Polar coordinates

We seek to transform the cartesian system

$$\dot{x} = f_1(x, y) \quad (1)$$
$$\dot{y} = f_2(x, y) \quad (2)$$

into a polar one via the transformation

$$x = r\cos\theta \quad (3)$$
$$y = r\sin\theta \quad (4)$$

To see how, differentiate (3)-(4) in time

$$\dot{x} = \cos\theta\, \dot{r} - r\sin\theta\, \dot{\theta} = f_1(r\cos\theta, r\sin\theta)$$
$$\dot{y} = \sin\theta\, \dot{r} + r\cos\theta\, \dot{\theta} = f_2(r\cos\theta, r\sin\theta)$$

These can be written in the compact matrix form

$$\underbrace{\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}}_{R} \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

The matrix R is easily inverted and one gets

$$\boxed{\begin{aligned} \dot{r} &= \cos\theta\, f_1 + \sin\theta\, f_2 \\ \dot{\theta} &= -\frac{\sin\theta}{r} f_1 + \frac{\cos\theta}{r} f_2 \end{aligned}} \quad (5)$$

EXAMPLE

$$\dot{x} = -y + xF(x^2+y^2) = f_1$$
$$\dot{y} = x + yF(x^2+y^2) = f_2$$

In polar, since $r=\sqrt{x^2+y^2}$

$$f_1 = -r\sin\theta + r\cos\theta\, F(r^2)$$
$$f_2 = r\cos\theta + r\sin\theta\, F(r^2)$$

Use (5) on previous page to simplify $\dot{r}$

$$\dot{r} = \cancel{-r\sin\theta\cos\theta} + r\cos^2\theta\, F(r^2) \cancel{+r\sin\theta\cos\theta} + r\sin^2\theta\, F(r^2)$$

$$\dot{r} = r(\cos^2\theta + \sin^2\theta)F(r^2)$$

$$\dot{r} = rF(r^2)$$

Similar calculations for $\dot{\theta}$ yields the system

$$\boxed{\begin{aligned}\dot{r} &= rF(r^2)\\ \dot{\theta} &= 1\end{aligned}}$$

note these eqns are decoupled!!

Periodic Solutions (period $T < \infty$)

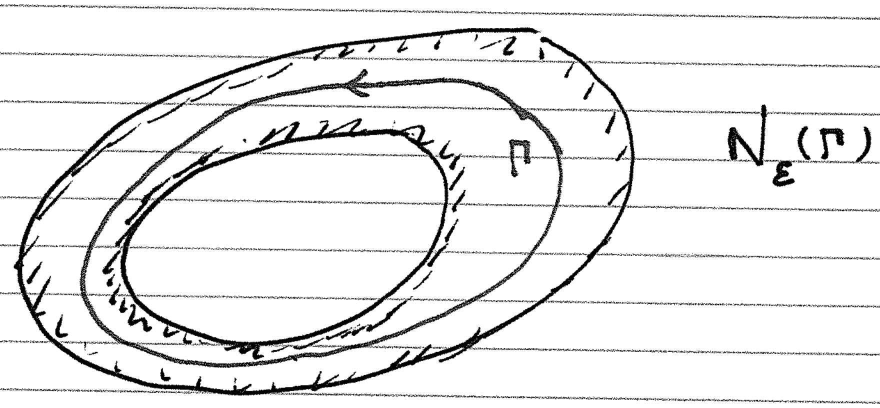
Linear systems $\dot{x} = Ax$ can't have "isolated" T-periodic solutions since if $x(t)$ is periodic the $c\,x(t)$ is as well $\forall c \in \mathbb{R}$.

Nonlinear systems can have isolated periodic solutions.

