Attribute Sampling Plans

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ATTRIBUTE SAMPLING PLANS

It is necessary in the ever competitive business world to ensure that companies deliver what they promise. If a company repeatedly produces sub-standard products then consumers will rarely, if ever, purchase products from that company. Therefore, it is necessary for a company to test its own products before they are relinquished to the consumer. However, it is not always clear how to best ensure the desired level of outgoing quality. It might seem best to test every unit produced. This plan is in many cases unfeasible while in some cases it is undesirable. For instance, if the testing of a unit necessitates the destruction of the unit or the cost of inspecting each unit is high relative to the value of the product, then testing all of the units is not possible. Also, if the inspection of the units is dependent on a tedious process, then mistakes would be made in some cases which would lead to rejecting good products and accepting bad products. Therefore, in many cases, some type of sampling inspection process would be very beneficial to the producer. This is not to say that sampling does not have drawbacks, because all sampling plans are imperfect by nature. When sampling it is possible to mistakenly reject good lots and accept bad lots. Also, in comparison to 100% inspection, sampling would produce less information about the product or process being tested and, while it decreases information produced, it simultaneously is more demanding with respect to planning and recordkeeping.

Having stated the disadvantages it is necessary to consider the advantages. Possibly the most important is that sampling is more cost effective than 100% inspection. Sampling requires fewer people to handle fewer units. This alone translates into savings in terms of manpower, less spent on the inspection of each lot, and decreases the handling of units which decreases the possibility that units will be damaged. Also, less testing generally incurs less inspection error. Sampling can be utilized even in the cases of destructive testing.

If an inpection sampling plan is adopted for use, it is then necessary to determine what exactly is to be measured. There are two types of plans based on two distinct types of measurements: sampling by attributes and sampling by variables. Sampling by attributes involves either counts of articles or events as the observed response. Sampling by variables involves a continuous measurement as the observed response. For example, if a company was interested in determining if a windshield of a car contained fewer than three minor flaws or if a rope could hold up to 100kgs, then the company would need to sample based on attributes. However, if a company was interested in the precise tensile strength of the rope then the company would need to sample based on a variable. Perhaps the distinctions above seem fairly trivial, but they are necessary to determine what sampling plan is preferred, or even if a certain sampling plan is applicable. In this paper, only attribute sampling will be discussed. First, single sampling plans will be discussed. Then the discussion is to be extended to double sampling and finally to sequential sampling.

SINGLE SAMPLING PLANS

Single sampling plans depend on one sample to determine if the inspected lot is to be rejected. Double and multiple sampling plans function under the assumption that on the first inspection the lot may fall into a nebulous area where the sample did not produce the support that is necessary to either accept or reject the lot. In this case, one or more additional samples from the lot are necessary to determine its fate.

Suppose the lot size N refers to the number of units in a lot. Then a lot sampling plan is generally defined by two characteristics. The first is n which is the number of units selected for testing from the N units in the lot. The second is c which is the maximum number of inspected defective units observed before the lot is rejected. If the sample of size n is taken and N is large, then the distribution of the sample of the number of defective units is, in fact Hypergeometric for any single N, but the process is approximately binomial with the parameters n and p where p is the proportion of defective units. The probability of finding d defective units is:

