Gauge Notes

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1 Introduction to E&M

1.1 Overview of Maxwell's Equations

The governing equations of electrodynamics are summarized in four equations known as Maxwell's Equations:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} - \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

where ρ and $\vec{J}$ are required to satisfy

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0.$$

The goal of this section is to provide context these equations to help us have a better understanding (or to serve as a review) of what these equations are describing. In deriving these equations, we follow a similar approach as that in [Gri13]

1.1.1 Coulomb's Law and the electric field: Gauss's Law

To begin, we start with a simple observation. If you take an aired up balloon and rub it through your hair, you can easily get the balloon to stick to the wall before it falls to the ground. Since there is no force from classical mechanics that would describe why the balloon would briefly stick to the wall, we are led to conclude that we have found a new force. Doing some simple experiments, one can come up with qualitative descriptions for this force which one typically does in a second undergraduate physics course. Doing a more meticulous experiment similar to Coulomb, one can determine the force of a stationary point charge with charge q on a stationary charge point charge with charge Q is given by

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{Qq}{||\vec{r}_2 - \vec{r}_1||^2} \left(\frac{\vec{r}_2 - \vec{r}_1}{||\vec{r}_2 - \vec{r}_1||} \right) = \frac{1}{4\pi\epsilon_0} \frac{Qq}{||\vec{r}_2 - \vec{r}_1||^3} (\vec{r}_2 - \vec{r}_1)$$

where $\vec{r}_1$ and $\vec{r}_2$ are the positions of charge q and Q, respectively, and ϵ_0 is an experimental constant called the permittivity of free space. This equation is called Coulomb's Law.

If we view the charge Q in Coulomb's Law as a test charge, a charge that we are free to move and change the amount of charge, then we can write the force of q on Q as

$$\vec{F}(\vec{r}, Q) = Q\vec{E}(\vec{r}) \qquad \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{||\vec{r} - \vec{r}_1||^3} (\vec{r} - \vec{r}_1)$$

were $\vec{r}$ is the position in space of the charge Q. The vector field $\vec{E}$ is called the electric field.

In the case we have multiple charges $q_1, \ldots, q_n$ applying a force on Q, we utilize the Superposition Principle which tells us that the total force on Q is given by the sum of the force from each q_i on Q:

$$\begin{aligned}\vec{F}(\vec{r}, Q) &= \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{Qq_i}{||\vec{r} - \vec{r_i}||^3}(\vec{r} - \vec{r_i}) \\ &= Q\sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_i}{||\vec{r} - \vec{r_i}||^3}(\vec{r} - \vec{r_i}) \\ &= Q\sum_{i=1}^{n} \vec{E_i}(\vec{r}) = Q\vec{E}(\vec{r})\end{aligned}$$

As we can see, we also take sum the individual electric fields to obtain the electric field for all n charges.

In the case we have a continuous distribution of charge $\vec{r}$ over a region R, then the electric field becomes an integral were the integral element is the charge:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_R \frac{1}{||\vec{r} - \vec{r}(q)||^3}(\vec{r} - \vec{r}(q))dq.$$

Typically, one does not work with such an integral, instead, one works with a charge density ρ which allows for a change of variable to obtain

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_R \frac{\rho(\vec{r}')}{||\vec{r} - \vec{r}'||^3}(\vec{r} - \vec{r}')dV$$

Since ρ is zero outside of R, then one can change the domain of integration to all $\mathbb{R}^3$.

In this form, we can compute the divergence to obtain

$$\begin{aligned}\nabla \cdot \vec{E}(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \rho(\vec{r}')\nabla \cdot (\frac{\vec{r} - \vec{r}'}{||\vec{r} - \vec{r}'||^3})dV \\ &= \frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \rho(\vec{r}')(4\pi\delta(\vec{r} - \vec{r}'))dV = \frac{\rho(\vec{r})}{\epsilon_0}\end{aligned}$$

Since ∇ differentiates only with respect to the spatial coordinates of $\vec{r}$, then even if $\vec{E}$ is time dependent we still obtain the same result. Thus, we arrive at the first of four in Maxwell's equations called Gauss's Law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

1.1.2 Lorentz's Force Law and the continuity equation

To obtain the other equations, we need to talk about magnetic fields. In a similar manner to the Coulomb force, there are simple experiments one can do to obtain quantitative descriptions of the magnetic force. Conducting more meticulous experiments, we know the force of a magnet with magnetic field $\vec{B}$ on a charge Q moving with velocity $\vec{v}$ is given by the Lorentz Force Law:

$$\vec{F} = Q(\vec{v} \times \vec{B}).$$

Therefore, in the case we also have an electric field $\vec{E}$, the total force on Q is given by

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$$

The interesting behavior of magnetic fields come from working with current carrying wires. In this case, we integrate along the wire to find the force is given as

$$\vec{F} = \int \vec{v} \times \vec{B} dq = \int \lambda(\vec{v} \times \vec{B}) dl = \int (\vec{I} \times \vec{B}) dl$$

where λ is the charge distribution along the wire and $\vec{I} = \lambda\vec{v}$ is the current in the wire.

Since carrying current wires have flowing charges, then we want to make sure the local charge is conserved which can be realized mathematically as relating the volume current density to the time change of the charge density. If we have a current $\vec{I}$, we can measure how the flux changes per perpendicular cross section $A_\perp$. Thus we define the volume current density

$$\vec{J} = \frac{d\vec{I}}{dA_\perp}.$$

If we are given a charge density ρ which moves with velocity $\vec{v}$, then our equation for the volume current density is given by

$$\vec{J} = \rho\vec{v}$$

Using the derivative definition of the volume current density, know the total current crossing through a surface S is given by

$$I = \int_S \vec{J} \cdot d\vec{a} = \int_{\mathcal{V}} (\nabla \cdot \vec{J}) dV$$

where the second equality is from Stokes's Theorem. For local charge to be conserved, the the flow of charge through the surface should decrease the remaining charge inside:

$$\int_{\mathcal{V}} (\nabla \cdot \vec{J}) dV = -\frac{d}{dt} \mathrm{Q}_{\text{enc}}.$$

For a charge density ρ, the charge inside a volume $\mathcal{V}$ is given by

$$Q_{\text{enc}} = \int_{\mathcal{V}} \rho dV$$

so that

$$\int_{\mathcal{V}} (\nabla \cdot \vec{J}) dV = -\frac{d}{dt} Q_{\text{enc}} = -\frac{d}{dt} \int_{\mathcal{V}} \rho dV = -\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} dV$$

Since we want this to hold for any volume, then we conclude that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0.$$

This gives us the additional equation to Maxwell's equations called the continuity equation.

1.1.3 Biot-Savart Law and Ampere's Law with Maxwell's correction

Though the Lorentz force tells us the force of a magnetic field on a charge Q, we still do not know a way to compute for magnetic field generated by a source such as a current carrying wire. To find the field $\vec{B}$, we consider the case of steady currents which is characterized by

$$\frac{\partial \rho}{\partial t} = 0$$

so that we also have, by the continuity equation,

$$\nabla \cdot \vec{J} = 0$$

In this regime, we can experimentally determine the magnetic field given by a steady current carrying wire of current $\vec{I}$ which is given by a line integral detailed in the Biot-Savart Law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_\gamma \frac{\vec{I} \times (\vec{r} - \vec{r}(l))}{||\vec{r} - \vec{r}(l)||^3} dl$$

where γ is the path of the current in the direction of the current's flow, $\vec{r}(l)$ is the position of the wire in space, and μ_0 is a experimental constant call the permeability of free space. Given a single straight wire, we can compute the integral for any closed loop in the perpendicular cross section to find

$$\oint \vec{B} \cdot d\vec{l} = \pm \mu_0 ||\vec{I}||.$$

where the $\pm$ depends on the flow direction of the current. Thus, for a bundle of wires, each wire contributes a signed magnitudes of its current so that

$$\int_S (\nabla \times \vec{B}) \cdot d\vec{a} = \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} = \mu_0 \int_S \vec{J} \cdot d\vec{a}$$

where S is any surface bounded by the loop and the first equality is from Stokes' Theorem. Since this holds for any surface, then

$$\nabla \times \vec{B} = \mu_0 \vec{J}.$$

This result is known as Ampere's Law, but does not represent the complete story. You can see in Maxwell's equation that there is an additional term in corresponding equation which does not appear in this equation. To see where the issue pops up, we must leave the realm of steady state currents. Recall, steady currents allowed us to say

$$\nabla \cdot \vec{J} = 0.$$

Without this assumption, we run into a problem as

$$0 = \nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J}$$

To fix the divergence, we utilize the continuity equation and Gauss's Law to obtain

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \vec{E}) = -\nabla \cdot (\epsilon_0 \frac{\partial \vec{E}}{\partial t})$$

Using this, one can guess and argue on physical grounds (which is what Maxwell did) that the fixed Ampere's Law is given by

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Note, since $\nabla \times$ and $\nabla \cdot$ differentiate with respect to the spatial coordinates and not the time coordinate, then this result still holds when $\vec{B}$, $\vec{J}$, ρ, and $\vec{E}$ are time dependent. Therefore we have found our second equation in Maxwell's equations.

1.1.4 Divergence of magnetic fields

We can generalize the Biot-Savart law for volume currents as

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}')}{||\vec{r}-\vec{r}'||^3} dV.$$

Since

$$\nabla \cdot \left(\frac{\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}')}{||\vec{r}-\vec{r}'||^3}\right) = \left(\frac{\vec{r}-\vec{r}'}{||\vec{r}-\vec{r}'||^3}\right) \cdot (\nabla \times \vec{J}(\vec{r}')) - \vec{J}(\vec{r}') \cdot \left(\nabla \times \frac{\vec{r}-\vec{r}'}{||\vec{r}-\vec{r}'||^3}\right) = 0 - 0 = 0,$$

then we find

$$\nabla \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \nabla \cdot \left(\frac{\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}')}{||\vec{r}-\vec{r}'||^3}\right) dV = 0$$

Again, since $\nabla\cdot$ only differentiates with respect to the spatial coordinates, then this result is still true if we introduce time dependence. Thus we obtain our third equation of Maxwell's:

$$\nabla \cdot \vec{B} = 0.$$

1.1.5 The electromotive force and Faraday's Law

For the final equation, we need to introduce the electromotive force (EMF). There are two versions of EMF. The first is electrical EMF one typically sees in circuits. For an electric field $\vec{E}$, the EMF in a circuit due to the electric field is given by

$$\mathcal{E}_{\text{electric}} = \oint \vec{E} \cdot d\vec{l}$$

where we integrate over the circuit. Typically, this EMF is written as

$$\mathcal{E}_{\text{electric}} = \oint \vec{f} \cdot d\vec{l}$$

where $\vec{f} = \vec{f_s} + \vec{E}$ and $\vec{f_s}$ is the electric field generated by a source(s) (such as a battery in a circuit), but we have no need for this term in this discussion. The other type of EMF is motional EMF which comes up when you move wires through magnetic fields (such as generators). For a circuit moving in a magnetic field $\vec{B}$, the EMF is given by a flux rule:

$$\mathcal{E}_{\text{magnetic}} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{a} = \int_S -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

where Φ is the flux of $\vec{B}$ through the circuit and S is a surface bounded by the loop.

In studying these EMFs, Michael Faraday conducted simple experiments:

1. Take of loop of wire and move it through a magnetic field. One will find that a current is generated.

2. Take a magnet and move it across a loop of wire. One will find a current is generated.

3. Take a loop and place it in a magnetic field. With this setup, vary the strength of the magnetic field. One will still find a current is still generated.

The results in the first experiment are explained by the flux rule for motional EMF. The second experiment's results tells us the EMF is from the electric field which is also equal to the time rate of change of the magnetic flux:

$$\int_S \nabla \times \vec{E} \cdot d\vec{a} = \oint \vec{E} \cdot d\vec{l} = \int_S -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}.$$

where the first equality holds from Stokes' Theorem. Since the equations above must be true for any surface S, then we conclude

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

which is called Faraday's Law and is the final equation for Maxwell's equations. Note the third experiment tell us that whenever the magnetic flux through a circuit wire changes, there is an induced current in the loop.

1.2 Scalar and Vector Potentials for Maxwell's Equations

Now that we have familiarized ourselves with Maxwell's equations, we introduce two fundamental quantities: the scalar potential and the vector potential. Though, we first begin with a review of the de Rham complex of $\mathbb{R}^3$.