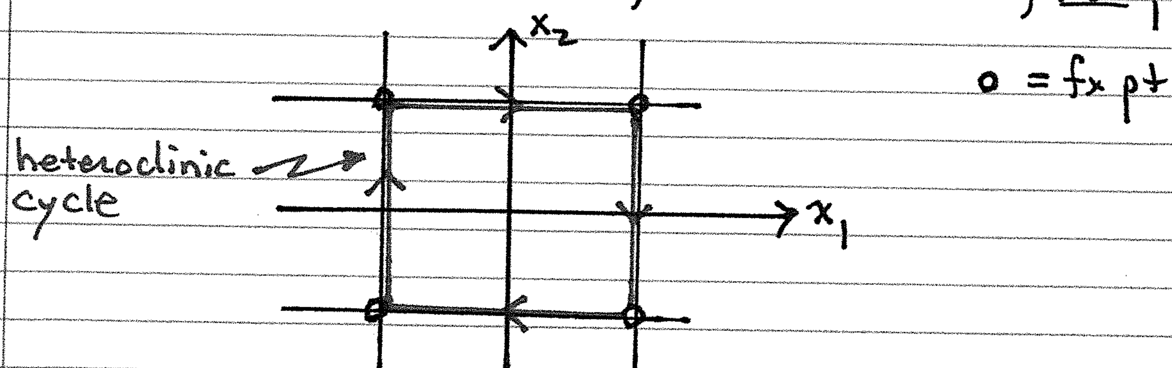
Defn Let Γ be a closed trajectory in $\mathbb{R}^2$ that is a T-periodic solution of $\dot{x} = f(x)$. Define

$$N_\varepsilon(\Gamma) = \{ x \in \mathbb{R}^2 : \|x - y\| < \varepsilon ,\ y \in \Gamma \}$$

We say Γ is isolated if $\exists \varepsilon$ such that $N_\varepsilon(\Gamma)$ contains no other periodic orbits



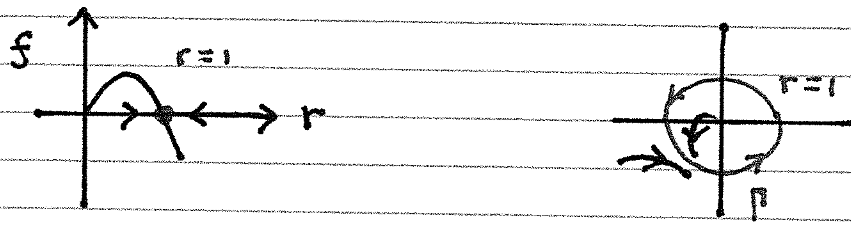
Remark: Heteroclinic cycles are closed, not periodic



Defn Γ is a limit cycle in $\mathbb{R}^2$ if it is an isolated T-periodic solution of $\dot{x} = f(x)$

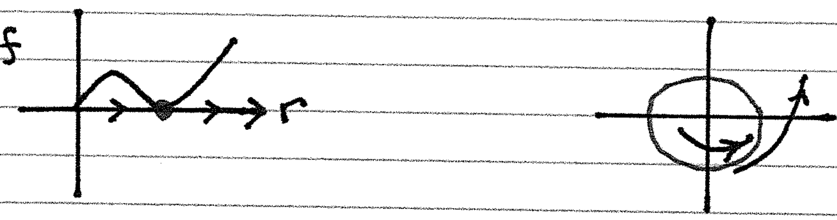
Limit cycle stability examples (polar)

$$\dot{r} = f(r) \qquad \dot{\theta} = 1$$

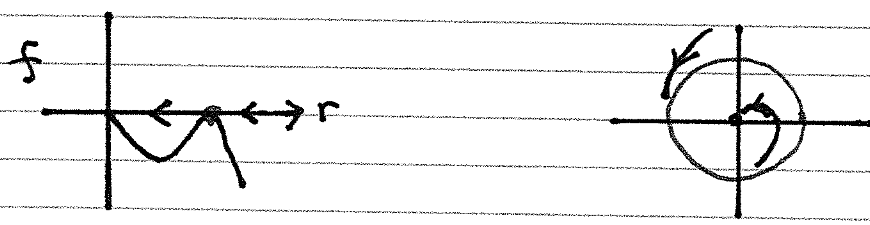


stable

attracting

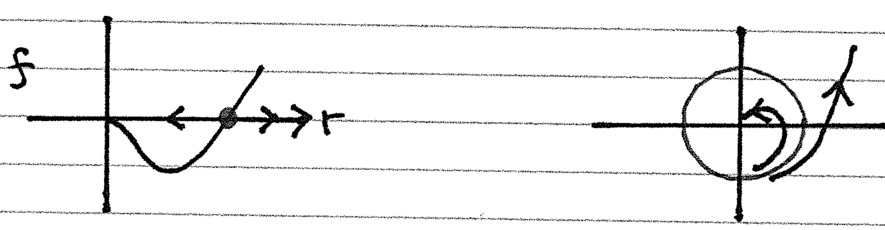


unstable
(half stable)



unstable
(half stable)

[origin stable fx pt]



unstable.

