Planar System introductory

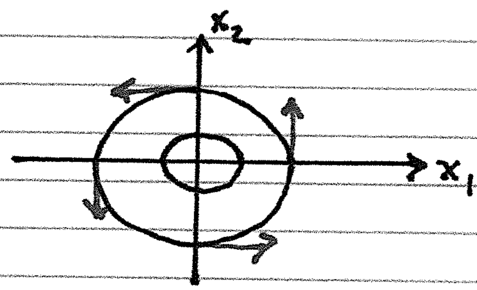
(1) $\qquad \dot{x} = f(x) \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

and the vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$

Solutions $x(t)$ of (1) define curves or trajectories in $\mathbb{R}^2$. Geometrically, (1) states that trajectories are everywhere tangent to $f(x)$

EXAMPLE Consider the linear (center) system

$$\dot{x} = f(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$$



x_1	x_2	$f(x)$
1	0	$(0,1)^T$
0	-1	$(-1,0)^T$
-1	0	$(0,-1)^T$
0	-1	$(-1,0)^T$

EXISTENCE AND UNIQUENESS

Theorem If f_i and $\frac{\partial f_i}{\partial x_j}$ are continuous on an open connected set $D \subset \mathbb{R}^2$ and $x_0 \in D$ then $\exists T > 0$ such that

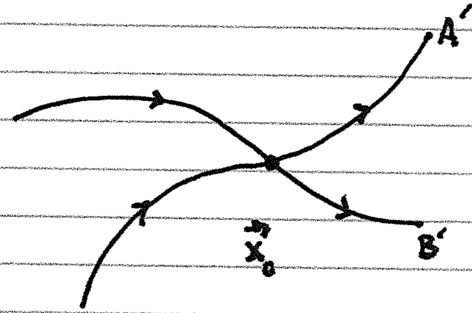
$$\dot{x} = f(x), \quad x(0) = x_0$$

has a unique solution $x(t)$ that exists $\forall t \in (-T, T)$.

Consequences of uniqueness

(i) trajectories cannot cross

(ii) blowup (when it occurs) is out of the phase portrait field



This can't happen since then

$$\dot{x} = Ax \qquad x(0) = \vec{x}_0$$

has two solns!

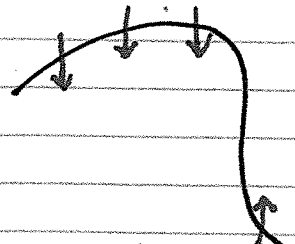
Nullclines

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

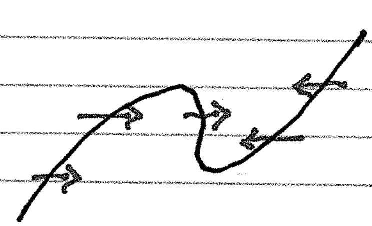
Equilibria occur at (x,y) pairs where f and g vanish simultaneously. Here we define

$f(x,y)=0$	x-nullcline(s)	$\dot{x}=0$
$g(x,y)=0$	y-nullcline(s)	$\dot{y}=0$

Since $\dot{x}=0$, $\dot{y}=0$ on nullclines the flow direction is horizontal and vertical, respectively.