1.2.1 Review of the de Rham complex for $\mathbb{R}^3$

Recall that for any smooth manifold M we have the de Rham complex $\Omega^*(M)$ which forms a cochain complex where coboundary map is the exterior derivative. For $M = \mathbb{R}^3$, the cochain complex is only four terms long given by

$$C^\infty(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3)$$

We also have a standard Riemannian metric on $\mathbb{R}^3$ called the flat metric. Letting x^1, x^2, x^3 denote the global coordinates on $\mathbb{R}^3$, the flat metric is given by

$$g = \sum_{i,j=1}^{n} \delta_{ij} dx^i \otimes dx^j \qquad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

This metric induces a linear isomorphism from covector fields on $\mathbb{R}^3$ and vector fields on $\mathbb{R}^3$ given by

$$g_\# : \Omega^1(\mathbb{R}^3) \to \mathfrak{X}(\mathbb{R}^3) \qquad dx^i \mapsto \frac{\partial}{\partial x^i}$$

Furthermore, since $\mathbb{R}^3$ is orientable, then, with respect to this metric, we have canonical top form called the Riemannian volume form which, in the coordinates, is given by

$$\omega_g = \sqrt{\det([\delta_{ij}])} dx^1 \wedge dx^2 \wedge dx^3 = dx^1 \wedge dx^2 \wedge dx^3.$$

Using this top form and the interior multiplication, we obtain an isomorphism

$$i\omega_g : \mathfrak{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3) \qquad i\omega_g(X) = i_X\omega_g$$

Additionally, $\Omega^3(\mathbb{R}^3) \cong C^\infty(\mathbb{R}^3)$ as $\bigwedge^3(T^*\mathbb{R}^3) \to \mathbb{R}^3$ is a trivial line bundle due to $\mathbb{R}^3$ being orientable. Under all these identifications we obtain the following where the vertical maps are isomorphism:

$$\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \\
\big\downarrow \text{id} & & \big\downarrow g_\# & & i\omega_g \big\uparrow & & \big\downarrow \\
C^\infty(\mathbb{R}^3) & & \mathfrak{X}(\mathbb{R}^3) & & \mathfrak{X}(\mathbb{R}^3) & & C^\infty(\mathbb{R}^3)
\end{array}$$

Taking the inverse of $i\omega_g$ as well as the induced maps along the bottom which are given, from left to right, by ∇, $\nabla\times$, and $\nabla\cdot$, we obtain an isomorphism of cochain complexes:

$$\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
C^\infty(\mathbb{R}^3) & \xrightarrow{\nabla} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\nabla\times} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\nabla\cdot} & C^\infty(\mathbb{R}^3)
\end{array}$$

Note, the same holds if we instead were working with an open subset of $\mathbb{R}^3$. Since $\mathbb{R}^3$ is contractible, we know the cochain complex

$$C^\infty(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3)$$

is exact. Thus, under the isomorphisms, we know

$$C^\infty(\mathbb{R}^3) \xrightarrow{\nabla} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\nabla\times} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\nabla\cdot} C^\infty(\mathbb{R}^3)$$

is an exact sequence. Again, the same holds true if we are working on an open subset of $\mathbb{R}^3$ that is contractible.

1.2.2 Constructing scalar and vector potentials for Maxwell's equations

Now lets go back to our electric and magnetic fields. From Maxwell's equations, we know

$$\nabla \cdot \vec{B} = 0,$$

so we would like utilize the exact sequence we constructed to say there exists $\vec{A} \in \mathfrak{X}(\mathbb{R}^3)$ such that $\vec{B} = \nabla\times\vec{A}$, but there is a slight issue: $\vec{B}$ is defined on $\mathbb{R}^4 \cong \mathbb{R}\times\mathbb{R}^3$ not $\mathbb{R}^3$. To get around this issue, we define for each $t \in \mathbb{R}$, $\vec{B}_t : \mathbb{R}^3 \to \mathbb{R}^3$ where $\vec{B}_t(\vec{x}) = \vec{B}(t,\vec{x})$. Since $\nabla\cdot$ only differentiates the spatial coordinates, then

$$\nabla \cdot \vec{B} = 0 \implies \forall t \in \mathbb{R}, \nabla \cdot \vec{B}_t = 0$$

Thus, for each $t \in \mathbb{R}$, we know there exists $\vec{A}_t \in \mathfrak{X}(\mathbb{R}^3)$ such that $\vec{B}_t = \nabla \times \vec{A}_t$. Define $\vec{A} : \mathbb{R}^4 \to \mathbb{R}^3$ where $\vec{A}(t,\vec{x}) = \vec{A}_t(\vec{x})$. Since $\nabla\times$ does not differentiate with respect to time, we have

$$(\nabla \times \vec{A})(x,t) = (\nabla \times A_t)(\vec{x}) = \vec{B}_t(\vec{x}) = \vec{B}(t,\vec{x})$$

so that $\vec{B} = \nabla \times \vec{A}$ as desired. Using

$$\vec{B} = \nabla \times \vec{A} \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

we obtain

$$\begin{aligned} 0 = \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= \nabla \times \vec{E} + \frac{\partial}{\partial t}(\nabla \times \vec{A}) \\ &= \nabla \times \vec{E} + \nabla \times \frac{\partial \vec{A}}{\partial t} \\ &= \nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) \end{aligned}$$

Via a similar argument as the magnetic field, we know there exists $\phi \in C^\infty(\mathbb{R}^4)$ such that

$$-\nabla\phi = \vec{E} + \frac{\partial \vec{A}}{\partial t}$$

which tells us

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

Note the same holds true even if $\vec{B}$ and $\vec{E}$ are defined on $I \times U$ where I is an open interval and U is a contractible open subset of $\mathbb{R}^3$.

One should beware that ϕ and A are not unique. This should come as no surprise as exact forms cannot be uniquely written. For example, we have for any $\chi \in C^\infty(\mathbb{R}^4)$

$$\phi' = \phi - \frac{\partial \chi}{\partial t} \qquad \vec{A}' = \vec{A} + \nabla\chi$$

are two functions which generate the same $\vec{E}$ and $\vec{B}$ fields. For contractible domains, this turns out to completely characterize the degeneracy of ϕ and $\vec{A}$ (see Exercise 1)

Definition 1.2.1.
Let $\rho : \mathbb{R}^4 \to \mathbb{R}$ and $\vec{J} : \mathbb{R}^4 \to \mathbb{R}^3$ be smooth functions satisfying the continuity equation.

- A pair $\phi, \vec{A}$ where $\phi : \mathbb{R}^4 \to \mathbb{R}$ and $\vec{A} : \mathbb{R}^4 \to \mathbb{R}^3$ is called a potential if

  $$\vec{B} = \nabla \times \vec{A} \qquad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

  satisfies Maxwell's equations with charge density ρ and volume current density $\vec{J}$.

- Given a potential $\phi, \vec{A}$, the map ϕ is called a scalar potential and the vector field $\vec{A}$ is called a vector potential.

- Two potentials $\phi, \vec{A}$ and $\phi', \vec{A}'$ are related by a gauge transformation $\chi \in C^\infty(\mathbb{R}^4)$ if

  $$\phi' = \phi - \frac{\partial \chi}{\partial t} \qquad \vec{A}' = \vec{A} + \nabla\chi$$

Through a quick verification, we know that the relation of being related by a gauge transformation is an equivalence relation on the set of potentials. Therefore we can make the following definition.

Definition 1.2.2.
An electromagnetic field on $\mathbb{R}^4$ is an equivalence class of potentials on $\mathbb{R}^4$.

1.2.3 Importance of the potentials

Using the potentials, we know we only need to check that the equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon} \quad \text{and} \quad \nabla \times \vec{B} - \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

are satisfied as the other two equations are always satisfied due to how we defined $\vec{B}$ and $\vec{E}$ in terms of ϕ and $\vec{A}$. However, the potentials also play an important role outside of finding $\vec{B}$ and $\vec{E}$ to satisfy Maxwell's equations. Later on, we will consider Maxwell's equations in the realm of special relativity, and we will need a means of computing $\vec{E}$ and $\vec{B}$ such that our fields are invariant under Lorentz transformations. Directly computing Lorentz invariant electric and magnetic fields would be challenging as there are four equations we need to consider. However, with the potentials, we only have two equations that need to be invariant under Lorentz transformations. In addition to being our access point for doing special relativistic E&M, the potentials will also provide a frame work for considering field theories. Finally, the potentials also contain information about the magnetic and electric field that can describe non-trivial behavior even if the magnetic or electric field is zero such as in the Aharono-Bohm effect.

1.3 Lagrangian for Maxwell's equations

Having familiarized ourselves with Maxwell's equations as well as the potentials, we complete our overview by determining the lagrangian whose Euler-Lagrange equations are Maxwell's equations. We first begin with a brief overview of the lagrangian and the development of the Euler-Lagrange equations from classical mechanics. We then look at the Euler-Lagrange equation with scalar fields and vector field. Afterwards, we will take a look at the reverse problem to help determine if Maxwell's equations are the equations of motion for some lagrangian. We then conclude by taking the electrodynamics lagrangian and verify the equations of motion for it are Maxwell's equations.

1.3.1 Lagrangian in classical mechanics

In classical mechanics, we can determine the trajectory of a particle by applying Newton's Second Law:

$$\vec{F} = m\vec{a} = m\ddot{\vec{x}}$$

Alternatively, if we know the kinetic and potential energy, we can apply Hamilton's Principle which states that a particle will travel along the path $\vec{x} : [t_0, t_1] \to \mathbb{R}^n$ from $\vec{a} \in \mathbb{R}^n$ to $\vec{b} \in \mathbb{R}^n$ such that

$$\int_{t_0}^{t_1} K(\dot{\vec{x}}(t)) - V(\vec{x}(t), t)dt$$

is minimized. The difference

$$K(\dot{\vec{x}}(t)) - V(\vec{x}(t), t) = \mathscr{L}(t, \vec{x}(t), \dot{\vec{x}}(t)) = \mathscr{L}(t, x^1(t), \dot{x}^1(t), \ldots, x^n(t), \dot{x}^n(t))$$

where $\vec{x} = (x^1, \ldots, x^n)$ is called the Lagrangian. To solve this problem, we define

$$C^\infty_{\vec{a},\vec{b}}([t_0, t_1], \mathbb{R}^n) = \{q \in C^\infty([t_0, t_1], \mathbb{R}^n) : q(t_0) = \vec{a}, q(t_1) = \vec{b}\},$$

to define the action functional

$$S : C^\infty_{\vec{a},\vec{b}}([t_0, t_1], \mathbb{R}^3) \to \mathbb{R} \qquad S(q) = \int_{t_0}^{t_1} \mathscr{L}(q^1(t), \dot{q}^1(t), \ldots, q^n(t), \dot{q}^n(t), t)dt.$$

This action functional has q as a critical point if and only if

$$\frac{\partial \mathscr{L}}{\partial q^i} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}^i} = 0 \quad i = 1, 2, \ldots, n$$

called the Euler Lagrange equations.

1.3.2 Lagrangian for fields

In the classical mechanics, the position functions and velocities determines how the system evolves over time. In field theory, the fields and the changing of the fields with respect to space time coordinates determine how the field changes over space-time. Suppose we have smooth function $\phi : \mathbb{R}^4 \to \mathbb{R}$, then our Lagrangian will be a function

$$\mathscr{L}(x^1, x^2, x^3, t, \phi, \frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}, \frac{\partial \phi}{\partial x^3}, \frac{\partial \phi}{\partial t})$$

where x^1, x^2, x^3 **are independent of** t and are the independent variables (along with t). Thus we obtain an action functional which integrates the lagrangian over space time which we want to minimize. Since we are varying the scalar field ϕ, then, applying Calculus of Variations, yields that ϕ is a critical point of the action functional if and only if

$$\frac{\partial \mathscr{L}}{\partial \phi} - \frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial(\frac{\partial \phi}{\partial t})} - \sum_{i=1}^{3} \frac{\partial}{\partial x^i}\frac{\partial \mathscr{L}}{\partial(\frac{\partial \phi}{\partial x^i})} = 0$$

In the case we have multiple scalars functions, then we obtain such an equation for each scalar. If we have a vector field $\vec{A} : \mathbb{R}^4 \to \mathbb{R}^3$ with $\vec{A} = (A^1, A^2, A^3)$, then we obtain such an equation for each A^i.

1.3.3 Finding a lagrangian from the equations of motion

So far, we have consider a Lagrangian and determined the equations of motion using the Euler-Lagrange equations. If we start with some differential equations, such as Maxwell's equations, then we would like to know two items:

1. Under what conditions does there exists a lagrangian whose Euler-Lagrange equations are the equations we started with.

2. Under such conditions, how do we determine such a lagrangian.

For two degrees of freedom, the first question has an answer in the following theorem.