$$P(d) \approx \frac{n!}{d!(n-d)!}p^d(1-p)^{n-d}. \tag{1}$$

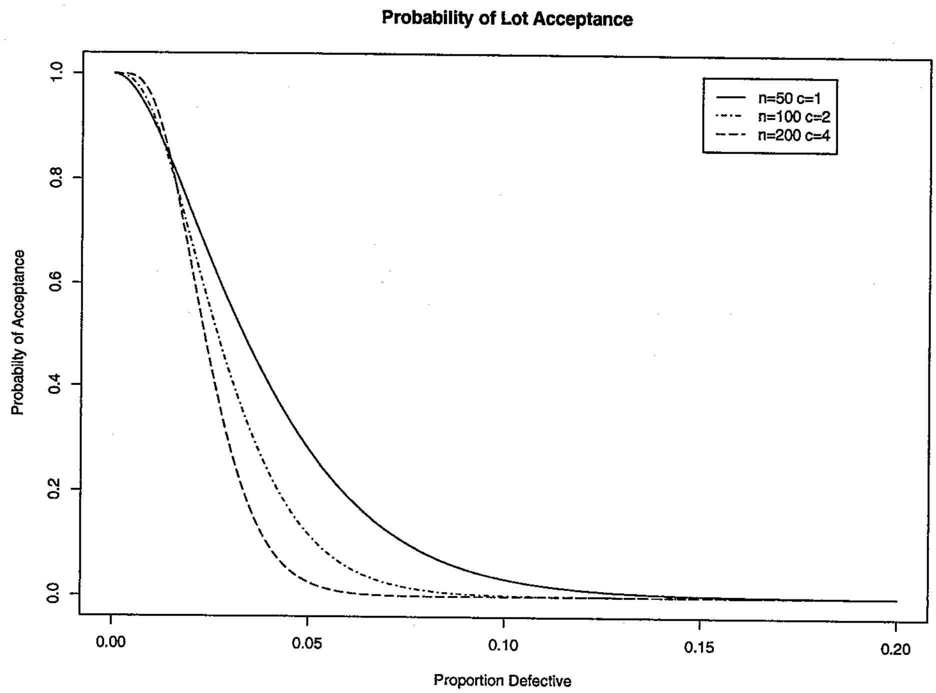
From that equation it is easy to see that the probability, P_a, of accepting any given lot is the probability that $d \leq c$ which is:

$$P_a = \sum_{d=0}^{c} \frac{n!}{d!(n-d)!}p^d(1-p)^{n-d}. \tag{2}$$

From this equation it is possible to describe multiple curves, known as operating-characteristic (OC) curves. Each OC curve is a plot of P_a vs. p for a given n and c. There are two types of OC curves: A and B. Type-A curves rely on the hypergeometric distribution to find the sampling distribution of the exact number of defective units from a single lot of finite size. Type-A curves are generally not used because the producer is interested in sampling from a process and does not intend to produce only one lot. Type-B curves are used to calculate the exact sampling distribution probability of lot acceptance. Any P_a calculated on a type-B curve will yield a higher value than it would if the type-A curve is used. However, it is still effective to use a type-B curve as an estimate of the type-A curve. Further, if the sampling fraction $\frac{n}{N}, \leq 0.1$, the type-A and type-B curves are almost identical (Grant p. 439). All following discussions will utilize and refer to the type-B curve. Because different OC curves can be formed by varying n and c when given the same lot fraction defective, it is necessary to determine how strict the producer wishes to be with respect to accepting lots. An ideal sampling plan would have a curve that would proceed horizontally at $P_a = 1$ until it crossed the point where the lot would be considered unacceptable, and then it would proceed vertically down to $P_a = 0$. This step function, however, is unobtainable in practice. In practice it is necessary for the producer to specify desirable points on the OC curve. This allows the producer to minimize the risk that the consumer will obtain defective merchandise while simultaneously protecting the profit level of the company. There are three main points of interest on an OC curve. The producers, by nature, would be interested in knowing what level of quality is needed in order to achieve a certain level of acceptance. For example, a producer would want to know what level of quality (at the unit level) they need to produce in order to have lots accepted 90% of the time. The consumer, on the other hand, would be most interested at a point they define as an acceptable quality level, AQL. The AQL is the lowest quality that the consumer would regard as acceptable for a process average. The consumer would also be interested in the lot tolerance percent defective, LTPD. The LTPD indicates the lowest level of quality that a consumer is willing to accept. Both the AQL and the LTPD are not characteristics of the OC curve, they are merely measures taken from the OC curve based on which consumers can compare similar products. For example, if a consumer if going to buy batteries and is faced with two different brands that have similar specifications, a rational consumer would select the batteries with the higher AQL or LTPD.