x-nullcline $f=0$

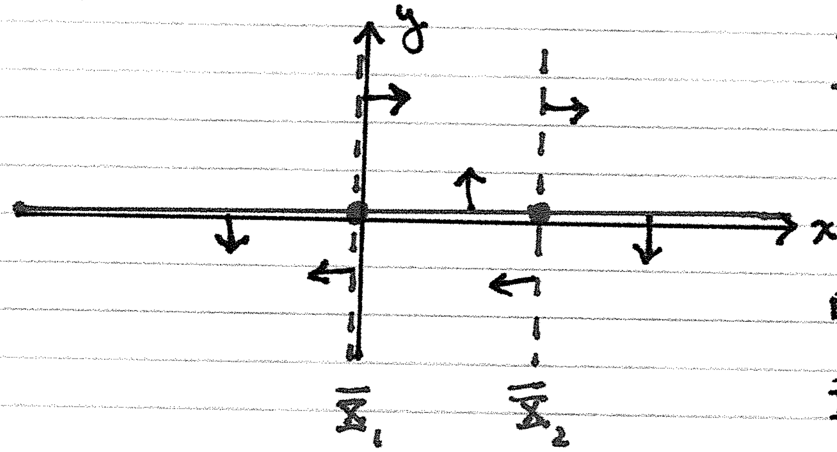


y-nullcline

EXAMPLE

$$\dot{x} = y$$
$$\dot{y} = x(1-x)$$

Can draw the x,y-nullclines and located fixed points



——— $\dot{x}=0$

- - - - $\dot{y}=0$

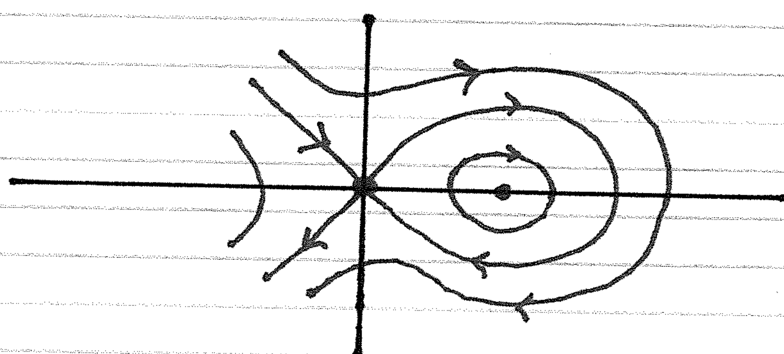
Fixed points

$\bar{X}_1=(0,0)$ $\bar{X}_2=(1,0)$

Shows some directions on nullclines. For $\dot{x}=0$ the vector field

$$\vec{F}(x,y) = \begin{pmatrix} y \\ x(1-x) \end{pmatrix} = \begin{pmatrix} 0 \\ x(1-x) \end{pmatrix}$$

whose horizontal direction depends on the sign of $x(1-x)$.
Can't deduce complete portrait but it is:



Trajectories are the level sets of the function $E(x,y)$ defined below

$\bar{X}_1$ saddle $\bar{X}_2$ center

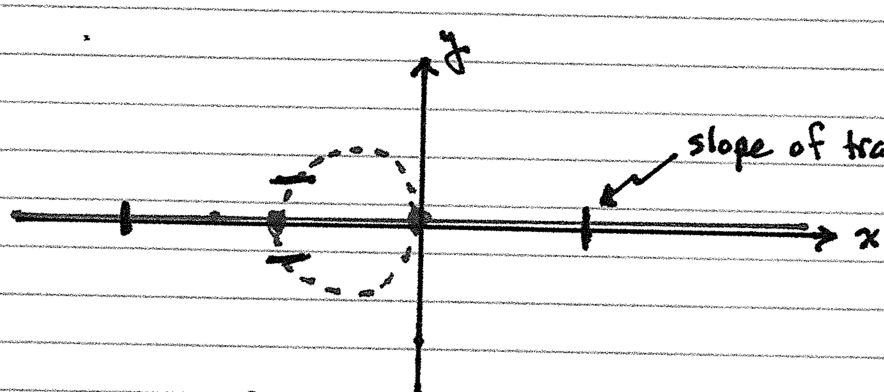
$$\boxed{E(x,y) = \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{2}y^2} \qquad \dot{E} = 0$$

EXAMPLE

$$\dot{x} = y$$
$$\dot{y} = (x+1)^2 + y^2 - 1$$

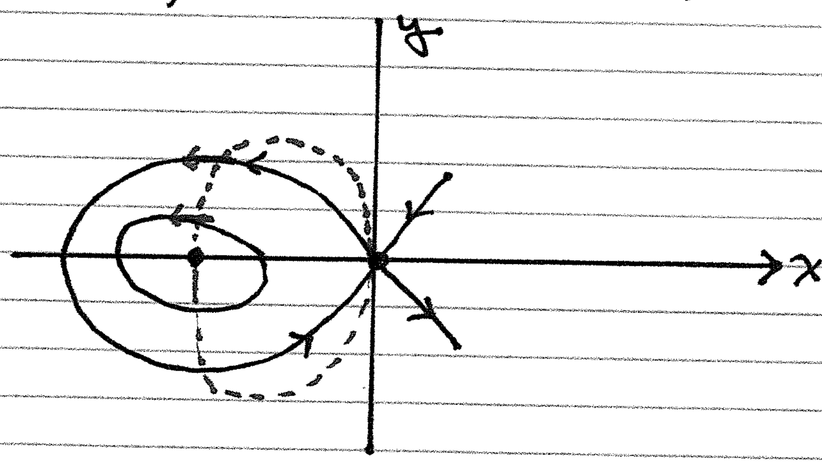
Here we have the nullclines and equilibria