Theorem 1.3.1 (Douglas, 1941)**.**
Suppose for $1 \leq i, j \leq n$, we have the second order differential equations

$$\ddot{u}^i = f^i(u^j, \dot{u}^j)$$

for some times $[0, T]$. Define for each $1 \leq i, j \leq n$,

$$\Phi^i_j = \frac{1}{2}\frac{d}{dt}\frac{\partial f^i}{\partial \dot{u}^j} - \frac{\partial f^i}{\partial u^j} - \frac{1}{4}\sum_{k=1}^{n}\frac{\partial f^i}{\partial \dot{u}^k}\frac{\partial f^k}{\partial \dot{u}^j}$$

and let $\Phi = [\Phi^i_j]$. There exists a lagrangian $L : [0, T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ such that the Euler-Lagrange equations are

$$\ddot{u}^i = f^i(u^j, \dot{u}^j) \quad i = 1, \ldots, n$$

if and only if there exists a symmetric, invertible matrix g where $g_{ij}(u, \dot{u})$ satisfying the Helmholtz conditions:

1. $g\Phi = (g\Phi)^T$.

2. *For $1 \leq i, j \leq n$,*
$$0 = \frac{dg_{ij}}{dt} + \frac{1}{2}\sum_{k=1}^{n}\frac{\partial f^k}{\partial \dot{u}^i}g_{kj} + \frac{\partial f^k}{\partial \dot{u}^j}g_{ki}$$

3. *For $1 \leq i, j, k \leq n$,*
$$\frac{\partial g_{ij}}{\partial \dot{u}^k} = \frac{\partial g_{ik}}{\partial \dot{u}^j}.$$

Using Douglas's Theorem, the Helmholtz conditions allows one to construct a lagrangian as a integral equation. More information for this construction can be found in [GK07].

1.3.4 The lagrangian for Maxwell's equations

An alternative to solving a pretty gnarly integral equation is to guess the lagrangian. Using the symmetry of Maxwell's equations, one can make an educated guess that the Lagrangian would be

$$\mathscr{L}(x^1, x^2, x^3, t, \phi, \vec{A}) = \frac{1}{2}(\epsilon_0||\vec{E}||^2 + \frac{1}{\mu_0}||\vec{B}||^2) - \rho\phi + \vec{J}\cdot\vec{A}$$

Varying ϕ, we obtain the Euler-Lagrange equation

$$\frac{\partial \mathscr{L}}{\partial \phi} - \frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial(\frac{\partial \phi}{\partial t})} - \sum_{i=1}^{3}\frac{\partial}{\partial x^i}\frac{\partial \mathscr{L}}{\partial(\frac{\partial \phi}{\partial x^i})} = 0.$$

Recall $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$, so the Euler-Lagrange equation becomes

$$-\rho - 0 - \epsilon_0\sum_{i=1}^{3} -\frac{\partial}{\partial x^i}(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A^i}{\partial t}) = -\rho + \epsilon_0\nabla\cdot\vec{E} = 0$$

which implies Gauss's Law:

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

Varying $\mathscr{L}$ with respect to each scalars A^1, A^2, and A^3 in the vector potential $\vec{A}$, one can obtain Ampere's Law.

1.4 Exercises

Lecture 1 Problem Set

1. Use Maxwell's equations to derive the continuity equation.

2. Let U be an open, contractible subset of $\mathbb{R}^4$. Let $\phi, \phi' \in C^\infty(U)$ be time dependent scalar fields, and let $\vec{A}, \vec{A}' : U \to \mathbb{R}^3$ be C^∞ time dependent vector fields. Show that if $\phi, \vec{A}$ and $\phi', \vec{A}'$ generate the same electric and magnetic field, then there exists a gauge transformation relating $\phi, \vec{A}$ to $\phi', \vec{A}'$.

3. The following questions provide you the opportunity to compute some Euler-Lagrange equations.

   (a) Let $\phi \in C^\infty(\mathbb{R}^4)$, and let $\vec{A} : \mathbb{R}^4 \to \mathbb{R}^3$ be a C^∞ vector field. Let

   $$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla\times\vec{A}.$$

   Show the Euler-Lagrange equations from the lagrangian

   $$\mathscr{L}(t, \vec{x}, \dot{\vec{x}}) = \frac{1}{2}m(\dot{\vec{x}}\cdot\dot{\vec{x}}) - Q\phi(t, \vec{x}) + Q\dot{\vec{x}}\cdot\vec{A}(t, \vec{x})$$

   where Q is a real constant and $\vec{x} = (x^1, x^2, x^3)$ implies the Lorentz Force Law: $\vec{F} = Q(\vec{E} + \dot{\vec{x}}\times\vec{B})$.

(b) (From [BF92])Consider the real lagrangian density

$$\mathscr{L} = \frac{\hbar}{2m}(\nabla\phi)\cdot(\nabla\phi^*) + V\phi\phi^* - \frac{i\hbar}{2}(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t})$$

where ϕ and ϕ^* are independent of one another and are functions of space-time, V is a function of space-time, and $\hbar$ is a constant. Show the Euler-Lagrange equations from this lagrangian density implies Schrodinger's Equation:

$$-\frac{\hbar}{2m}\nabla^2\phi + V\phi = i\hbar\frac{\partial\phi}{\partial t} \qquad -\frac{\hbar}{2m}\nabla^2\phi^* + V\phi^* = -i\hbar\frac{\partial\phi^*}{\partial t}$$

(c) Show the Euler-Lagrange equations from the electrodynamics lagrangian density for the scalar fields from the vector potential implies Ampere's Law.

4. (From [BF92]) If one starts with equations of motion and determines an appropriate lagrangian density, then it makes sense to ask if the lagrangian density is unique. The answer turns out to be no. Suppose we have a lagrangian density

$$\mathscr{L} = \mathscr{L}(x_k, \phi_j, \frac{\partial\phi_j}{\partial x_k}) \qquad k = 1,\ldots,n \quad j = 1,\ldots,m$$

where $x_1,\ldots,x_n$ are the independent variables and $\phi_1,\ldots,\phi_m$ are the scalar fields dependent on $x_1,\ldots,x_n$. Suppose $f = (f_1,\ldots,f_n)$ is a $\mathbb{R}^n$ valued-function where $f_k = f_k(\phi_1,\ldots,\phi_m)$. Show the lagrangian density

$$\mathscr{L}' = \mathscr{L} + \sum_{k=1}^{n}\frac{\partial f_k}{\partial x^k}$$

generates the same Euler-Lagrange equations as $\mathscr{L}$.

2 Electrostatics

2.0.1 Setup

For electrostatics, we make the following assumptions:

$$\vec{J} = 0 \qquad \vec{A} = 0$$
$$\frac{\partial \rho}{\partial t} = 0 \qquad \frac{\partial \phi}{\partial t} = 0$$

Under these assumptions, the continuity equation is always satisfied:

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 + 0 = 0.$$

Since $\vec{A} = 0$ and $\vec{B} = \nabla \times \vec{A}$, then we know $\vec{B} = 0$. Furthermore, we have

$$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t} = -\nabla \phi.$$

Therefore we know Faraday's Law is always satisfied, the divergence of $\vec{B}$ is always zero, and Ampere's Law is always satisfied. Thus, the only equation that we need to actually check is Gauss's Law:

$$\frac{\rho}{\epsilon_0} = \nabla \cdot \vec{E} = -\nabla \cdot (\nabla \phi) = -\nabla^2 \phi.$$

Thus, for electrostatics, we need to our scalar field ϕ to satisfy Poisson's equation.

2.0.2 Aside on the Divergence Theorem for oriented Riemannian manifolds

An important tool for working with Poisson's equation is the Divergence Theorem. For orientable manifolds, this result follows from Stokes' theorem. For non-orientable manifold, the result still holds were we are integrating densities.

To see the orientable case, suppose (M, g) is a C^∞ Riemannian manifold with boundary of dimension n. Since ∂M is a C^∞ embedded submanifold of M, then the inclusion map inc : $\partial M \to M$ is an immersion. Therefore we can pullback the Riemannian metric, denote as $\tilde{g}$, to make ∂M into a Riemannian manifold. Now, suppose M is orientable, then we can take the orientating top form as the canonical Riemannian volume form ω_g. To orientate the boundary ∂M, we equip ∂M with the boundary orientation. Recall, the boundary orientation is given by $i_X \omega$ where X is any outward pointing vector field along ∂M and ω is any top form in the equivalence class of the orientation on M. Since we have a Riemannian metric, we can take X such that the vector field is normal along the boundary. In fact, there is a unique such vector field which we denote as N. Taking $\omega = \omega_g$, we obtain

$$\omega_{\tilde{g}} = i_N(\omega_g)$$

where $\omega_{\tilde{g}}$ is the canonical Riemannian volume form on ∂M with respect to the boundary orientation.

Since M is a Riemannian manifold, we have, similar to $\mathbb{R}^3$, an isomorphism $i\omega_g : \mathfrak{X}(M) \to \Omega^{n-1}(M)$ where $X \mapsto i_X\omega_g$. Such differential form pullback nicely under inc as

$$\text{inc}^*(i_X\omega) = \langle X, N\rangle_g \omega_{\tilde{g}}.$$

Furthermore, taking the exterior of such differential forms provides an generalization of the divergence operator $\nabla\cdot$ on $\mathbb{R}^3$:

$$\text{div} : \mathfrak{X}(M) \to \Omega^n(M) \qquad X \mapsto d(i_X\omega_g)$$

Applying Stokes' Theorem with theses identifications yields the Divergence Theorem for oriented Riemannian manifolds.

Theorem 2.0.1 (Divergence Theorem (orientable case))**.**
Let (M, g) be an oriented Riemannian C^∞ manifold of dimension n with boundary. For any $X \in \mathfrak{X}(M)$ with compact support,

$$\int_M div(X) = \int_{\partial M} \langle X, N\rangle_g \omega_{\tilde{g}}$$

where $\tilde{g}$ is the induced metric on ∂M, N is the unique outward pointing normal vector field along ∂M, and $\omega_{\tilde{g}}$ is the canonical Riemannian volume form on ∂M.

Proof.
By Stokes' Theorem, we have

$$\int_M \text{div}(X) = \int_M d(i_X\omega_g) = \int_{\partial M} \text{inc}^*(i_X\omega_g) = \int_{\partial M} \langle X, N\rangle_g \omega_{\tilde{g}}.$$

□

For $\mathbb{R}^n$, one only needs to work on an open, bounded subset U whose boundary is a C^1 manifold, and, instead of vector fields, one works with $f \in C^1(U)$ functions f so that the result reads

$$\int_U \nabla f dx^1 \dots dx^n = \int_{\partial U} f\hat{n}dA$$

where $\vec{n}$ is an outward pointing normal unit normal vector field along ∂U.

2.0.3 Uniqueness of solutions to Poisson's equation

Using the Divergence Theorem for $\mathbb{R}^3$, we can prove uniqueness of solutions to Poisson's equation when ϕ vanishes at infinity, $||\vec{x}||\phi$ is bounded, and $||\vec{x}|| * ||\nabla\phi||$ vanishes at infinity.

Theorem 2.0.2.
For any given charge density $\rho : \mathbb{R}^3 \to \mathbb{R}$, there exists at most one solution to

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

such that

1. $\lim_{||\vec{x}|| \to \infty} \phi = 0.$

2. $||\vec{x}||\phi$ *is bounded*

3. $\lim_{||\vec{x}|| \to \infty} ||\vec{x}|| * ||\nabla\phi|| = 0$

Proof.

Suppose ϕ and ϕ' are two solutions satisfying the three conditions, then $\psi = \phi - \phi'$ is a solution to $\nabla^2\psi = 0$ which satisfies the three conditions. Since

$$0 = \psi(\nabla^2\psi) = \nabla \cdot (\psi\nabla\psi) - ||\nabla\psi||^2,$$

then for any ball of radius R, we have

$$0 = \int_{r \le R} \nabla \cdot (\psi\nabla\psi) - ||\nabla\psi||^2 dx^1 dx^2 dx^3$$

Let $\hat{n}$ be a unit normal vector field along the boundary of the ball of radius R, then the Divergence Theorem tells us

$$\int_{r \le R} \nabla \cdot (\psi\nabla\psi) dx^1 dx^2 dx^3 = \int_{r=R} \psi\hat{n} \cdot \nabla\psi dA.$$

Therefore

$$|\int_{r=R} \psi\hat{n} \cdot \nabla\psi dA| \le 4\pi R^2 ||\psi||_R * ||\nabla\psi||_R$$

where $||\psi||_R$ is the maximum of ψ on the surface similarly for $||\nabla\psi||_R$. Using condition (2), we know there exists $C > 0$ such that $R||\psi||_R < C$ for all $R > 0$. Condition (3) tells us

$$\lim_{R \to \infty} R||\nabla\psi||_R = 0.$$

Thus

$$\lim_{R \to \infty} 4\pi R^2 ||\psi||_R * ||\nabla\psi||_R \le \lim_{R \to \infty} 4\pi C R ||\nabla\psi||_R = 0$$

which implies

$$\lim_{R \to \infty} \int_{r \le R} \nabla \cdot (\psi\nabla\psi) dx^1 dx^2 dx^3 = \lim_{R \to \infty} \int_{r=R} \psi\hat{n} \cdot \nabla\psi dA = 0$$

Therefore

$$0 = \lim_{R \to \infty} \int_{r \le R} \nabla \cdot (\psi\nabla\psi) - ||\nabla\psi||^2 dx^1 dx^2 dx^3 = -\int_{\mathbb{R}^3} ||\nabla\psi||^2 dx^1 dx^2 dx^3$$

so that $||\nabla\psi||^2 = 0$ for all $\mathbb{R}^3$. Hence $\nabla\psi = 0$ so that ψ is a constant on $\mathbb{R}^3$. Condition (1) for ψ implies the constant is zero. Hence $\psi = \phi - \phi' = 0$ so that $\phi = \phi'$ as desired.