It is common to design a sampling plan based on a specified OC curve. This can be done by first defining two points. The first point is (p_1, α) where p_1 is defined as the point where the lot quality level should be accepted with a relatively high probability and α, is the probability of rejecting the lot if it is, in fact, satisfactory. The second point of interest is (p_2, β) where p_2 is the point where the lots with that quality level should be accepted with relatively low probability and β is the probability of accepting the lot when it is "bad." It is important to note that in cases when p_1 is defined as the AQL and p_2 is the LPTD then $1 - \alpha$ is referred to as the the producer's risk, and β is the consumer's risk.

Once these two points are defined it is then necessary to know how to modify the sampling scheme to produce an OC curve that does indeed go through these two points. Given that the binomial distribution defines the OC curve, it is necessary to note that the only two variables that can be set by the sampler are n and c. If the ratio between n and c is kept constant and n is reduced, the resulting OC curve will flatten out. A flatter OC curve will result in larger $1 - \alpha$ and β which will by definition increase the risk of the producer and consumer. If c is changed without changing n, then the OC curve will move left or right with little deformation with respect to the shape of the curve. From this it follows that a reduction in c increases $1 - \alpha$ while β decreases. Therefore, when $c = 0$ the producers risk is extremely high which would make it desirable to form the sampling plan in such a way that at least one defective unit is acceptable. At this stage is also helpful to note that more consistent results are obtained when n is held constant as opposed to varying it with respect to N. The relationship between n and c are illustrated in the following plot.



Once a sampling plan is devised, it is necessary to decide how to handle lots that did not pass inspection. There are different methods for dealing with rejected lots, most of which involve 100% inspection. After the lot is inspected and all of the defective units are removed, some sort of rectification scheme needs to be implemented. The most common rectification scheme is to simply replace the defective units with acceptable units. Once this rectification has been implemented it is possible to assess the average outgoing quality, AOQ. In deriving the AOQ it is necessary to assume three things. First, there are no defective units in the sample of size n because those found have been replaced with acceptable units. Second, if the lot has been rejected, the remaining units, $N - n$ in the lot contain no defectives. And third, if the lot was not rejected, the lot contains $p(N - n)$ defective units. Therefore, the $AOQ = P_a p \frac{N-n}{N}$. Note when N is large relative to n the $AOQ \approx P_a p$. The maximum value of the AOQ, called the average outgoing quality limit (AOQL) represents the worst average quality that would result from this rectification scheme teamed with the sampling scheme. The AOQL assures the consumer that on average the lots will not have a worse quality level than the AOQL, but this does not hold for any single lot.

For the producer it is necessary, in order to properly implement a sampling scheme, to know the average total inspected per lot, ATI. Because every sample unit will be inspected, at least n units will be inspected. If the lot is rejected then remaining $N - n$ units will also be inspected. Therefore, the $ATI = (1 - P_a)(N - n) + n$. Because this is an average, it does not imply that any single lot will have ATI units inspected.

DOUBLE SAMPLING PLANS

An alternative to a single sampling plan is a double sampling plan. A double sampling plan is defined by six characteristics:

- n_1 is the size of the first sample
- c_1 is the acceptance number for the first sample
- d_1 is the number of defectives found in the first sample
- n_2 is the size of the second sample
- c_2 is the combined acceptance number for both samples
- d_2 is the number of defectives found in the second sample

In double attribute sampling, an initial sample is taken and inspected for defective units. If $d_1 \leq c_1$ then the lot is accepted or if $d_1 > c_2$ the lot is rejected. However, if $c_1 < d_1 \leq c_2$ then a second sample is taken and is inspected for defective units. Then if $d_1 + d_2 \leq c_2$ the lot is accepted, or if $d_1 + d_2 > c_2$ the lot is rejected. For example, suppose $n_1 = 100$, $c_1 = 1$, and after inspection, d_1 is revealed to be two. This would result in a failure to accept the lot based on this initial sample. Then, the company decides to take a second sample where $n_2 = 300$, $c_2 = 4$, and upon inspection $d_2 = 0$. In this case, $d_1 + d_2 = 2$ which is less than c_2. Therefore, the lot is accepted.