Periods of Periodic orbits - linear system

(1) $$\dot{x} = Ax \qquad x \in \mathbb{R}^2$$

Recall the characteristic polynomial for (1) is

$$P(\lambda) = \lambda^2 - \text{Tr}A\,\lambda + \det A = 0$$

For centers, $\text{Tr}A = 0$ and $\det A > 0$ so

(2) $$\lambda = \pm i\sqrt{|\det A|} = \pm i\omega$$

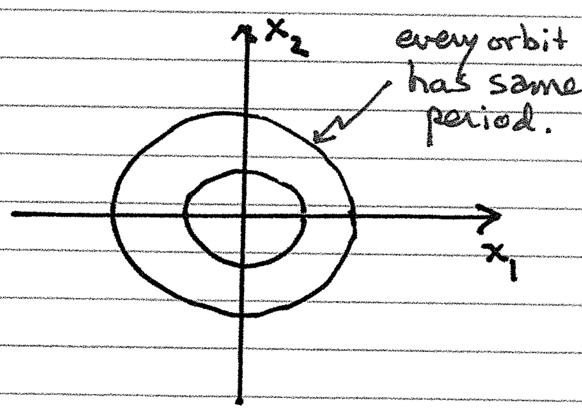
where ω is the frequency. Solutions involve $\sin(\omega t)$, $\cos(\omega t)$ so the period of each orbit is

(3) $$T = \frac{2\pi}{\omega}$$

EXAMPLE

$$\dot{x} = \begin{bmatrix} 1 & -5 \\ 2 & -1 \end{bmatrix} x$$

Here $\det A = 9$ so that $\omega = 3$ and



$$T = \frac{2\pi}{3}$$

More often in nonlinear systems the period varies from orbit to orbit.

Simple Hamiltonian system $\ddot{u} + f(u) = 0$

(1) $\dot{x}_1 = x_2$

(2) $\dot{x}_2 = -f(x_1)$

For such Hamiltonian systems if $(x_1(t), x_2(t))$ is a solution so is $(x_1(-t), -x_2(-t))$ reversible. Thus, closed orbits have an upper and lower part:

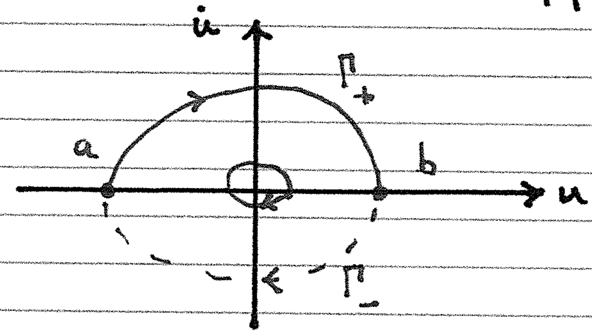


FIG 1

To find the period we recall the first integral

(3) $$E = \tfrac{1}{2}\dot{u}^2 + V(u) \qquad V(u) = \int^u f(s)\,ds$$

where E is a constant determined by init. cond. Given FIG 1, when $\dot{u} = 0$, $E = V(a)$ so that

$$\left(\frac{du}{dt}\right)^2 = 2\left(V(a) - V(u)\right)$$

separating and integrating the period of $\Gamma_+ \cup \Gamma_-$ is

(4) $$T = 2\int_a^b \frac{ds}{\sqrt{2(V(a) - V(s))}}$$

Most often this is not a doable integral

EXAMPLE $\quad \ddot{u} + u = 0$

We already know $T = 2\pi$ from linear theory.