□

2.0.4 Aside on Generalized Functions and Distributions

Many charge distributions can be written using the Dirac Delta function such as a point charge or a charge along the surface of a sphere. Therefore we take this moment to discuss generalized functions and distributions which the Dirac Delta function is an example of.

Definition 2.0.3.
Let $U \subset \mathbb{R}^n$ be an open set. Denote the set of smooth real valued function on U with compact support as $C_c^\infty(U)$.

- We call elements of $C_c^\infty(U)$ test functions.
- The real vector space $\mathcal{D}(U) := C_c^\infty(U)$ under pointwise addition and scalar multiplication is called the space of all test functions on U.
- For each $\alpha \in \mathbb{N}_0^n$, let $|\alpha| = \alpha_1 + \ldots + \alpha_n$.
- For each $\alpha \in \mathbb{N}_0^n$, define
$$D^\alpha = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x^2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n}$$
- For each $m \in \mathbb{N}_0$, define
$$||\cdot||_m : \mathcal{D}(U) \to \mathbb{R} \qquad ||\phi||_m = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| < m} ||D^\alpha \phi||_\infty$$
where $||\cdot||_\infty$ is the L^∞-norm with respect to the Lebesgue measure on U.
- A sequence $(\phi_i)_{i\in\mathbb{N}}$ in $\mathcal{D}(U)$ converges to ϕ in $\mathcal{D}(U)$ if there exits a compact subset $K \subset U$ such that $\text{supp}(\phi_i) \subset K$ for each $i \in \mathbb{N}$ and $\lim_{i\to\infty} ||\phi_i - \phi||_m = 0$ for each $m \in \mathbb{N}_0$.
- A sequence $(\phi_i)_{i\in\mathbb{N}}$ in $\mathcal{D}(U)$ is Cauchy if there exists a compact subset $K \subset U$ such that $\text{supp}(\phi_i) \subset K$ for each $i \in \mathbb{N}$ and for each $\epsilon > 0$ and each $m \in \mathbb{N}_0$, there exits $N \in \mathbb{N}$ such that $||\phi_i - \phi_j||_m < \epsilon$ whenever $i, j \geq N$.

With respect to this convergence, we can endow $\mathcal{D}(U)$ with the uniform convergence topology. Thus we can make the following definition.

Definition 2.0.4.
Let $U \subset \mathbb{R}^n$ be an open subset.

- A distribution or generalized function on U is continuous $\mathbb{R}$-linear map $T : \mathcal{D}(U) \to \mathbb{R}$.
- We denote the vector space of all distribution on U as $\mathcal{D}'(U)$.

Checking continuity with respect to the topology on $\mathcal{D}(U)$ would not be a pleasant experience. Thankfully, for linear functionals, we have the following equivalent notions of continuity for maps from $\mathcal{D}(U)$ to $\mathbb{R}$.

Theorem 2.0.5.
Let U be an open subset of $\mathbb{R}^n$, and let $T : \mathcal{D}(U) \to \mathbb{R}$ be a linear map. Then the following are equivalent:

1. *T is continuous.*
2. *T is sequentially continuous.*
3. *T is sequentially continuous at the zero function.*
4. *For every compact set K of U, there exists $n \in \mathbb{N}_0$ and $C > 0$ such that $|T(\phi)| \leq C||\phi||_n$ for all $\phi \in \mathcal{D}(U)$ with supp$(\phi) \subset K$.*

The parts of the proof for this statement can be found in chapter 5 of [AB53].

Example 2.0.6.

1. Let $U \subset \mathbb{R}^n$, and fix $p \in U$. Consider $\delta_p : \mathcal{D}(U) \to \mathbb{R}$ where $\delta_p(\phi) = \phi(p)$. Using (4) in the prior theorem shows δ_p is indeed continuous. The details are left as an exercises. Hence δ_p is distribution called the Dirac Delta function.
2. Let $U \subset \mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a locally integrable function; that is, f is measurable and for each compact subset $K \subset U$, $\int_K |f(x)|dx < \infty$. Define $\Lambda_f : \mathcal{D}(U) \to \mathbb{R}$ where

$$\Lambda_f(\phi) = \int_U f(x)\phi(x)dx.$$

We claim Λ_f is a distribution. First, since the integral is linear, then Λ_f is a linear functional. Therefore it remains to see that Λ_f is continuous which is left as an exercises.

It turns out the map from locally integrable functions on U to distribution on U where $f \mapsto \Lambda_f$ is almost everywhere one-to-one in the sense that $\Lambda_f = \Lambda_g$ if and only if $f = g$ almost everywhere. Again, a proof of this claim can be found in chapter 5 of [AB53].

One important tool for test functions is differentiation. Indeed, $D^\alpha : \mathcal{D}(U) \to \mathcal{D}(U)$ is a linear map which is sequentially continuous for each $\alpha \in \mathbb{N}_0^n$. Thus, given $T \in \mathcal{D}'(U)$, we can define

$$(D^\alpha)^* T = T \circ D^\alpha : \mathcal{D}(U) \to \mathbb{R}$$

which is a distribution on U. Note, in the case $f \in C_c^\infty(U)$, then

$$D^\alpha \Lambda_f = \Lambda_{D^\alpha f}$$

which means that for each $\phi \in \mathcal{D}(U)$,

$$D^\alpha \int_U f(x)\phi(x)dx = \int_U (D^\alpha f)(x)\phi(x)dx$$

Using integration by parts, we know

$$\int_U (D^\alpha f)(x)\phi(x)dx = (-1)^{|\alpha|} \int_U f(x)(D^\alpha \phi)(x)dx.$$

Therefore we can generalize the notion to the derivative of a distribution as follows.

Definition 2.0.7.
Let $U \subset \mathbb{R}^n$ be open. Given $T \in \mathcal{D}'(U)$ and $\alpha \in \mathbb{N}_0^n$, we define

$$D^\alpha T : \mathcal{D}(U) \to \mathbb{R} \qquad (D^\alpha T)(\phi) = (-1)^{|\alpha|}(T \circ D^\alpha)(\phi)$$

2.0.5 Green's functions for differential operators

Now that we had our detour into generalized functions, we return to solving Poisson's equation. To solve such an equation, we utilize Green's function. To motivate Green's function, we follow a similar approach in [BF92]. Suppose we have a differential operator L such as $L = \frac{d^n}{dx^n}$ or $L = \nabla^2$. For a fixed function g, we can ask for which function(s) f, if any exist, satisfy

$$Lf = g.$$

If L was a n-by-n matrix which was invertible, then we can easily solve this problem by computing L^{-1} and have $f = L^{-1}g$. Since L is an operator on an infinite dimensional function space, it is a challenge to find a left hand inverse of L. To help in this process, let us suppose that L is an operator on a Hilbert space. In particular, let's consider the space of square integrable functions on $\mathbb{R}^n$. Suppose L has a complete, orthonormal basis in terms of eigenfunctions of L; that is, there exists a $\{\phi_n : n \in \mathbb{N}\}$ such that

$$L\phi_n = \lambda_n \phi_n \quad n \in \mathbb{N}$$

and the closure of the span of $\{\phi_n : n \in \mathbb{N}\}$ is the entire Hilbert space. Since f and g are vectors in our Hilbert space, then we can write

$$f = \sum_{i=1}^{\infty} a_i \phi_i \qquad g = \sum_{j=1}^{\infty} b_j \phi_j$$

where a_i and b_j are real numbers. Therefore

$$Lf = \sum_{i=1}^{\infty} a_i L\phi_i = \sum_{i=1}^{\infty} \lambda_i a_i \phi_i = \sum_{j=1}^{\infty} b_i \phi_n$$

which implies

$$\sum_{i=1}^{\infty} (\lambda_i a_i - b_i)\phi_i = 0.$$

By linearly independence of our ϕ_i's, we know the coefficients must be zero. In the case $\lambda_i \neq 0$, then we can solve for a_i

$$a_i = \frac{b_i}{\lambda_i}.$$

Otherwise, if $\lambda_i = 0$, then $b_i = 0$. Using the inner product on our Hilbert space (which is given by integration) and the orthogonality of the ϕ_i's , we know

$$b_i = \langle g, \phi_i \rangle = \int g(x')\phi_i(x')dx'.$$

Thus, we have

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{1}{\lambda_i} \langle \phi_i, g \rangle \phi_i = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \phi_i(x) \int \phi_i(x') g(x') dx' \\ &= \int \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \phi_i(x)\phi_i(x') g(x') dx' \\ &= \int \left(\sum_{n=1}^{\infty} \frac{\phi_i(x)\phi_i(x')}{\lambda_i} \right) g(x') dx' \\ &= \int G(x,x') g(x') dx' \end{aligned}$$

where the function $G(x, x')$ is called the Green's function for L. We can view $G(x, x')$ as a distribution of a locally integrable function. Thus, it makes sense to consider what is the distribution $LG(x, x')$? The claim is that

$$LG(x, x') = \delta(x - x').$$

To see this, we show $LG(x, x')$ is the same linear functional. By the properties of L locally integrable functions, we know

$$\begin{aligned} LG(x,x') &= L \sum_{i=1}^{\infty} \frac{\phi_i(x)\phi_i(x')}{\lambda_i} = \sum_{i=1}^{\infty} \frac{(L\phi)\phi(x')}{\lambda_i} \\ &= \sum_{i=1}^{\infty} \phi_i(x)\phi_i(x') \end{aligned}$$

so that

$$\begin{aligned} \int LG(x,x')g(x')dx &= \int \left(\sum_{i=1}^{\infty} \phi_i(x)\phi_i(x') \right) g(x')dx' \\ &= \sum_{i=1}^{\infty} \phi_i(x) \int \phi_i(x')g(x')dx' \\ &= \sum_{i=1}^{\infty} \phi_i(x) \langle \phi_i, g \rangle = g(x) \end{aligned}$$

Hence

$$LG(x, x') = \delta(x - x')$$

as claimed.

Therefore to solve $Lf = g$, our goal is to determine the Green's function for L which is characterized by

$$LG(x, x') = \delta(x - x')$$

so that a solution f to $Lf = g$ is given by

$$f(x) = \int G(x, x')g(x')dx'$$

If L admits a complete, orthonormal basis for the Hilbert space in terms of eigenfunctions, then we can write down the Green's function. If the operator L does not admit such a basis, then we have some work to do to determine $G(x, x')$.

2.0.6 Green's function and the Poisson equation

Let's now specialize to the case of Poisson's equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

To solve this, the goal remains the same: determine the Green's function. We start by considering the case that we want ϕ to vanish at infinity. To impose this boundary condition, we can simply impose it on our Green's function as, after all,

$$\phi(x) = \frac{-1}{\epsilon_0} \int G(x, x')\rho(x')dx'$$

which will vanish at infinity if $G(x, x')$ vanishes at infinity in x. Therefore we are looking for $G(x, x')$ such that

$$\nabla^2 G(x, x') = \delta(x - x') \quad \text{and} \quad G(x, x') \to 0 \text{ as } x \to \infty$$

It turns out that we can easily identify this Green's function by ansatz as

$$G(x, x') = \frac{-1}{4\pi||x - x'||}$$

so that

$$\phi(x) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x')}{||x - x'||} dx'$$

Example 2.0.8 (point charge)**.**

Consider a point charge q at the origin which has a charge distribution given by $\rho(x) = q\delta(x)$. Then

$$\phi(x) = \frac{1}{4\pi\epsilon_0} \int \frac{q\delta(x')}{||x - x'||} dx' = \frac{1}{4\pi\epsilon_0} \frac{q}{||x||}.$$

Applying $\vec{E} = -\nabla\phi$, we find

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{||\vec{r}||^3} \vec{r}$$

which agrees with the electric field of a point charge that we defined from Coulomb's Law.