The single most important advantage that double attribute sampling has over single attribute sampling is that it has the potential to reduce the number of units that are inspected. It is possible to sample so that the initial sample size for double sampling is smaller than the sample required in single sampling while still providing the same level of protection against faulty units in both cases. Hence, there is a definite advantage if the initial sample passes inspection or exceeds the acceptance number for both samples. If the lot is not accepted, there is potential benefit if not all of the second sample needs to be inspected in order to classify the lot as unacceptable, which is known as curtailment. However, there are two main disadvantages to double samping plans. The first disadvantage being that on occasion more units will be inspected under a double sampling plan in order to provide the same level of protection as a single sampling plan. The second disadvantage is that double attribute sampling is more complex which is necessarily harder to keep track of administratively.

A double sampling plan, like the single sampling plan, can be represented by an OC curve. In double sampling $P_a = P_a^1 + P_a^2$. Where P_a^1 is the probability of accepting the lot based on the first sample, and P_a^2 is the probability of accepting the lot based on the second sample. In general, the equations for P_a^1 and P_a^2 are:

$$P_a^1 = \sum_{d_1=0}^{c_1} \frac{n_1!}{d_1!(n_1-d_1)!} p^{d_1}(1-p_1)^{n_1-d_1}. \tag{3}$$

$$P_a^2 = P[d_1 = c_1+1]P[d_2 \leq c_2 - d_1] + P[d_2 = c_1+2]P[d_2 \leq c_2 - c_1] + \ldots P[d_1 = c_2]P[d_2 = 0]. \tag{4}$$

Using the case with $n_1 = 100$, $c_1 = 1$, $n_2 = 300$, and $c_2 = 4$ as an example:

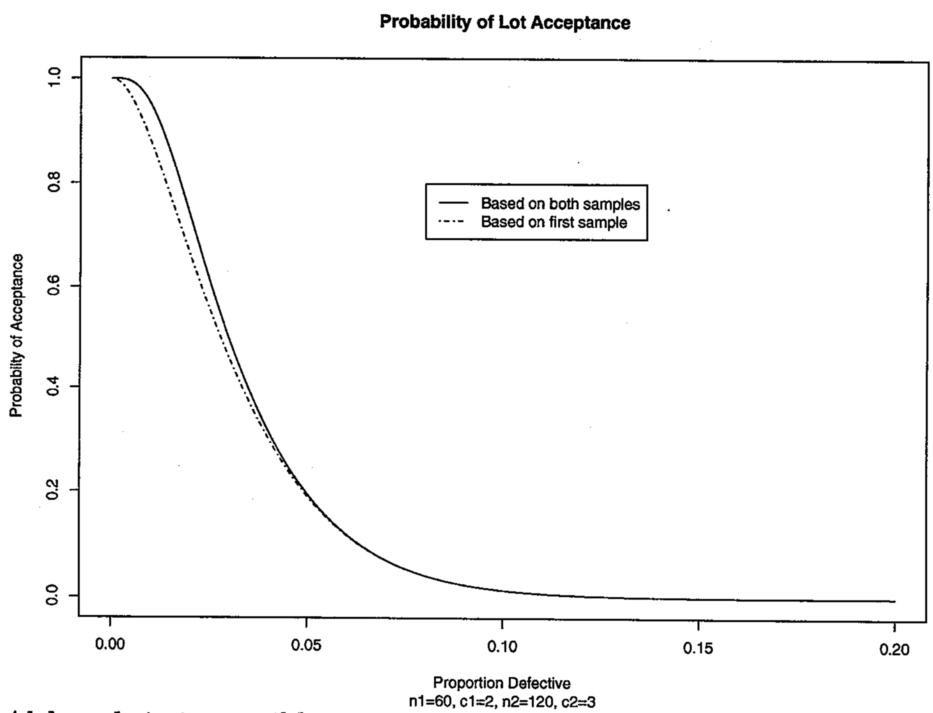
$$P_a^1 = \sum_{d_1=0}^{1} \frac{100!}{d_1!(100-d_1)!} p^{d_1}(1-p)^{100-d_1}. \tag{5}$$