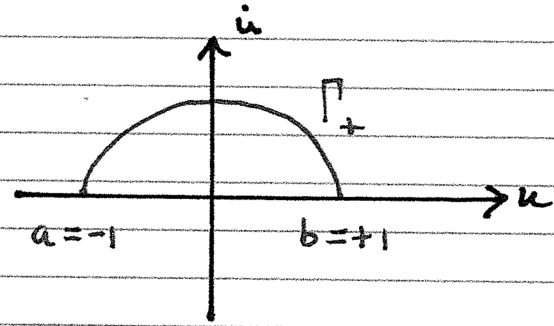
Here the potential function is

$$V(u) = \int^{u} s\,ds = \tfrac{1}{2}u^2$$

so that

$$E = \tfrac{1}{2}\dot{u}^2 + \tfrac{1}{2}u^2$$

For the $E = \frac{1}{2}$ orbit, $\dot{u}^2 + u^2 = 1$. Thus, for $\dot{u} = 0$, $u = \pm 1$.



$E = \frac{1}{2}$

Given $V(a) = \frac{1}{2}$ we get

$$T = 2\int_a^b \frac{ds}{\sqrt{2(V(a) - V(u))}} = 2\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$$

an integral which can be evaluated

$$T = 2\arcsin x\Big|_{-1}^{1} = 2\pi \qquad \square$$

Gradient Systems

Are one type of systems that cannot have periodic orbits

Defn: $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ is a gradient system if $\exists\, V(x)$ such that $f(x) = -\nabla V(x)$

Theorem: Gradient systems cannot have periodic orbits.

Pf/ Suppose $x(t) = x(t+T)$ $\forall t$ is a periodic orbit. Then note

$$
\begin{aligned}
\Delta V &= V(x(T)) - V(x(0)) \\
&= \int_0^T \frac{dV}{dt}\, dt \\
&\qquad \text{chain rule} \\
&= \int_0^T \nabla V \cdot \dot{x}\, dt \\
&\qquad \dot{x} = -\nabla V \\
&= -\int_0^T \|\dot{x}\|^2\, dt \\
&< 0
\end{aligned}
$$

This contradicts the supposition since were $x(t)$ a T-periodic orbit then $\Delta V = 0$. $\square$

EXAMPLE Show the following has no periodic orbits

$$\dot{x} = f_1(x,y) = 3x^2 - 1 - e^{2y}$$
$$\dot{y} = f_2(x,y) = -2x e^{2y}$$

To do so we show its a gradient sys: $\dot{x} = -\nabla V$

$$(1) \qquad V_x = -3x^2 + 1 + e^{2y}$$
$$(2) \qquad V_y = \underline{2xe^{2y}}$$

Integrate (1) in x

$$(3) \qquad V = -x^3 + x + xe^{2y} + \phi(y)$$

for some function $\phi(y)$. Use (3) in (2):

$$\cancel{2x}e^{2y} + \phi'(y) = \underline{\cancel{2x}e^{2y}}$$

Thus $\phi'(y) = 0 \Rightarrow \phi(y) = c$. wlog $c = 0$ so

$$V(x,y) = x - x^3 + xe^{2y}$$

is called the <u>potential function</u>

Since $f(x) = -\nabla V$, the original sys. can't have periodic orbits.

Hamiltonian vs. Gradient

Hamiltonian systems can have periodic. Gradient systems can't. Sometimes a system can be both! Suppose

$$(1) \qquad \dot{x} = f_1 = -\frac{\partial H}{\partial y} = -\frac{\partial V}{\partial x}$$

$$(2) \qquad \dot{y} = f_2 = +\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial y}$$

To be Hamiltonian one must have $\vec{\nabla} \cdot \vec{f} = 0$. This yields the necessary condition

$$\nabla^2 V = V_{xx} + V_{yy} = 0$$

EXAMPLE (Saddle)

$$\dot{x} = -x$$
$$\dot{y} = y$$

is both Hamiltonian and gradient with

$$V = \tfrac{1}{2}(x^2 - y^2)$$

$$H = xy$$

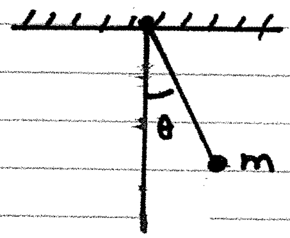
Nonlinear oscillations with dissipation

Many simple physical problems have an equation

$$\ddot{u} + G(u, \dot{u}) + f(u) = 0$$

kinetic term — damping or dissipation — potential term

EXAMPLE Damped nonlinear pendulum



$$\ddot{\theta} + k\dot{\theta} + \sin\theta = 0 \qquad (1)$$

has a friction term $k\dot{\theta}$ at the pendulum pivot.