Now let's consider the case we are working on a bounded domain $U \subset \mathbb{R}^3$ and want to impose a boundary condition along ∂U given by a function ψ. The way we construct ϕ becomes a bit more complicated but follows the same premise: determine the Green's function. Recall, the Divergence Theorem tells us that for a vector field $\vec{X}$ on U and a C^1 function f that

$$\int_U f\nabla \cdot \vec{X} + \nabla f \cdot \vec{X} dV = \oint_{\partial U} f\vec{X} \cdot \hat{n} dA.$$

Taking $\vec{X} = G\nabla\phi - \phi\nabla G$ and f as the constant function at one, the integrals become

$$\frac{-1}{\epsilon_0}\int_U G(x,x')\rho(x')dV - \phi(x) = \oint_{\partial U} G(x,x')D_{\hat{n}}\phi - \phi(x')D_{\hat{n}}GdA$$

so that

$$\phi(x) = \frac{-1}{\epsilon_0}\int_U G(x,x')\rho(x')dV + \oint_{\partial U} G(x,x')D_{\hat{n}}\phi - \phi(x')D_{\hat{n}}GdA.$$

Now if we impose $G(x,x')$ vanishes on the boundary and we want $\phi|_{\partial U} = \psi$, then ϕ is determined by

$$\phi(x) = \frac{-1}{\epsilon_0}\int_U G(x,x')\rho(x')dV + \oint_{\partial U} \psi(x')D_{\hat{n}}GdA$$

which is completely in terms of the Green's function, the charge density, and our boundary condition ψ. To have $G(x,x')$ vanish at the boundary and satisfy $\nabla^2 G(x,x') = \delta(x,x')$ we can take

$$G(x,x') = \frac{-1}{4\pi||x-x'||} + F(x,x')$$

where we pick F to be the function that makes G vanish on ∂U as well as satisfies $\nabla^2 F(x,x') = 0$ on U.

2.1 Exercises

1. Let $p \in \mathbb{R}$.
   (a) Verify the Dirac Delta function at p, denoted as δ_p, is a distribution on $\mathbb{R}$.
   (b) Verify the heavy side function at p, denoted as H_p, is locally integrable on $\mathbb{R}$.
   (c) Verify $\frac{d}{dx}\Lambda_{H_p} = \delta_p$.
2. Verify that if G is the Green's function for a continuous partial differential operator L, then a solution to $Lf = g$ is indeed $f(x) = \int G(x,x')g(x')dx'$.
3. Verify taking $\vec{X} = G\nabla\phi - \phi\nabla G$ and f as the constant function at one in the Divergence Theorem does in fact yield the equation for ϕ in terms of the the Green's function G, charge density ρ, and boundary condition ψ.
4. The following problem will have you prove the uniqueness for solutions to the Dirichlet Boundary Value Problem. Let $U \subset \mathbb{R}^n$ be open such that $\overline{U}$ is compact. Fix a continuous functions $f : \partial U \to \mathbb{R}$ and $g : U \to \mathbb{R}$. Let $\phi : \overline{U} \to \mathbb{R}$ be a function such that ϕ is C^2 on U and ϕ is continuous on ∂U. Show that if $\phi|_{\partial U} = f$ and $\nabla^2\phi = g$ on U, then ϕ is unique. (Hint: consider an approach similar to showing Poisson's equation has unique solutions using the Divergence Theorem)

3 Electrodynamics

3.1 The Equations of Electrodynamics

3.1.1 The local charge-current conservation

In an open set $U \subset \mathbb{R} \times \mathbb{R}^3$ we consider a charge density $\rho(t, \mathbf{x})$ and a current density $\mathbf{J}(t, \mathbf{x})$ satisfying the charge-current conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \text{ in } U,$$

which, upon using the divergence theorem, is equivalent to

$$\frac{d}{dt} \int_{\mathcal{V}} \rho \, dV = - \int_{\partial \mathcal{V}} \mathbf{J} \cdot \mathbf{dS}$$

for all $\mathcal{V} \subset U$. The above equation explains the local nature of the charge-current conservation. We then ask, for the above charge density and current density, if we can find an electric field $\mathbf{E}(t, \mathbf{x})$ and a magnetic field $\mathbf{B}(t, \mathbf{x})$ satisfying Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0, \, \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \, \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \, \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}.$$

Observe that the charge-current conservation is a necessary condition of the last two Maxwell's equations (1.4 Exercises, Problem 1). Also, if there are potentials ϕ and $\mathbf{A}$ in U such that

$$\mathbf{B} = \nabla \times \mathbf{A}, \, \mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t},$$

then the first two Maxwell's equations hold: the converse is true in case that U is simply connected (the last sentence in 1.2.1). Indeed, using the sign convention (-,+,+,+), we see that the 2-form on U

$$\begin{aligned} F &= \frac{1}{c} \mathbf{E} \cdot d\mathbf{r} \wedge dt + \mathbf{B} \cdot (d\mathbf{r} \times d\mathbf{r}) \\ &= \frac{1}{c} (E_x \, dx + E_y \, dy + E_z \, dz) \wedge dt + B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy \end{aligned}$$

is closed in view of the first two Maxwell's equations. Then, applying the Poincaré lemma to the simply connected region U, we get a 1-form A with $A = -\frac{\phi}{c} \, dt + A_x \, dx + A_y \, dy + A_z \, dz$ such that $F = dA$, which yields the above expressions of $\mathbf{B}$ and $\mathbf{E}$ in terms of the potentials.

3.1.2 Einstein summation convention and the Levi-Civita symbol

Here we are using indices ranging over $1, 2, 3$ and our vectors are from $\mathbb{R}^3$. In the following equations we will use the summation convention: a pair of repeated indices means summation as in

$$a_i b_i = \sum_{i=1}^{3} a_i b_i.$$

Also we introduce the Levi-Civita symbol:

$$\epsilon_{ijk} = \delta_{i1}\delta_{j2}\delta_{k3} + \delta_{j1}\delta_{k2}\delta_{i3} + \delta_{k1}\delta_{i2}\delta_{j3} - \delta_{i1}\delta_{k2}\delta_{j3} - \delta_{k1}\delta_{j2}\delta_{i3} - \delta_{j1}\delta_{i2}\delta_{k3}.$$

Notice that $\epsilon_{ijk} = 0$ unless i, j, k are distinct. Also, we find that $(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk} A_j B_k$ and $(\nabla \times \mathbf{B})_i = \epsilon_{ijk} \partial_j B_k$. A useful identity is

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.$$

We see that both sides are zero unless $i \neq j$ and $l \neq m$. In the left hand side the summation over k contributes for exactly one k when $k \neq i, j$ and $k \neq l, m$, which implies $\{i, j\} = \{l, m\}$. In that case both sides are 1 or -1 depending on whether $i = l, j = m$ or $i = m, j = l$. We will use the following identities, which are left as exercises:

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$$
$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{2}\nabla |\mathbf{E}|^2 - (\mathbf{E} \cdot \nabla) \cdot \mathbf{E}.$$

3.1.3 Conservation of energy and momentum

In order to consider conservation of energy and momentum let's define the energy density, the momentum density and the stress tensor of our electromagnetic fields by

$$\mathcal{E} = \frac{1}{2}\left(\epsilon_0 |\mathbf{E}|^2 + \frac{1}{\mu_0}|\mathbf{B}|^2\right), \ \mathcal{P} = \epsilon_0 \mathbf{E} \times \mathbf{B}, \ \Theta_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \delta_{ij}\mathcal{E}.$$

Also, we define the Poynting vector by $\mathcal{S} = c^2 \mathcal{P} = \mathbf{E} \times \mathbf{B}/\mu_0$ since $c^2\epsilon_0\mu_0 = 1$, and we check that it is the energy flux. Using Maxwell's equations and the vector identity $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$, we have

$$\begin{aligned}
\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathbf{S} &= \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0}\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} + \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}) \\
&= \epsilon_0 \mathbf{E} \cdot \left[c^2(\nabla \times \mathbf{B} - \mu_0 \mathbf{J})\right] + \frac{1}{\mu_0}\mathbf{B} \cdot (-\nabla \times \mathbf{E}) + \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}) \\
&= -\mathbf{E} \cdot \mathbf{J} + \frac{1}{\mu_0}\left[\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{E}) + \nabla \cdot (\mathbf{E} \times \mathbf{B})\right] \\
&= -\mathbf{E} \cdot \mathbf{J}.
\end{aligned}$$

Now, if we postulate that the total energy $\mathcal{E}_{\text{tot}} = \mathcal{E} + \mathcal{E}_{\text{matter}}$ is to be conserved, that is,

$$\frac{\partial \mathcal{E}_{\text{tot}}}{\partial t} + \nabla \cdot \mathcal{S} = 0,$$

then, we obtain $\partial \mathcal{E}_{\text{matter}}/\partial t = \mathbf{E} \cdot \mathbf{J}$: the electromagnetic field adds energy per volume to the matter at this rate.

Similarly from the identity

$$\frac{\partial \mathcal{P}_i}{\partial t} - \partial_j \Theta_{ij} = -[\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}]_i,$$

we see that the electromagnetic field exerts a force per unit volume on matter, that is, $\partial \mathcal{P}_{\text{matter}}/\partial t = \mathbf{f} =$

$\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$, which is nothing but the Lorentz force law. Indeed,

$$\begin{aligned}
\frac{\partial \mathcal{P}_i}{\partial t} - \partial_j \Theta_{ij} &= \epsilon_0 \left(\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \right)_i - \epsilon_0 \partial_j (E_i E_j) - \frac{1}{\mu_0} \partial_j (B_i B_j) + \delta_{ij} \partial_j \mathcal{E} \\
&= \epsilon_0 c^2 \left([(\nabla \times \mathbf{B}) - \mu_0 \mathbf{J}] \times \mathbf{B} \right)_i - \epsilon_0 \left[\mathbf{E} \times (\nabla \times \mathbf{E}) \right]_i - \epsilon_0 \left(E_i \nabla \cdot \mathbf{E} + (\mathbf{E} \cdot \nabla) E_i \right) \\
&\quad - \frac{1}{\mu_0} \left(B_i \nabla \cdot \mathbf{B} + (\mathbf{B} \cdot \nabla) B_i \right) + \epsilon_0 \partial_i \left(\frac{1}{2} |\mathbf{E}|^2 \right) + \frac{1}{\mu_0} \partial_i \left(\frac{1}{2} |\mathbf{B}|^2 \right) \\
&= -[\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}]_i,
\end{aligned}$$

where we used Maxwell's equations $\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \nabla \cdot \mathbf{B} = 0$ and a vector identity we mentioned in the above.

3.1.4 Maxwell's equations in Lorenz gauge

In terms of potentials the last two Maxwell's equations become

$$\begin{aligned}
-\nabla^2 \phi - \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} &= \frac{\rho}{\epsilon_0} \\
-\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) + \frac{1}{c^2} \frac{\partial}{\partial t} \nabla \phi + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} &= \mu_0 \mathbf{J}.
\end{aligned}$$

Define the d'Alembertian

$$\Box = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2$$

and let χ be a solution of the equation $\Box \chi = -s$, where

$$s = \frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A}.$$

Then, with the gauge transformation

$$\phi' = \phi - \frac{\partial \chi}{\partial t}, \ \mathbf{A}' = \mathbf{A} + \nabla \chi,$$

we find that ϕ' and $\mathbf{A}'$ satisfy the Lorenz gauge condition, that is,

$$\frac{1}{c^2} \frac{\partial \phi'}{\partial t} + \nabla \cdot \mathbf{A}' = 0.$$

Dropping primes let's assume that our potentials ϕ and $\mathbf{A}$ already satisfy the Lorenz gauge condition. Then, the above Maxwell's equations in potentials, along with the gauge condition, are written in the same form as

$$\Box \phi = -\frac{\rho}{\epsilon_0}, \ \Box \mathbf{A} = -\mu_0 \mathbf{J}, \ \frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0.$$

With $x^\mu = (ct, \mathbf{x})$, $A^\mu = (\phi/c, \mathbf{A})$, $J^\mu = (c\rho, \mathbf{J})$ and $\partial_\mu = (\partial/\partial x^0, \partial/\partial \mathbf{x})$ we can write the above as

$$\Box A^\mu = -\mu_0 J^\mu, \ \ \partial_\mu A^\mu = 0.$$

3.2 Retarded Green's Function

3.2.1 Fourier transform

Let's review the Fourier transform briefly. Let $g : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ be in the Schwartz space $\mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$, that is, g is smooth and any partial derivative $D^\alpha g$ vanishes faster than $|(t, \mathbf{x})|^{-m}$ for any $m \in \mathbb{N}$ as $|(t, \mathbf{x})|$ tends to infinity. Define its Fourier transform $\hat{g} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ by

$$\hat{g}(\omega, \mathbf{k}) = \left(\frac{1}{\sqrt{2\pi}}\right)^4 \int_{\mathbb{R}\times\mathbb{R}^3} g(t, \mathbf{x}) e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}} \, dt \, d^3x$$

Then it follows that $\hat{g} \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$ and

$$g(t, \mathbf{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^4 \int_{\mathbb{R}\times\mathbb{R}^3} \hat{g}(\omega, \mathbf{k}) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} \, d\omega \, d^3k :$$

this amounts to the Fourier inversion theorem.