$\dot{x} = 0$	$y = 0$	x-axis	——
$\dot{y} = 0$	$(x+1)^2 + y^2 = 1$	circle	- - - -



$\bar{X}_1 = (-2, 0)$ $\bar{X}_2 = (0,0)$

Numerically one can show the portrait is



Linearization about fixed points

Seek to understand phase portaits near fixed points in nonlinear problems

(1) $$\dot{x} = f(x) \qquad x(t) \in \mathbb{R}^2$$

Suppose $\bar{x}$ is a fixed point of (1), namely

(2) $$0 = f(\bar{x}) \qquad \bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

Let $\varepsilon \ll 1$ be some small number and define $u(t)$ via

(3) $$x(t) = \bar{x} + \varepsilon u(t)$$

Since $\dot{x}(t) = \varepsilon \dot{u}(t)$, eqn (1) becomes (Taylor Series)

$$\varepsilon \dot{u} = f(\bar{x} + \varepsilon u)$$

$$\varepsilon \dot{u} = \cancelto{0}{f(\bar{x})} + \varepsilon \underbrace{Df(\bar{x})}_{\text{Jacobian}} + \frac{1}{2!} \varepsilon^2 u^T \underbrace{H_f}_{\text{Hessian}}(\bar{x}) u + O(\varepsilon^3)$$

hence

(4) $$\dot{u} = Df(\bar{x}) + \tfrac{1}{2} \varepsilon u^T H_f(\bar{x}) u + O(\varepsilon^2)$$

wher

$$Df(\bar{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x=\bar{x}} \qquad \text{Jacabian}$$

$$u^T H_f(\bar{x}) \equiv \begin{pmatrix} u^T H_{f_1}(\bar{x}) u \\ u^T H_{f_2}(\bar{x}) u \end{pmatrix}$$

The term in (4) involving the Hessians H_{f_k} is small. Assuming we can ignore it when ε gets small we arrive at the following <u>linearized system</u> about $\bar{x}$

<u>Linearization of $\dot{x} = f(x)$ about $\bar{x}$</u>

$$\boxed{\dot{y} = Ay = Df(\bar{x})\, y}$$

The question is: when does the linearized system well approximate $x(t)$ near $\bar{x}$?

<u>Definition</u> A fixed point of $\dot{x} = f(x)$, $x \in \mathbb{R}^2$ is <u>hyperbolic</u> if all eigenvalues of $Df(\bar{x})$ have nonreal parts. Otherwise, $\bar{x}$ is <u>nonhyperbolic</u>.

<u>EXAMPLE</u> $\dot{x} = Ax$ is a linear system

$\det A = 0 \Rightarrow \bar{x}$ nonhyperbolic line of fxpt

$\det A > 0$, $\operatorname{Tr} A = 0 \Rightarrow \bar{x}$ nonhyperbolic center

Otherwise $\Rightarrow \bar{x}$ hyperbolic

EXAMPLE <u>Multiple fixed points</u> $\bar{X}_k = (\bar{x}_k, \bar{y}_k)$

$$\dot{x} = y \qquad = f(x,y)$$
$$\dot{y} = x - x^2 \qquad = g(x,y)$$

Easily verified the fixed points are

$$\bar{X}_1 = (0,0) \qquad \bar{X}_2 = (1,0)$$

Compute the Jacobian

$$Df(x,y) = \begin{bmatrix} 0 & 1 \\ 1-2x & 0 \end{bmatrix}$$

Evaluate at fixed points

$$Df(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \bar{X}_1 \text{ hyperbolic}$$

$$Df(1,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \bar{X}_2 \text{ not hyperbolic}$$

The latter is true since its eigenvalues are purely imaginary $\lambda = \pm i$.

EXAMPLE

$$\dot{x} = y \qquad \dot{y} = x - x^3 \qquad Df = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix}$$

"Wronskian"

