} + \mathcal{C}_3 \text{ (green)}$$

Piecewise Curve Integration Example II

Example

Calculate $\int_{\mathcal{C}} x\,dy - y\,dx$ where $\mathcal{C}$ is the boundary of the quarter-ellipse $x^2 + 4y^2 = 16$, $x, y \geq 0$, in counterclockwise orientation.



Parametrize $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$

$$\begin{aligned}
\mathcal{C}_1 &: \mathbf{c}_1(t) = \langle t, 0\rangle, \quad 0 \leq t \leq 4 \\
\mathcal{C}_2 &: \mathbf{c}_2(t) = \langle 4\cos t, 2\sin t\rangle, \quad 0 \leq t \leq \pi/2 \\
\mathcal{C}_3 &: \mathbf{c}_3(t) = \langle 0, 2 - t\rangle, \quad 0 \leq t \leq 2
\end{aligned}$$

Piecewise Curve Integration Example III

Example

Calculate $\int_{\mathcal{C}} x\,dy - y\,dx$ where $\mathcal{C}$ is the boundary of the quarter-ellipse $x^2 + 4y^2 = 16$, $x, y \geq 0$, in counterclockwise orientation.

Integrate over $\mathcal{C}_1$

$$\mathbf{c}_1(t) = \langle t, 0 \rangle, \quad 0 \leq t \leq 4$$

$$dx = x'(t)\,dt = dt, \quad dy = y'(t)\,dt = 0.$$

$$\int_{\mathcal{C}_1} x\,dy - y\,dx = \int_0^4 t \cdot 0 - 0 \cdot 1\,dt = 0.$$

Piecewise Curve Integration Example IV

Example

Calculate $\int_{\mathcal{C}} x\,dy - y\,dx$ where $\mathcal{C}$ is the boundary of the quarter-ellipse $x^2 + 4y^2 = 16$, $x, y \geq 0$, in counterclockwise orientation.

Integrate over $\mathcal{C}_2$

$$\mathbf{c}_2(t) = \langle 4\cos t, 2\sin t \rangle, \quad 0 \leq t \leq \pi/2$$

$$dx = x'(t)\,dt = -4\sin t\,dt, \quad dy = y'(t)\,dt = 2\cos t\,dt.$$

$$\begin{aligned}
\int_{\mathcal{C}_2} x\,dy - y\,dx &= \int_0^{\pi/2} (4\cos t)(2\cos t) - (2\sin t)(-4\sin t)\,dt \\
&= \int_0^{\pi/2} 8\cos^2 t + 8\sin^2 t\,dt = \int_0^{\pi/2} 8\,dt = 4\pi.
\end{aligned}$$

Piecewise Curve Integration Example V

Example

Calculate $\int_{\mathcal{C}} x\,dy - y\,dx$ where $\mathcal{C}$ is the boundary of the quarter-ellipse $x^2 + 4y^2 = 16$, $x, y \geq 0$, in counterclockwise orientation.

Integrate over $\mathcal{C}_3$

$$\mathbf{c}_3(t) = \langle 0, 2 - t \rangle, \quad 0 \leq t \leq 2$$

$$dx = x'(t)\,dt = 0, \quad dy = y'(t)\,dt = -\,dt.$$

$$\int_{\mathcal{C}_3} x\,dy - y\,dx = \int_0^2 0 \cdot (-1) - (2-t) \cdot 0\,dt = 0.$$

Piecewise Curve Integration Example V

Example

Calculate $\int_{\mathcal{C}} x\,dy - y\,dx$ where $\mathcal{C}$ is the boundary of the quarter-ellipse $x^2 + 4y^2 = 16$, $x, y \geq 0$, in counterclockwise orientation.

Add up the results

$$\begin{aligned}\int_{\mathcal{C}} x\,dy - y\,dx &= \int_{\mathcal{C}_1} x\,dy - y\,dx + \int_{\mathcal{C}_2} x\,dy - y\,dx + \int_{\mathcal{C}_3} x\,dy - y\,dx \\ &= 0 + 4\pi + 0 = 4\pi.\end{aligned}$$