In 2.0.4 we defined distributions. Now, we define a tempered distribution T as a linear functional on $\mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$ such that $\langle T, \phi_n \rangle \to 0$ whenever ϕ_n is a sequence in $\mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$ satisfying that, for all n and α, the sequence $|D^\alpha \phi_n||(t, \mathbf{x})|^m$ converges to 0 uniformly as n tends to infinity. Observe that any tempered distribution is a distribution, in particular, delta functions are all tempered distributions. For a tempered distribution T on $\mathbb{R} \times \mathbb{R}^3$ one can define its derivatives $D^\alpha T$ and Fourier transform $\hat{T}$ by $\langle D^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle$ and $\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle$. These turn out to be tempered distributions. Also, the Fourier inversion formula holds for the tempered distributions and for the delta function:

$$\delta(t)\delta(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}\times\mathbb{R}^3} \frac{1}{(2\pi)^2} e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} \, d\omega \, d^3k.$$

3.2.2 Green's function method

We had the equation for the scalar potential $\Box\phi = -\rho/\epsilon_0$, which is readily solved by

$$\phi(t, \mathbf{x}) = \int G(t, \mathbf{x}; t', \mathbf{x}') \frac{\rho}{\epsilon_0}(t', \mathbf{x}') \, d^3x' dt',$$

if G is a Green's function, that is,

$$\Box_{(t,\mathbf{x})} G(t, \mathbf{x}; t', \mathbf{x}') = -\delta(t - t')\delta(\mathbf{x} - \mathbf{x}').$$

Because of the form of the delta function on the right hand side we try $G(t, \mathbf{x}; t', \mathbf{x}') = g(t - t', \mathbf{x} - \mathbf{x}')$. Since

$$\Box g(t, \mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}\times\mathbb{R}^3} \left(\frac{\omega^2}{c^2} - k^2\right) \hat{g}(\omega, \mathbf{k}) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} \, d\omega \, d^3k$$
$$-\delta(t)\delta(\mathbf{x}) = -\frac{1}{(2\pi)^2} \int_{\mathbb{R}\times\mathbb{R}^3} \frac{1}{(2\pi)^2} e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} \, d\omega \, d^3k,$$

we have

$$\left(\frac{\omega^2}{c^2} - k^2\right) \hat{g}(\omega, \mathbf{k}) = -\frac{1}{4\pi^2.}$$

When we use the Fourier inversion to get $g(t, \mathbf{x})$ we first inverse transform with respect to t to get

$$\tilde{g}(t, \mathbf{k}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{g}(\omega, \mathbf{k}) e^{-i\omega t}\, d\omega = -\left(\frac{1}{\sqrt{2\pi}}\right)^5 \int_{\mathbb{R}} \frac{c^2 e^{-i\omega t}}{(\omega - kc)(\omega + kc)}\, d\omega.$$

Notice that the integrand has poles at $\omega = \pm kc$. Thus to get a physically relevant result we use the retarded regularization to write

$$\tilde{g}(t, \mathbf{k}) = \lim_{\substack{\epsilon \downarrow 0 \\ R \uparrow \infty}} -\left(\frac{1}{\sqrt{2\pi}}\right)^5 \int_{C(\epsilon, R)} \frac{c^2 e^{-i\omega t}}{(\omega - kc)(\omega + kc)}\, d\omega,$$

where $C(\epsilon, R)$ is a contour going about $\pm kc$ in clockwise half circle with radius ϵ above the x-axis and closing in a clockwise(counterclockwise) in a half circle with radius R for $t > 0 (t < 0)$. This results in

$$\tilde{g}(t, \mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \frac{c \sin(kct)}{k}$$

for $t > 0$ and zero for $t < 0$. Hence, with $x = |\mathbf{x}|$ we have

$$\begin{aligned}
g_{\text{ret}}(t, \mathbf{x}) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{c \sin(kct)}{k} e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3k \\
&= \frac{1}{8\pi^3} \int_0^\infty k^2\, dk \int_0^{2\pi} d\varphi \int_0^\pi d\theta\, \frac{c \sin(kct)}{k} e^{ikx\cos\theta} \sin\theta \\
&= \frac{1}{4\pi^2} \int_0^\infty k^2\, dk\, \frac{e^{ikx} - e^{-ikx}}{ikx} \frac{c \sin(kct)}{k} \\
&= -\frac{c}{8\pi^2 x} \int_0^\infty dk\, (e^{ikx} - e^{-ikx})(e^{ickt} - e^{-ickt}) \\
&= -\frac{c}{8\pi^2 x} \int_{-\infty}^\infty dk\, (e^{ik(x+ct)} - e^{ik(x-ct)}) \\
&= -\frac{c}{4\pi x} (\delta(x + ct) - \delta(x - ct)) \\
&= \frac{1}{4\pi x} \delta\left(t - \frac{x}{c}\right)
\end{aligned}$$

for $t > 0$ and $\tilde{g}_{\text{ret}}(t, \mathbf{x}) = 0$ for $t < 0$. Therefore we have

$$G_{\text{ret}}(t, \mathbf{x}; t', \mathbf{x}') = \begin{cases} 0 & \text{if } t < t', \\ \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \delta\left(t - t' - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right) & \text{if } t > t'. \end{cases}$$

And we get retarded solution corresponding to the charge density ρ and the current density J:

$$\begin{aligned}
\phi(t, \mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \frac{[\rho(t', \mathbf{x}')]_{\text{ret}}}{\mathbf{x} - \mathbf{x}'|}\, d^3x' \\
\mathbf{A}(t, \mathbf{x}) &= \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{[\mathbf{J}(t', \mathbf{x}')]_{\text{ret}}}{|\mathbf{x} - \mathbf{x}'|}\, d^3x',
\end{aligned}$$

where $[\rho(t', \mathbf{x}')]_{\text{ret}} = \rho(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}')$. One easily checks that the above solution satisfies the Lorenz gauge condition

$$\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0.$$

3.2.3 Radiation zone approximation

Suppose that $\rho(t, \mathbf{x}) = 0$ if $|\mathbf{x}| > d$. And assume that ρ describes periodically moving particles with frequency ω. Assume that the velocity of particles is much less than c. We want to find a good approximation of $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$ for $x = |\mathbf{x}| \gg d, c/\omega$. For $|\mathbf{x}'| \leq d$ we have approximations

$$|\mathbf{x} - \mathbf{x}'| \approx x - \frac{\mathbf{x} \cdot \mathbf{x}'}{x}, \quad \frac{1}{|\mathbf{x} - \mathbf{x}'|} \approx \frac{1}{x} + \frac{\mathbf{x} \cdot \mathbf{x}'}{x^3},$$

thus,

$$\begin{aligned}
[\mathbf{J}(t', \mathbf{x}')]_{\text{ret}} &= \mathbf{J}(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}') \\
&\approx \mathbf{J}\left(t - \frac{x}{c} + \frac{\mathbf{x} \cdot \mathbf{x}'}{xc}, \mathbf{x}'\right) \\
&\approx \mathbf{J}\left(t - \frac{x}{c}, \mathbf{x}'\right) + \dot{\mathbf{J}}\left(t - \frac{x}{c}, \mathbf{x}'\right) \frac{\mathbf{x} \cdot \mathbf{x}'}{xc}.
\end{aligned}$$

Integrating $\partial_j(J_j x_i) = -\frac{\partial \rho}{\partial t} x_i + J_i$ and using $\rho(\mathbf{J}, \mathbf{x}) = 0$ for $|\mathbf{x}| > d$, we get

$$\int \mathbf{J}(t - x/c, \mathbf{x}')\, d^3x' = \frac{d}{dt} \int \rho(t - x/c, \mathbf{x}')\mathbf{x}\, d^3x = \dot{\mathbf{p}}(t - x/c),$$

where $\mathbf{p}$ is the electric dipole moment. For the vector potential we get

$$\begin{aligned}
\mathbf{A}(t, \mathbf{x}) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(t - |\mathbf{x} - \mathbf{x}'|/c), \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x' \approx \frac{\mu_0}{4\pi x} \int \mathbf{J}\left(t - \frac{x}{c}, \mathbf{x}'\right)\, d^3x' \\
&= \frac{\mu_0}{4\pi x} \dot{\mathbf{p}}(t - x/c).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{B} = \nabla \times \mathbf{A} &\approx \nabla \times \frac{\mu_0}{4\pi x} \dot{\mathbf{p}}(t - x/c) \\
&= -\frac{\mu_0}{4\pi x^2} \hat{\mathbf{x}} \times \dot{\mathbf{p}}(t - x/c) - \frac{\mu_0}{4\pi xc} \hat{\mathbf{x}} \times \ddot{\mathbf{p}}(t - x/c) \\
&\approx -\frac{\mu_0}{4\pi xc} \hat{\mathbf{x}} \times \ddot{\mathbf{p}}(t - x/c),
\end{aligned}$$

where we used the assumption $x \gg c/\omega$. Also, from

$$\dot{\mathbf{E}} = c^2 \nabla \times \mathbf{B} \approx \frac{\mu_0}{4\pi x} \hat{\mathbf{x}} \times (\hat{\mathbf{x}} \times \dddot{\mathbf{p}}(t - x/c))$$

we get

$$\mathbf{E} \approx \frac{\mu_0}{4\pi x} \hat{\mathbf{x}} \times (\hat{\mathbf{x}} \times \ddot{\mathbf{p}}(t - x/c)).$$

From the above approximations we get the *Larmor formula* for the energy flux

$$\mathcal{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \frac{c}{\mu_0} |\mathbf{B}|^2 \hat{\mathbf{x}} \approx \frac{\mu_0}{16\pi^2 x^2 c} \left| \hat{\mathbf{x}} \times \ddot{\mathbf{p}}\left(t - \frac{x}{c}\right) \right|^2 \hat{\mathbf{x}}$$

and the radiated power

$$P(t) = \int_{dS^2} \mathcal{S} \cdot d\mathbf{S} \approx \frac{\mu_0}{16\pi^2 d^2 c} \int_{dS^2} \left| \hat{\mathbf{x}} \times \ddot{\mathbf{p}}\left(t - \frac{x}{c}\right) \right|^2 dS.$$

3.3 Plane Waves

3.3.1 Initial value theorems

In order to study plane waves we need a couple theorems on the initial value problems. The proof of Theorem A can be found in Wald.

Theorem A (Initial Value Theorem for the Wave Equations) Let $f(t, \mathbf{x})$ be an arbitrary smooth function on spacetime, and let $\chi_1(\mathbf{x})$ and $\chi_2(\mathbf{x})$ be arbitrary smooth functions on space. Then there exists a unique smooth solution $\psi(t, \mathbf{x})$ of $\Box\psi = -f$ such that $\psi(0, \mathbf{x}) = \chi_1(\mathbf{x})$ and $\frac{\partial \psi}{\partial t}(0, \mathbf{x}) = \chi_2(\mathbf{x})$.

Let $\rho(t, \mathbf{x})$ and $\mathbf{J}(t, \mathbf{x})$ satisfy $\partial\rho/\partial t + \nabla \cdot \mathbf{J} = 0$. We want to find ϕ and $\mathbf{A}$ satisfying the equations $\Box\phi = -\rho/\epsilon_0$, $\Box\mathbf{A} = -\mu_0\mathbf{J}$ and $\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$. With

$$\Psi = \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A}$$

we find

$$\Box\Psi = \frac{1}{c^2}\frac{\partial}{\partial t}\Box\phi + \nabla \cdot \Box\mathbf{A} = -\frac{1}{c^2}\frac{\partial \rho}{\epsilon_0 \partial t} - \mu_0 \nabla \cdot \mathbf{J} = -\mu_0\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}\right) = 0.$$

Thus, by Theorem A the Lorenz gauge condition $\Psi = 0$ is satisfied if ϕ and $\mathbf{J}$ are chosen so that $\Psi = 0$ and $\partial\Psi/\partial t = 0$ at $t = 0$. The first can be obtained by a gauge transformation. Since

$$\frac{\partial \Psi}{\partial t} = \frac{\rho}{\epsilon_0} - \nabla \cdot \mathbf{E},$$

we will have to assume $\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$ and $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ at $t = 0$. Thus, we have the following theorem.