$\dot{x} = 0$ nullcline $\quad y = 0$

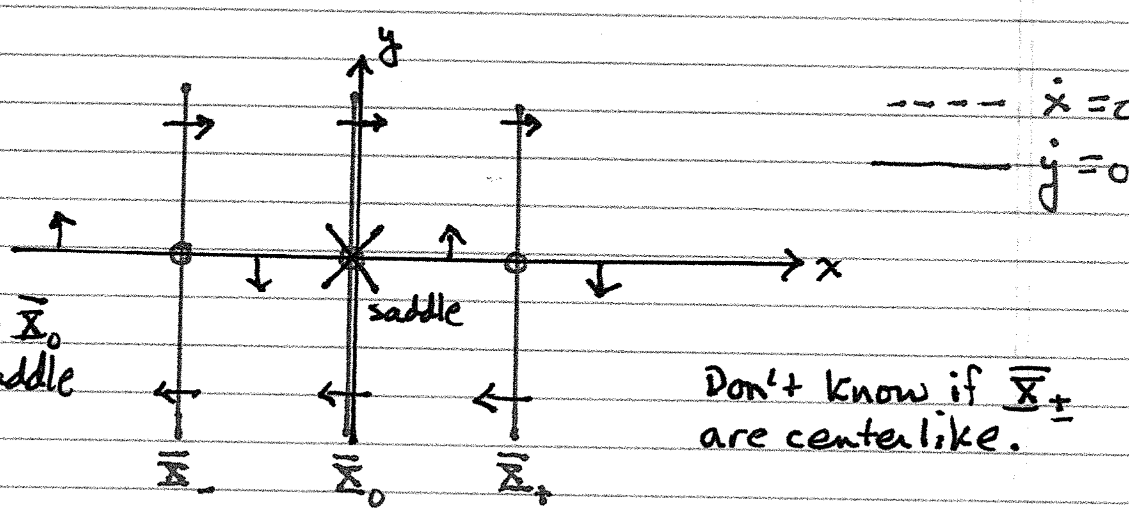
$\dot{y} = 0$ nullcline $\quad x = 0, \pm 1$

Hence have three fixed points $\bar{X}_0 = (0,0)$ and $\bar{X}_\pm = (\pm 1, 0)$

$$Df(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \lambda = \pm 1 \qquad \vec{\xi} = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \qquad \text{hyperbolic}$$

$$Df(\pm 1, 0) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \qquad \text{nonhyperbolic center}$$

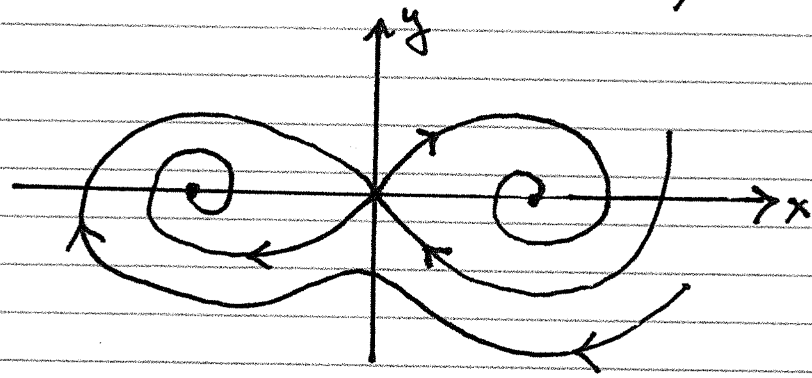
(theory fails)



locally $\bar{X}_0$ is a saddle

Don't know if $\bar{X}_\pm$ are centerlike.

Numerical solutions show they are spirals!



EXAMPLE Failure in linear stability

$$\dot{x} = f(x,y) = -y + \lambda x(x^2+y^2)$$

$$\dot{y} = g(x,y) = x + \lambda y(x^2+y^2)$$

After some calculations we can show $\bar{X} = (0,0)$ is nonhyperbolic

$$Df(\bar{X}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \forall \lambda \in \mathbb{R}$$

This looks like a center but ain't. To see this note

$$(1) \qquad x\dot{x} + y\dot{y} = \lambda(x^2+y^2)^2$$

Define $z = x^2+y^2$ (radius r squared) then (1) $\Rightarrow$

$$\boxed{\dot{z} = 2\lambda z^2}$$

$\lambda > 0 \Rightarrow z(t) \to \infty$ $\bar{X}$ unstable

$\lambda = 0 \Rightarrow$ linear center $\bar{X}$ neutral stable

$\lambda < 0 \Rightarrow z(t) \to 0$ $\bar{X}$ asymptotically stable.

Moral: when fixed point is nonhyperbolic almost anything can happen.