$$P[d_1 = 2, d_2 \leq 2] = \frac{100!}{2!98!} p^2(1-p)^{98} \times \sum_{d_2=0}^{2} \frac{400!}{d_2!(400-d_2)!} p^{d_2}(1-p)^{400-d_2} \tag{6}$$

$$P[d_1 = 3, d_2 \leq 1] = \frac{100!}{3!97!} p^3(1-p)^{97} \times \sum_{d_2=0}^{1} \frac{400!}{d_2!(400-d_2)!} p^{d_2}(1-p)^{400-d_2} \tag{7}$$

$$P[d_1 = 4, d_2 = 0] = \frac{100!}{4!96!} p_1^4(1-p)^{96} \times \frac{400!}{0!(400-0)!} p^0(1-p)^{400} \tag{8}$$

When equations 5, 6, 7, and 8 are summed it yields the P_a for a double attribute sampling plan which can be compared to the P_a for a single sampling plan. As is the case with single attribute sampling, it is possible to produce a sampling plan to match a desired OC curve where the defining points are $(p_1, 1-\alpha)$ and (p_2, β). The following graph illustrates the probability of accepting the lot based on the first sample versus accepting the lot based on both samples.



Although it is possible to create a reference table given equations 3 and 4, there are, however, many tables in existence so this exercise is often unneeded (Duncan 189). In the aforementioned tables, there are five values listed: R, c_1, c_2, $P = .95$, and $P = .1$. In order to select the right table, it is first necessary to know the relationship between n_1 and n_2, i.e.

to know what value of $k = \frac{n_2}{n_1}$. Once the right table is found, the plans are listed according to their correspondence with $R = \frac{p_2}{p_1}$. Also, the selected table will aid in the calculation of n_1 and therefore n_2. The columns labeled $P = .95$ and $P = .1$ correspond to the approximate values for pn_1 at $\alpha = .05$ and $\beta = .1$ respectively. The value for $P = .95$ should be used in the equation $n_1^* = pn_1/p_1$ to calculate n_1^* where n_1^* is the size of the first sample when α is held at 0.05. Since the table would have been selected by the value of k, the value of n_2 can be calculated from n_1.

If the OC curves for single and double attribute sampling are essentially the same, then a reasonable way to choose between them is by comparing the average sample number, ASN. For single sampling, the sample number is always constant. Double sampling methods are not so obvious. If the first sample is accepted or there are sufficient defective units to reject the entire lot then there will be only n_1 units inspected. The probability that this event occurs, P_1 ,can be expressed as

$$P_1 = P[d_1 \leq c_1] + P[d_1 > c_2]. \tag{9}$$

If a second sample is required and curtailment is not used, then n_1+n_2 units will be inspected. Which yields:

$$ASN = n_1 P_1 + (n_1 + n_2)(1 - P_1) = n_1 + n_2(1 - P_1). \tag{10}$$

However, if the producer decides to implement curtailment, the equation becomes even more involved. There will always be n_1 units inspected. Now, two different cases must be considered. The first would occur if, in the course of inspecting the second sample, insufficient defectives are found and the lot is not rejected. This number can is expressed as:

$$\sum_{i=c_1+1}^{c_2} P(n_1, i) n_2 P_1(n_2, c_2 - i) \tag{11}$$

where $P(n_1, i)$ is defined to be the binomial probability of finding exactly i defective units out of a sample of n_1 units. $P_1(n_2, c_2)$ is the binomial probability of finding $c_2 - i$ defective units in a sample of size n_2. The second case would occur if the second sample did contain enough defective units to disqualify the sample. The contribution to the ASN from this case is(Burr 7):

$$\sum_{i=c_1+1}^{c_2} \frac{c_2 - i + 1}{p} P(n_1, i) P_2(n_2 + 1, c_2 - i + 2). \tag{12}$$