Theorem B (Initial Value Theorem for Maxwell's Equations) Let $\rho(t, \mathbf{x})$ and $\mathbf{J}(t, \mathbf{x})$ be arbitrary, smooth specifications of the charge density and current density on spacetime, subject to a charge-current conservation $\partial\rho/\partial t + \nabla \cdot \mathbf{J} = 0$. Let $\mathbf{E}_0(\mathbf{x})$ and $\mathbf{B}_0(\mathbf{x})$ be arbitrary smooth vector fields on space satisfying

$$\nabla \cdot \mathbf{E}_0(\mathbf{x}) = \frac{1}{\epsilon_0}\rho(0, \mathbf{x}), \ \ \nabla \cdot \mathbf{B}_0(\mathbf{x}) = 0.$$

Then there exists a unique smooth solution $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$ to Maxwell's equations with the property that $\mathbf{E}(0, \mathbf{x}) = \mathbf{E}_0(\mathbf{x})$ and $\mathbf{B}(0, \mathbf{x}) = \mathbf{B}_0(\mathbf{x})$.

Indeed, if $\mathbf{E}_0(\mathbf{x})$ and $\mathbf{B}_0(\mathbf{x})$ satisfy the above conditions, then ϕ and $\mathbf{A}$ at $t = 0$ are determined uniquely up to gauge transformation so that they satisfy $\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$ and $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ at $t = 0$. Thus Theorem B follows from Theorem A.

3.3.2 Plane waves

In the Lorenz gauge with $\rho = 0$ and $\mathbf{J} = 0$ Maxwell's equations are

$$\Box\phi = 0, \ \ \Box\mathbf{A} = \mathbf{0}, \ \ \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0.$$

We have the gauge freedom

$$\phi \to \phi' = \phi - \frac{\partial \chi}{\partial t}, \ \ \mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\chi,$$

where $\Box\chi = 0$. According to Theorem A two functions χ and $\partial\chi/\partial t$ can be specified arbitrarily at $t = 0$. We fix this so that the following holds:

$$\frac{\partial\chi}{\partial t}(0,\mathbf{x}) = \phi(0,\mathbf{x}), \quad \nabla^2\chi(0,\mathbf{x}) = \frac{1}{c^2}\frac{\partial\phi}{\partial t}(0,\mathbf{x}).$$

If $\frac{\partial\phi}{\partial t}(0,\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ rapidly, then we can assume $\frac{\partial\chi}{\partial t}(0,\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. This determines χ uniquely.

Using Theorem A again for ϕ', we see that $\phi' = 0$. Dropping primes, it suffices to solve $\Box\mathbf{A} = \mathbf{0}$ with $\nabla \cdot \mathbf{A} = 0$. Using the Fourier transform

$$\hat{\mathbf{A}}(t,\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int e^{-i\mathbf{k}\cdot\mathbf{x}}\mathbf{A}(t,\mathbf{x})\,d^3x,\ \mathbf{A}(t,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int e^{i\mathbf{k}\cdot\mathbf{x}}\hat{\mathbf{A}}(t,\mathbf{k})\,d^3k,$$

we obtain

$$\begin{aligned}\Box\mathbf{A} =& \frac{1}{(2\pi)^{3/2}}\int\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - k^2\right)e^{i\mathbf{k}\cdot\mathbf{x}}\hat{\mathbf{A}}(t,\mathbf{k})\,d^3k \\ \nabla\cdot\mathbf{A} =& \frac{1}{(2\pi)^{3/2}}\int e^{i\mathbf{k}\cdot\mathbf{x}} i\mathbf{k}\cdot\hat{\mathbf{A}}(t,\mathbf{k})\,d^3k,\end{aligned}$$

which implies

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - k^2\right)\hat{\mathbf{A}}(t,\mathbf{k}) = 0,\ \mathbf{k}\cdot\hat{\mathbf{A}}(t,\mathbf{k}) = 0.$$

The general solution is

$$\hat{\mathbf{A}}(t,\mathbf{k}) = \mathbf{c}_1(\mathbf{k})e^{-i\omega t} + \mathbf{c}_2(\mathbf{k})e^{i\omega t},$$

where $\omega = kc, \mathbf{c}_i\cdot\mathbf{k} = 0$. Since $\mathbf{A}(t,\mathbf{x})$ is real we have $\hat{\mathbf{A}}^*(t,\mathbf{k}) = \hat{\mathbf{A}}(t,-\mathbf{k})$, hence $\mathbf{c}_1^*(\mathbf{k}) = \mathbf{c}_2(-\mathbf{k})$. Thus, with $\mathbf{C}(\mathbf{k}) = \mathbf{c}_1(\mathbf{k})/(2\pi)^{3/2}$ we have

$$\begin{aligned}\mathbf{A}(t,\mathbf{x}) =& \frac{1}{(2\pi)^{3/2}}\int e^{i\mathbf{k}\cdot\mathbf{x}}[\mathbf{c}_1(\mathbf{k})e^{-i\omega t} + \mathbf{c}_1^*(-\mathbf{k})e^{i\omega t}]\,d^3k \\ =& \int \mathbf{C}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}\,d^3k\ + c.c.\end{aligned}$$

where c.c. means the complex conjugate and $\mathbf{k}\cdot\mathbf{C}(\mathbf{k}) = 0$.

By a plane electromagnetic wave we mean the solution

$$\phi = 0,\ \mathbf{A}(t,\mathbf{x}) = \mathbf{C}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}},\ \omega = kc, \mathbf{k}\cdot\mathbf{C} = 0, \mathbf{C}\in\mathbb{C}^3$$

for some wave vector $\mathbf{k}\in\mathbb{R}^3$.

3.3.3 Polarization

For a plane electromagnetic wave $\mathbf{A}(t,\mathbf{x}) = \mathbf{C}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ we have $\mathbf{k}\cdot\mathbf{C} = 0$, hence we have two polarization degree of freedom. The complex electric and magnetic fields are

$$\begin{aligned}\mathbf{E} =& -\frac{\partial\mathbf{A}}{\partial t} = i\omega\mathbf{C}(\mathbf{k})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} \\ \mathbf{B} =& \nabla\times\mathbf{A} = i\mathbf{k}\times\mathbf{C}(\mathbf{k})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}.\end{aligned}$$

Thus, vectors $\mathbf{E}, \mathbf{B}, \mathbf{k}$ are orthogonal and have the orientation of x, y, z-axes. Also, $|\mathbf{E}| = c|\mathbf{B}|$. Let's assume $\mathbf{k} = k\hat{\mathbf{z}}$ and write

$$2i\omega\mathbf{C} = (\alpha_x e^{i\beta_x}, \alpha_y e^{i\beta_y}, 0),$$

where α, β are real numbers. The the corresponding real solutions are

$$\mathbf{E} = \alpha_x \cos(kz - \omega t + \beta_x)\hat{\mathbf{x}} + \alpha_y \cos(kz - \omega t + \beta_y)\hat{\mathbf{y}}$$
$$c\mathbf{B} = -\alpha_y \cos(kz - \omega t + \beta_y)\hat{\mathbf{x}} + \alpha_x \cos(kz - \omega t + \beta_x)\hat{\mathbf{y}}.$$

We consider three cases.

Linearly polarized This is the case where $\beta_x = \beta_y$. Here $\mathbf{E}$ oscillates along the direction α and $\mathbf{B}$ oscillates along the direction β.

Circularly polarized This is the case where $\alpha_x = \alpha_y$ and $\beta_x = \beta_y \pm \pi/2$. Here $\mathbf{E}$ and $c\mathbf{B}$ maintain the same magnitude and they rotate. If $\beta_x = \beta_y + \pi/2$, then it is called a right circularly polarized. The other case is left circularly polarized.

Elliptically polarized This is the case which is neither linearly polarized nor circularly polarized. But, this is a linear combination of two linearly polarized plane waves one in x-direction, the other in y-direction. Also, this case is a linear combination of two circularly polarized plane waves, one in right-handed, the other in left-handed.

3.4 Exercises

1. Prove following identities:

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$$
$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{2}\nabla|\mathbf{E}|^2 - (\mathbf{E} \cdot \nabla) \cdot \mathbf{E}.$$

[Hint. Show that both sides have the same i-th components. Use the expressions of $(\mathbf{A} \times \mathbf{B})_i$ and $(\nabla \times \mathbf{A})_i$ in Levi-Civita symbol. Also use the identity about $\epsilon_{ijk}\epsilon_{klm}$.]

2. *Conservation of the total angular momentum of the electromagnetic fields* [Wald, Problem 5.2] The angular momentum density of the electromagnetic field is given by

$$\mathbf{l} = \mathbf{x} \times \mathcal{P} = \epsilon_0 \mathbf{x} \times (\mathbf{E} \times \mathbf{B}).$$

Consider a source-free ($\rho = 0, \mathbf{J} = \mathbf{0}$) solution to Maxwell's equations with $\mathbf{E}$ and $\mathbf{B}$ vanishing rapidly as $|\mathbf{x}| \to \infty$, so the total momentum

$$\mathbf{L} = \int \mathbf{l}\, d^3x$$

is well defined. Show that $\mathbf{L}$ is conserved (i.e., independent of time). [Hint. In the source free case we have $\partial\mathcal{P}_i/\partial t = \partial_j\Theta_{ij}$. Use the integration by parts and Stokes' theorem. One may assume that $|E_i| \leq C/|\mathbf{x}|^2$. Finally, notice that $\epsilon_{ijk} = -\epsilon_{jik}$ and $\Theta_{ij} = \Theta_{ji}$.]

3. *Force on a charge from a circularly moving charge* [Wald, Problem 5.3] A particle of charge q_1 moves with velocity v in a circular orbit of radius R about the origin in the x-y plane, such that its ϕ coordinate varies as $\phi = \omega t$, with $\omega = v/R$. Assume that $v \ll c$. Another particle of charge q_2 is at rest at point $\mathbf{x}$, where $|\mathbf{x}| \gg R$. To order $1/|\mathbf{x}|$, find the force $\mathbf{F}$ on the particle of charge q_2 at time t. [Hint. Use the approximation for $\mathbf{E}$ in radiation zone with $\mathbf{p}(t) = Rq_1(\cos \omega t, \sin \omega t, 0)$.]

4. *Radiation of electromagnetic energy from an oscillating charge* [Wald, Problem 5.6] A point charge of charge q and mass m is placed 1at the end of a spring with spring constant k. The charge is displaced in the z-direction by an amount α away from its equilibrium position and is then released to oscillate. Assume that the resulting motion is nonrelativistic, $v \ll c$.

(a) Assume that the charge oscillates harmonically with amplitude α. To order $1/r$ in distance from the charge and to leading order in v/c, what are the resulting electromagnetic potential $\phi, \mathbf{A}$?

(b) What is the radiated power?

(c) As a result of the radiation of electromagnetic energy, the maximum amplitude of oscillation, α, will, in fact, slowly decay with time. Find $\alpha(t)$. [Hint. Use Larmor formula and relate the radiated power to the damping coefficent of the damped harmonic oscillator.]

5. *Schwartz space and tempered distributions*

(a) Show that the Fourier transform is a bijection on the Schwartz space $\mathcal{S}(\mathbb{R}^4)$. Is it continuous?

(b) Show that any tempered distribution is a distribution. Is the delta function $\delta(\mathbf{x})$ on $\mathbb{R}^4$ a tempered distribution?

(c) Show that the Fourier inverse transform of the Fourier transform of a tempered distribution T is T itself.

4 Optics

4.1 Geometric Optics and the WKB Approximation

Recall that plane wave solutions to the source-free Maxwell's equations have the form

$$\boldsymbol{A}(t,\boldsymbol{x}) = \mathbf{C}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}e^{-i\omega t}$$

where $\mathbf{C}$,the amplitude, is constant over spacetime. In the context of optics we want to consider solutions for waves propagating through inhomogeneous media, that is, media with a spatially variable index of refraction. The index of refraction modulates the amplitude of the wave due to conservation of energy, in that a higher index of refraction causes the speed of propagation to slow which forces the amplitude to increase in order to conserve energy. Since this modulation occurs over space we want to consider solutions whose amplitudes are variable over space but not time. Solutions of this nature will have the form

$$\boldsymbol{A}(t,\boldsymbol{x}) = \alpha(\boldsymbol{x})e^{-i\omega t}$$

This will certainly complicate the analysis. But fear not, one way to simplify is the geometric optics or WKB (Wentzel-Kramers-Brillouin) approximation. This approximation method is based on the assumption that the spatial variability of the medium is much larger than the wavelength so that we can treat it as propagating through a locally homogeneous medium. This will simplify the mathematics, allowing us to split α into

$$\alpha(\boldsymbol{x}) = \mathfrak{C}(\boldsymbol{x})e^{iS(\boldsymbol{x})}$$

while still adhering to the physicality of the system under consideration, that is S is varying rapidly compared to $\mathfrak{C}$. The goal of this method is to get approximate solutions to Maxwell's equations which give us the notion of *light ray*, from which we can recover the technique and of ray tracing we all know and love from freshman physics.