A potential function (not potential energy) can be defined in the usual way

$$V(\theta) \equiv -\int f(\theta)\, d\theta = -\cos\theta$$

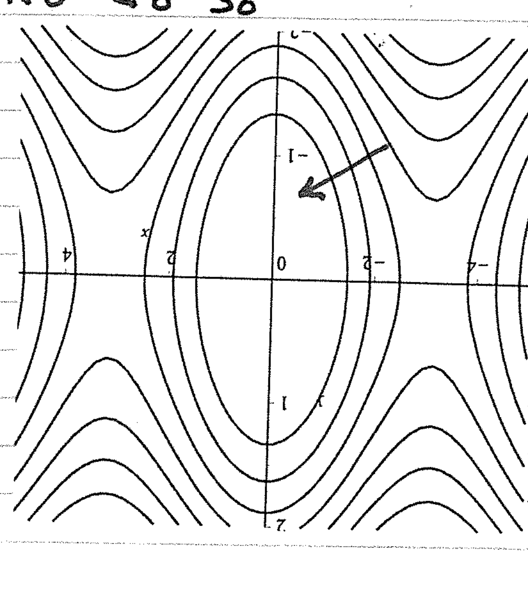
and hence

$$E(\theta, \dot{\theta}) \equiv \tfrac{1}{2}\dot{\theta}^2 + V(\theta) \qquad (2)$$

Using (1) we can show $\dot{E} = -k\dot{\theta}^2 < 0$ so that level sets of E are not trajectories (closed orbits).

$$\dot{E} = -k\dot{\theta}^2 < 0$$

Arrow shows direction of decreasing E on the illustrated level sets.



EXAMPLE Use energy arguments to show the following has no periodic orbits

$$(1) \qquad \ddot{x} + (\dot{x})^3 + x = 0$$

Define the energy

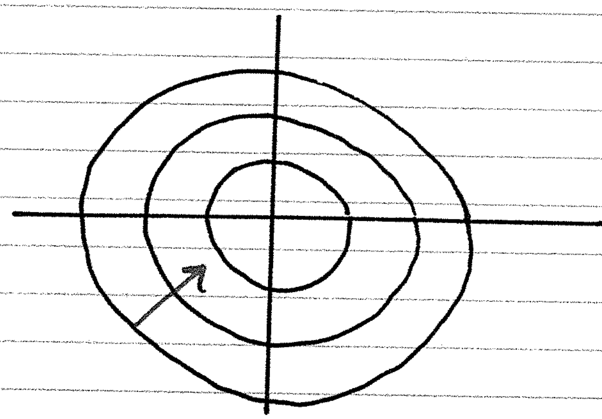
$$(2) \qquad E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2$$

Multiply (1) by $\dot{x}$ and reorder terms

$$\underbrace{\dot{x}\ddot{x} + \dot{x}x}_{\dot{E} \text{ given } (2)} = -(\dot{x})^4$$

Hence

$$\frac{dE}{dt} = -(\dot{x})^4 < 0$$



E level curves

Trajectories of (1) must move toward lower E values (arrow direction)

If there were a T-periodic ordit then $\Delta E = 0$, but

$$\Delta E = E(x(T)) - E(x(0)) = -\int_0^T (\dot{x})^4 dt < 0.$$

i.e. $\Delta E \neq 0$.

Liapunov Functions

Is a generalization of the energy idea.