This next problem is a template for future HW problem

$\dot{x} = 0$ ——

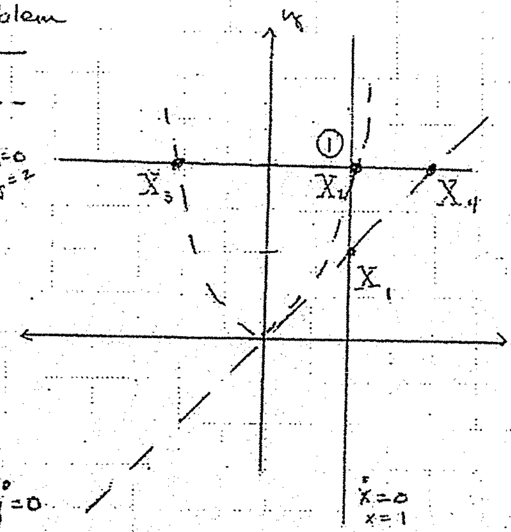
$\dot{y} = 0$ - - -

<u>Example</u> $\dot{x} = (y-2)(x-1) = f(x,y)$

$\dot{y} = (y-x)(y-2x^2) = g(x,y)$

$\dot{x}=0$ $x=1$ $y=2$ (x-nullclines) vertical

$\dot{y}=0$ $y=x$ $y=2x^2$ (y-nullclines) horizontal



There are 4 fixed points

$\dot{y}=0$

① It is possible for x nullclines to cross.
Here I have two x nullclines and a y-nullcline crossing.

$X_1 = (1,1)$
$X_2 = (1,2)$
$X_3 = (-1,2)$
$X_4 = (2,2)$

To classify fixed points we need the Jacobian

$$Df(X) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} y-2 & x-1 \\ 6x^2-4xy-y & -2x^2-x+2y \end{bmatrix}$$

<u>Aside</u>

$\dot{x} = xy - y - 2x + 2$

$\dot{y} = y^2 - 2x^2y - xy + 2x^3$

I take the $Df(X)$ and evaluate it at X_1, X_2, X_3 and X_4

	$Tr\, Df(X_K)$	$\det Df(X_K)$	Conclude
X_1	-2	1	stable node
X_2	1	0	nonhyperbolic
X_3	3	24	unstable spiral
X_4	-6	-6	saddle

Ex $\quad \dot{x} = (x-1)(x^2-2y+4)$

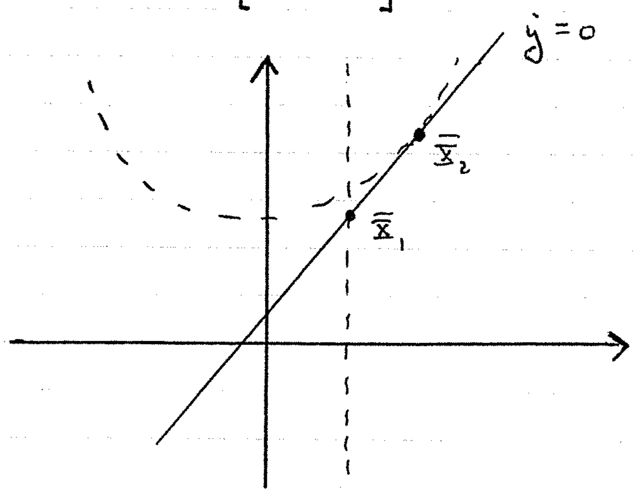
$\quad\quad \dot{y} = y-2x$

Equilibria

$\bar{X}_1 = (1, 2) \qquad \bar{X}_2 = (2, 4)$

$Df(\bar{X}_1) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad \left.\begin{aligned} Tr\, Df(\bar{X}_1) &= 2 > 0 \\ \det Df(\bar{X}_1) &= 1 > 0 \end{aligned}\right\}$ repeated root (UNSTABLE)

$Df(\bar{X}_2) = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \qquad \det Df(\bar{X}_1) = 0$ nonhyperbolic



NOTE:

$x = 1$ is invariant.

Defn: The flow $\phi(t, x_0)$ generated by $\dot{x} = f(x)$ is that function $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ which uniquely solves the IVP