Where $P_2(n_2+1, c_2-i+2)$ is the binomial probability of observing $c_2 = i+2$ defective units in a sample of size $n_2 + 1$. Combining the three equations yields:

$$ASN = n_1 + \sum_{i=c_1+1}^{c_2} P(n_1, i)[n_2[P_1(n_2, c_2 - i) + \frac{c_2 - i + 1}{p} P_2(n_2 + 1, c_2 - i + 2)]. \tag{13}$$

Calculating the AOQ for double attribute sampling is much like that for single sampling. It is calculated under the assumption that all defectives found in the sample or in 100%

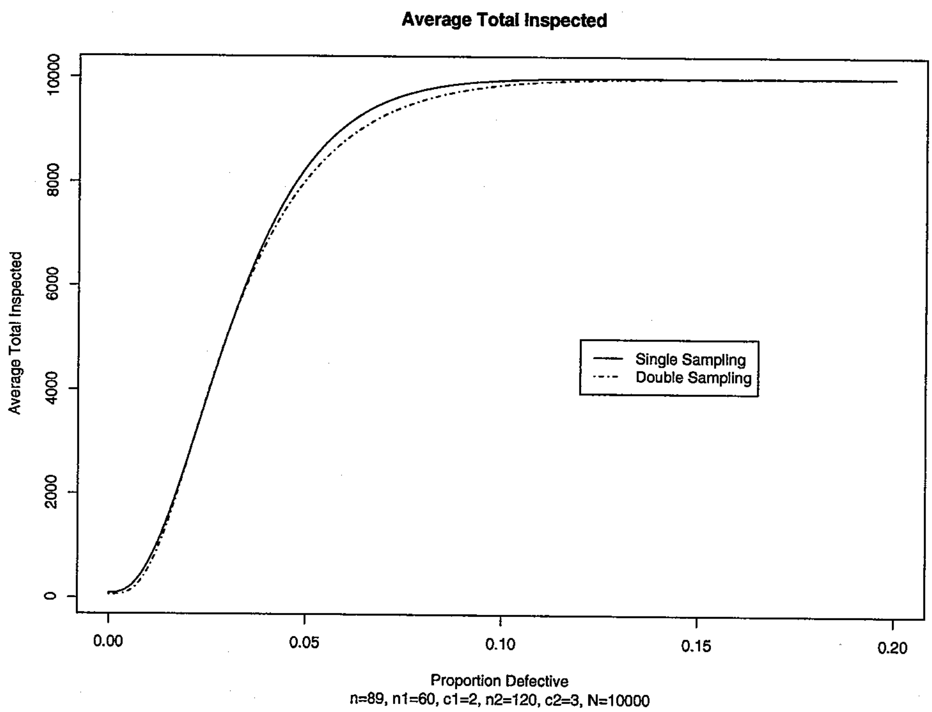
inspection are replaced with non-defective units.