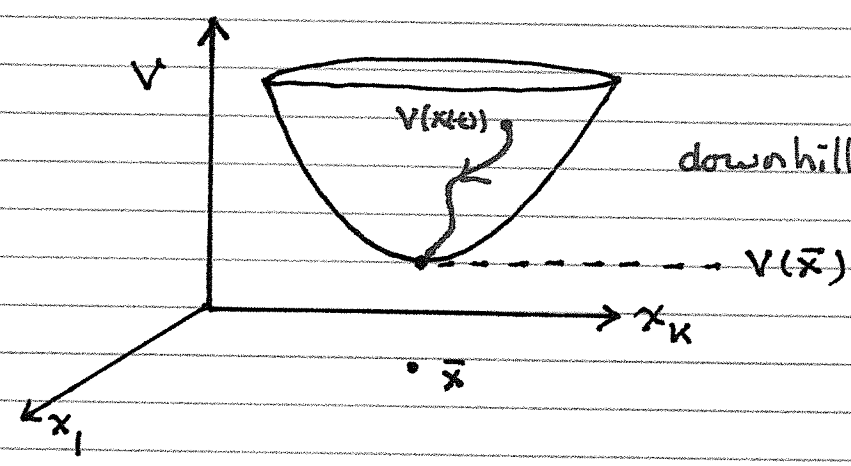
Defn Let $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ have a fixed point $\bar{x}$. By a Liapunov function of $\dot{x} = f(x)$ we mean a function $V: \mathbb{R}^n \to \mathbb{R}$ (smooth) such that

(a) $\quad V(x) > V(\bar{x}) \qquad x \neq \bar{x}$

(b) $\quad \dfrac{dV}{dt} = \nabla V \cdot \dot{x} < 0 \qquad \forall x(t) \neq \bar{x}$

Theorem If $\dot{x} = f(x)$ has a Liapunov function $V(x)$ then $\bar{x}$ is <u>globally asymptotically stable</u>

Pf idea See Jordan-Smith 1987 for formal proof



Remark: No systematic method for finding $V(x)$ when they exist.

If $\dot{x} = f(x)$ has more than one fx pt then $\nexists\, V(x)$

EXAMPLE $\dot{x} = -2x + 3y$
$\dot{y} = -6x - y^3$

Easy to show $\bar{x} = (0,0)$ is sole equilibria.

Guess: $V(x,y) = ax^2 + y^2$ for some $a \in \mathbb{R}$.

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$\dot{V} = 2ax(-2x+3y) + 2y(-6x-y^3)$$

(1) $$\dot{V} = -4ax^2 + (6a-12)xy - 2y^4$$

Chose $a = 2$ so middle term vanishes. Then

(a) $V(x,y) = 2x^2 + y^2 > V(0,0) \quad \forall (x,y) \neq \bar{x}$

(b) $\dot{V} = -8x(t)^2 - 2y(t)^4 < 0$

hence V is a Liapunov function and:

$\bar{x}$ is globally asymptotically stable $\square$

EXAMPLE Find a Liapunov function for

$$(1) \qquad \dot{x} = y - x^3$$

$$(2) \qquad \dot{y} = -x - y^3$$

of the form $V = ax^2 + by^2$ for some $a, b \in \mathbb{R}$.

$$\dot{V} = V_x \dot{x} + V_y \dot{y}$$

$$\dot{V} = 2ax(y - x^3) + 2by(-x - y^3)$$

$$\dot{V} = -2ax^4 + \underbrace{(2a - 2b)}xy - 2by^4$$

Choose $a = b$ to make this vanish (non unique)

Hence with the choice $a = 1$ and $b = 1$

$$V = x^2 + y^2$$

$$\dot{V} = -2(x^4 + y^4) < 0$$

hence V is a Liapunov function and (1)-(2) has no periodic solutions, and

$$(x(t), y(t)) \longrightarrow (0,0)$$

Weakening Liapunov requirements

Theorem: Let $V: \mathbb{R}^2 \to \mathbb{R}$ and U be some open connected subset such that