$$\dot{x} = f(x) \qquad x(0) = x_0$$

Therefore

$$(1) \quad \frac{\partial \phi}{\partial t} = f(\phi(t, x_0))$$

$$(2) \quad \phi(0, x_0) = x_0$$

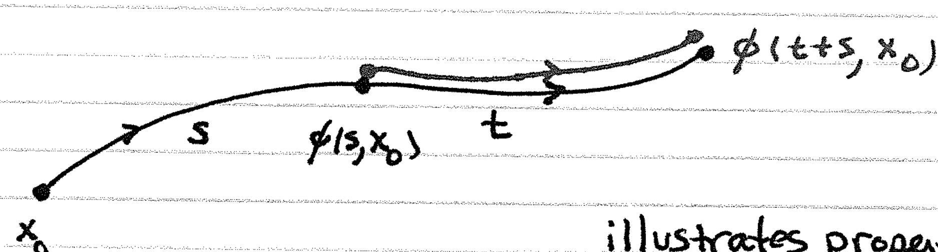
EXAMPLE Recall for $\dot{x} = Ax$ the solution is $x(t) = \Phi(t) x_0$ where $\Phi(t)$ is the fundamental matrix. Thus

$$\phi(t, x_0) = \Phi(t) x_0$$

Properties of flow functions

(a) $\phi(t+s, x_0) = \phi(t, \phi(s, x_0))$

(b) $\phi(-t, \phi(t, x_0)) = x_0$



illustrates property (a)

EXAMPLE Flow function for a nonlinear system

$$\dot{x} = -x^2 \qquad x(0) = x_0$$
$$\dot{y} = x + y \qquad y(0) = y_0$$

By direct solution techniques

$$x(t) = \frac{x_0}{1 + x_0 t} = \phi_1(t, x_0, y_0)$$

$$y(t) = y_0 e^t + e^t \int_0^t \frac{x_0}{1 + x_0 s} e^{-s} ds = \phi_2(t, x_0, y_0)$$

are the components of $\phi(t, x, y)$:

$$\phi(t, x_0, y_0) = \begin{pmatrix} \phi_1(t, x_0, y_0) \\ \phi_2(t, x_0, y_0) \end{pmatrix}$$

and $\phi : \underset{\substack{\uparrow \\ t}}{\mathbb{R}} \times \underset{\substack{\uparrow \\ (x_0, y_0)}}{\mathbb{R}^2} \longrightarrow \mathbb{R}^2$

Defn: A <u>homeomorphism</u> H is a continuous invertible map $H: \mathbb{X} \to \mathbb{Y}$. when continuously differentiable as well H is a <u>diffeomorphism</u>.

Theorem* Hartman-Grobman Theorem on $\mathbb{R}^2$

(1) $f: \mathbb{R}^2 \to \mathbb{R}^2$ cont. diff on open $E \subseteq \mathbb{R}^2$

(2) $\phi(t, x_0)$ flow fn for $\dot{x} = f(x)$ $\quad x(0) = x_0$

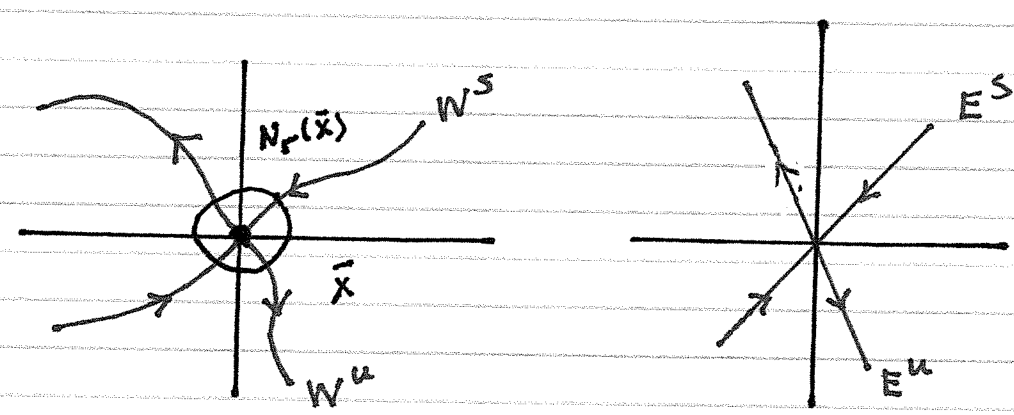
(3) $\bar{x} \in E$ hyperbolic fixed point

(4) $\psi(t, y_0)$ flow fn for $\dot{y} = Df(\bar{x})y$, $y(0) = y_0$

Then $\exists$ homeomorphism H defined on a neighbourhood $N_r(\bar{x})$ such that

$$\boxed{H(\phi(t, x_0)) = \psi(t, H(x_0))} \qquad \forall t \ni \phi(t, x_0) \in N_r(\bar{x})$$

For a saddle a casual picture looks like



$\dot{x} = f(x)$ $\qquad\qquad$ $\dot{y} = Df(\bar{x})\,y$

* preserves direction of flow.