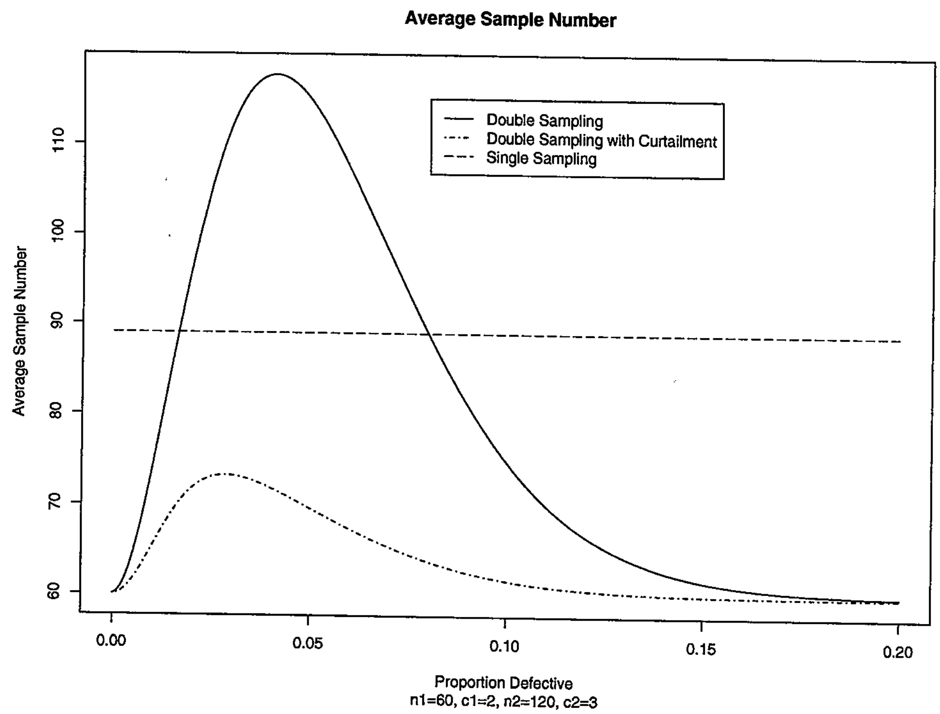
$$AOQ = \frac{[P_a^1(N - n_1) + P_a^2(N - n_1 - n_2)]p}{N} \tag{14}$$

The average total inspection curve is given by:

$$ATI = n_1 P_a^1 + (n_1 + n_2)P_a^2 + N(1 - P_a). \tag{15}$$

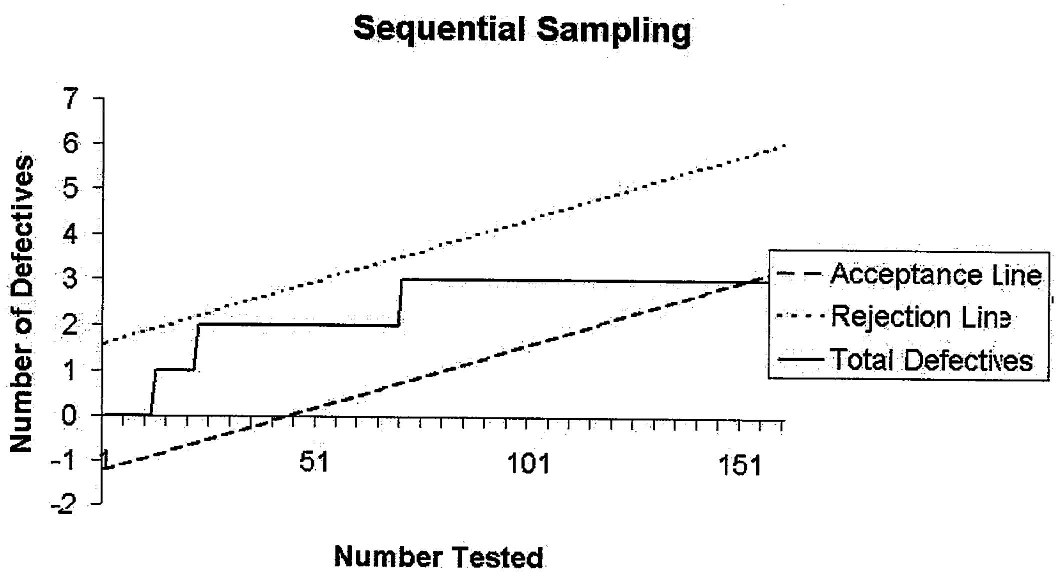
The following two graphs relate a single sampling plan to a double sampling plan having the same α and β levels.