(1) $V \in C^1(U)$

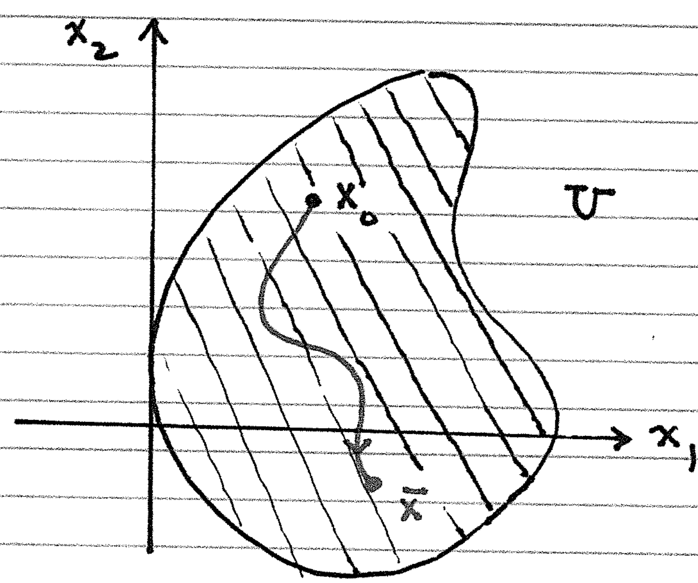
(2) $V(\bar{x}) = 0$ ($\bar{x}$ fxpt of $\dot{x} = f(x)$)

(3) $V(x) > 0$ $\forall x \in U$, $x \neq \bar{x}$

(4) $\dot{V} < 0$ $\forall x(t) \neq \bar{x}$, $x(t) \in U$

then $x(t) \to \bar{x}$ for all solutions of

$$\dot{x} = f(x) \qquad x(0) = x_0 \in U$$



EXAMPLE Fixed points are $P_0 = (0,0)$ and $P_1 = (-1,0)$

(1) $$\dot{x} = y$$
(2) $$\dot{y} = -(x + x^2) - \alpha y \qquad (\alpha > 0)$$

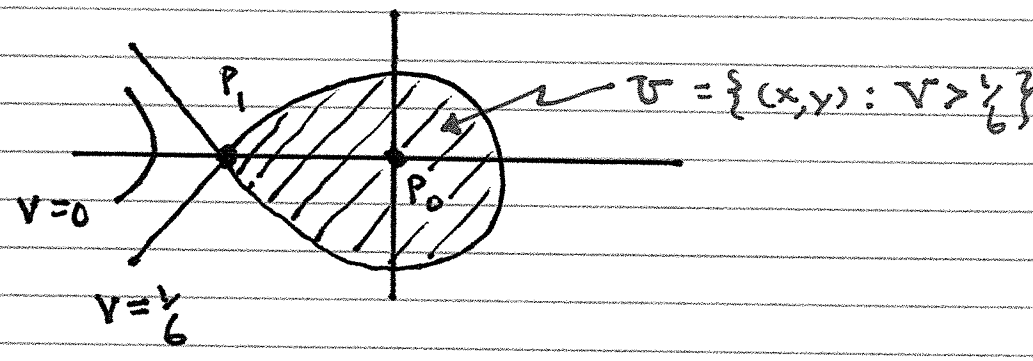
Guess $V(x,y) = E$ energy when $\alpha = 0$. (1)-(2) $\Rightarrow$

(3) $$V(x,y) = \tfrac{1}{2}y^2 + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3$$

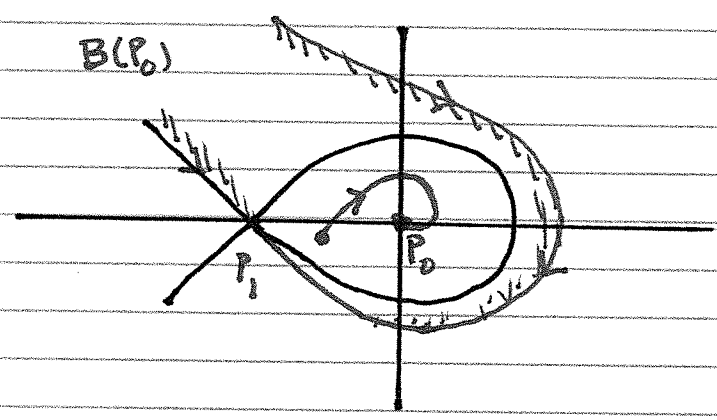
Use (1)-(2) to compute $\dot{V}$

(4) $$\dot{V} = -\alpha y^2 < 0$$

Look at level sets of V



By the theorem $(x(t), y(t)) \to P_0$ so long as initial condition is in $\mathcal{U}$



Basin of attraction $B(P_0)$, $V = \tfrac{1}{6}$ level set and trajectory in $\mathcal{U}$