SEQUENTIAL SAMPLING PLANS

Another common method exists for sampling attributes and accepting or rejecting the lot. It is known as sequential sampling and relative to the other two previously mentioned attribute sampling plans, it is very straightforward. In sequential sampling, samples are taken and evaluated until the acceptance or rejection line is crossed. In sequential sampling, the samples can either be single units, which is known as item-by-item sequential sampling, or it can be samples with size greater than one, which is called group sequential sampling. If the sample point falls on or below X_A, the acceptance line, then the lot is accepted. However if the sample point falls on or above X_R, the lot is rejected.



The sampling plan is dependent on the points $(p_1, 1-\alpha)$ and (p_2, β). P_a is dependent on p and can be expressed in terms of a new variable t:

$$P_a = \frac{\left(\frac{1-\beta}{\alpha}\right)^t - 1}{\left(\frac{1-\beta}{\alpha}\right)^t - \left(\frac{\beta}{1-\alpha}\right)^t} \tag{16}$$

Using different values for t can approximate the effect of different values for p. In order to obtain P_a when $p \approx p_1$, use $t = 1$. In order to evaluate the case when $p \approx p_2$ set $t = -1$. And, setting $t = 0$ would give an approximate midpoint between p_1 and p_2, $p \approx s$ where s defined in equation 20. After choosing these points, calculate the following four values (Montgomery 702,703):

$$h_1 = \frac{log(\frac{1-\alpha}{\beta})}{k} \tag{17}$$

$$h_2 = \frac{log(\frac{1-\beta}{\alpha})}{k} \tag{18}$$

$$k = log\left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right) \tag{19}$$

$$s = \frac{log(\frac{1-p_1}{1-p_2})}{k} \tag{20}$$

After calculating these four values for any given n, the number of units inspected, it is easy to calculate the boundary lines.

$$X_A = -h_1 + sn \tag{21}$$

$$X_R = h_2 + sn \tag{22}$$

In order to calculate the ASN it is necessary to calculate three values:

$$A = log\left(\frac{\beta}{1-\alpha}\right) \tag{23}$$

$$B = log\left(\frac{1-\beta}{\alpha}\right) \tag{24}$$

$$C = plog\left(\frac{p_2}{p_1}\right) + (1-p)log\left(\frac{1-p_2}{1-p_1}\right) \tag{25}$$

From these values it is easy to calculate the ASN:

$$ASN = P_a\left(\frac{A}{C}\right) + (1-P_a)\left(\frac{B}{C}\right) \tag{26}$$

The AOQ for sequential attribute sampling is:

$$AOQ \approx P_a p \tag{27}$$

The calculation of the ATI relies on two different numbers. The first number is $\frac{A}{C}$ which is the average number of units inspected when the lot is accepted. The second number is N which is the average number of units inspected when the lot is rejected. So, the ATI is:

$$ATI = P_a\left(\frac{A}{C}\right) + (1-P_a)N \tag{28}$$

CONCLUSION

Given the need for a sampling plan, there are many options available. The single sampling plan is predictable in the number of units that will be sampled and is administratively the easiest plan. Double sampling, under curtailment, requires fewer units be sampled. The cost saved by reducing the number of samples must then be weighed against the added administrative cost. Sequential sampling also generally entails fewer sampling units compared to single sampling. However, sequential sampling leaves open the possibility that the entire lot could be sampled before a decision is reached. There is a possiblity that the entire lot will need to be sampled. Although each sampling plan has drawbacks and advantages, They are all capable of aiding the manufacturer in reducing costs. Once the benefits and drawbacks are weighed for each sampling plan, the most favorable will emerge.