4.1.1 WKB approximation for standard wave equation

Let's consider the standard wave equation

$$\Box\psi = \frac{-1}{c^2}\psi + \nabla^2\psi = 0$$

(The additional constraints of Maxwell's equations merely forces the orthogonality of the amplitude to the direction of propagation, the approximation method is the same.)
We seek approximate solutions of the form

$$\psi(t,\boldsymbol{x}) = \alpha(\boldsymbol{x})e^{-i\omega t} = \mathfrak{C}(\boldsymbol{x})e^{iS(\boldsymbol{x})}e^{-i\omega t}$$

which means that $\alpha(\boldsymbol{x})$ must satisfy

$$\frac{\omega^2}{c^2}\alpha + \nabla^2\alpha = 0$$

the Helmholtz equation. This gives

$$\frac{\omega^2}{c^2}\mathfrak{C}(\boldsymbol{x})e^{iS(\boldsymbol{x})} + \nabla^2\mathfrak{C}(\boldsymbol{x})e^{iS(\boldsymbol{x})} = 0$$

which expands to

$$\left(-|\nabla S|^2\mathfrak{C} + \nabla^2\mathfrak{C} + \frac{\omega^2}{c^2}\mathfrak{C} + i\left((\nabla^2 S)\mathfrak{C} + 2\nabla S\cdot\nabla\mathfrak{C}\right)\right)e^{iS} = 0$$

Now, as we are considering approximate solutions where S is varying much more rapidly than $\mathfrak{C}$ we can drop the $\nabla^2\mathfrak{C}$ term. Since the real and imaginary parts both must be zero the approximate solution

$$\alpha(\boldsymbol{x}) = \mathfrak{C}(\boldsymbol{x})e^{iS(\boldsymbol{x})}$$

must satisfy

$$|\nabla S|^2 = \frac{\omega^2}{c^2}$$

and

$$(\nabla^2 S)\mathfrak{C} + 2\nabla S\cdot\nabla\mathfrak{C} = 0$$

The units of the first equation are (radians per distance)^2 or the angular wavenumber squared this motivates the notation

$$\nabla S := \boldsymbol{k}$$

Recall that for plane waves, the surfaces of constant phase, $S = \boldsymbol{k}\cdot\boldsymbol{x}$, are planes and $\boldsymbol{k}$ is orthogonal to the planes. In these solutions $\boldsymbol{k}$ and $\mathfrak{C}$ are constant while in the WKB approximation they are not so we can get surfaces of constant S that are not necessarily planes but could be curved. Nevertheless, $\nabla S = \boldsymbol{k}$ tells us $\boldsymbol{k}$ is still orthogonal to these surfaces. The criteria above can then be written

$$|\boldsymbol{k}|^2 = \frac{\omega^2}{c^2}$$

$$(\nabla\cdot\boldsymbol{k})\mathfrak{C} + 2(\boldsymbol{k}\cdot\nabla)\mathfrak{C} = 0$$

4.1.2 Light rays form integral curves

Let's consider the integral curves $\boldsymbol{x}(\tau)$ of the vector field $\boldsymbol{k}$ defined as

$$\frac{dx_i}{d\tau} = k_i$$

The claim is that these are light rays, in the sense that they should be straight lines, i.e.,

$$\frac{d^2x_i}{d\tau^2} = 0$$

Indeed, the change in $\boldsymbol{k}$ with respect to position is given $(\boldsymbol{k} \cdot \nabla)\boldsymbol{k}$, the i-th component of which is

$$[(\boldsymbol{k} \cdot \nabla)\boldsymbol{k}]_i = \sum_j k_j \partial_j k_i = \sum_j k_j \partial_j \partial_i S = \sum_j k_j \partial_i \partial_j S = \sum_j k_j \partial_i k_j = \frac{1}{2}\partial_i |\boldsymbol{k}|^2 = 0$$

using the definition $\boldsymbol{k} = \nabla S$ gives the second and fourth equalities, the symmetry of mixed partials gives the third equality, and the fifth comes from linearity of ∂_i. As $|\boldsymbol{k}|^2$ is constant the change in the i-th component is zero. Coming back to the integral curves we see that

$$[(\boldsymbol{k} \cdot \nabla)\boldsymbol{k}]_i = \left(\frac{d\boldsymbol{x}}{d\tau} \cdot \nabla\right) \frac{dx_i}{d\tau} = \frac{d^2 x_i}{d\tau^2} = 0$$

Hence, the integral curves are in fact straight.

4.1.3 Geometric optics in inhomogeneous media

In order to investigate propagation through a medium with spatially variable index of refraction we need to look at the WKB approximation of the modified wave equation

$$-\frac{n^2(\boldsymbol{x})}{c^2}\frac{\partial^2 \psi}{\partial t^2} + \nabla^2 \psi = 0$$

where $n(\boldsymbol{x})$ is the index of refraction.

The approximation proceeds exactly as above, again writing $\boldsymbol{k} = \nabla S$, to get

$$|\boldsymbol{k}|^2 = n^2(\boldsymbol{x})\frac{\omega^2}{c^2}$$

$$(\nabla \cdot \boldsymbol{k})\mathcal{C} + 2(\boldsymbol{k} \cdot \nabla)\mathcal{C} = 0$$

When we consider the integral curves corresponding to this type of solution we find

$$\left|\frac{d\boldsymbol{x}}{d\tau}\right| = n(\boldsymbol{x})\frac{\omega}{c}$$

and

$$(\boldsymbol{k} \cdot \nabla)\boldsymbol{k} = \frac{1}{2}\nabla|\boldsymbol{k}|^2 = \frac{\omega^2}{2c^2}\nabla n^2(\boldsymbol{x}) = \frac{\omega^2}{c^2} n(\boldsymbol{x})\nabla n(\boldsymbol{x}) = \frac{d^2\boldsymbol{x}}{d\tau^2}$$

That is, the light rays will no longer be straight but will curve in accordance with $n(\boldsymbol{x})$. This bending will be in the direction of, and proportional to ∇n. Hence, the rays will curve towards the greater index of refraction with greater curvature in regions of larger ∇n. This phenomenon is harnessed through gradient-index optics, where light can be focused using a gradient of refractive index rather than the shape of a lens.

Note that the trajectories solving

$$\frac{d^2\boldsymbol{x}}{d\tau^2} = \frac{\omega^2}{c^2} n(\boldsymbol{x})\nabla n(\boldsymbol{x})$$

are exactly the solutions to the Euler-Lagrange equations obtained by extremizing the action given by

$$\mathcal{S} = \int n(\boldsymbol{x})\sqrt{\left\|\frac{d\boldsymbol{x}}{d\lambda}\right\|^2}\, d\lambda$$

4.2 Interference

While the WKB (geometric optics) approximation provides a good description of the propagation of electromagnetic radiation in many scenarios it obviously cannot describe them all. Following Wald's terminology, *interference* refers to phenomena that require a sum of WKB solutions, while *diffraction* encompasses a variety of phenomena, including scattering and propagation through an aperture, but is not a strict classification, as many phenomena can be described in multiple ways.

4.2.1 Intensity of EM radiation

Our discussion of interference will be based on the intensity of incident radiation as this is the primary observable when the frequency of light is too high for individual oscillations to be distinguished. The intensity is

$$I(t) = \frac{1}{\mu_0}|\overline{\boldsymbol{E}\times\boldsymbol{B}}|$$

where

$$|\overline{\boldsymbol{E}\times\boldsymbol{B}}|(t) = \left|\frac{1}{2T}\int_{t-T}^{t+T}\boldsymbol{E}(t')\times\boldsymbol{B}(t')\,dt'\right|$$

is the time averaged Poynting flux over $T >> 1/\omega$.
From the WKB approximate solution

$$\boldsymbol{A}(t,\boldsymbol{x}) = \boldsymbol{\mathfrak{C}}(\boldsymbol{x})e^{iS(\boldsymbol{x})}e^{-i\omega t}$$

the electric and magnetic fields are given by

$$\boldsymbol{E} = -\frac{\partial\boldsymbol{A}}{\partial t} = i\omega\boldsymbol{\mathfrak{C}}(\boldsymbol{x})e^{iS(\boldsymbol{x})}e^{-i\omega t}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = i\boldsymbol{k}\times\boldsymbol{\mathfrak{C}}(\boldsymbol{x})e^{iS(\boldsymbol{x})}e^{-i\omega t}$$

from which we take the real parts $\mathrm{Re}[\boldsymbol{E}]$ and $\mathrm{Re}[\boldsymbol{B}]$. The intensity of the resulting electromagnetic radiation is then

$$I(t) = \frac{1}{\mu_0}|\overline{\mathrm{Re}[\boldsymbol{E}]\times\mathrm{Re}[\boldsymbol{B}]}|$$

Substituting the relations

$$\mathrm{Re}[\boldsymbol{E}] = \frac{1}{2}(\boldsymbol{E}+\boldsymbol{E}^*), \quad \mathrm{Re}[\boldsymbol{B}] = \frac{1}{2}(\boldsymbol{B}+\boldsymbol{B}^*)$$

we have

$$I = \frac{1}{4\mu_0}\left|\overline{\boldsymbol{E}\times\boldsymbol{B}+\boldsymbol{E}\times\boldsymbol{B}^*+\boldsymbol{E}^*\times\boldsymbol{B}+\boldsymbol{E}^*\times\boldsymbol{B}^*}\right|$$

From properties of the cross product we can pull out the time dependent scaling factors

$$\boldsymbol{E}\times\boldsymbol{B} = (e^{-i\omega t})^2\Big((i\omega\boldsymbol{\mathfrak{C}}(\boldsymbol{x})e^{iS(\boldsymbol{x})})\times(i\boldsymbol{k}\times\boldsymbol{\mathfrak{C}}(\boldsymbol{x})e^{iS(\boldsymbol{x})})\Big)$$

$$\boldsymbol{E}^*\times\boldsymbol{B}^* = (e^{i\omega t})^2\Big((i\omega\boldsymbol{\mathfrak{C}}(\boldsymbol{x})e^{-iS(\boldsymbol{x})})\times(i\boldsymbol{k}\times\boldsymbol{\mathfrak{C}}(\boldsymbol{x})e^{-iS(\boldsymbol{x})})\Big)$$

$$\boldsymbol{E}^* \times \boldsymbol{B} = (e^{-i\omega t+i\omega t})\Big((i\omega\mathfrak{C}(\boldsymbol{x})e^{-iS(\boldsymbol{x})}) \times (i\boldsymbol{k} \times \mathfrak{C}(\boldsymbol{x})e^{iS(\boldsymbol{x})})\Big)$$
$$\boldsymbol{E} \times \boldsymbol{B}^* = (e^{-i\omega t+i\omega t})\Big((i\omega\mathfrak{C}(\boldsymbol{x})e^{iS(\boldsymbol{x})}) \times (i\boldsymbol{k} \times \mathfrak{C}(\boldsymbol{x})e^{-iS(\boldsymbol{x})})\Big)$$

Since $\boldsymbol{E} \times \boldsymbol{B}$ and $\boldsymbol{E}^* \times \boldsymbol{B}^*$ both oscillate on the order of $e^{\pm 2i\omega t}$ their time average for $T >> 1/\omega$ can be ignored. Where as $\boldsymbol{E}^* \times \boldsymbol{B}$ and $\boldsymbol{E} \times \boldsymbol{B}^*$ are time independent. Hence,

$$I = \frac{1}{4\mu_0}\left|\overline{\boldsymbol{E}^* \times \boldsymbol{B} + \boldsymbol{E} \times \boldsymbol{B}^*}\right| = \frac{1}{4\mu_0}|\boldsymbol{E}^* \times \boldsymbol{B} + \boldsymbol{E} \times \boldsymbol{B}^*|$$

Since $(\boldsymbol{E}^* \times \boldsymbol{B})^* = \boldsymbol{E} \times \boldsymbol{B}^*$ the intensity is

$$\frac{1}{4\mu_0}|2(\boldsymbol{E}^* \times \boldsymbol{B})| = \frac{\omega|\boldsymbol{k}|}{2\mu_0}|\mathfrak{C}|^2 = \frac{\omega^2}{2\mu_0 c}|\mathfrak{C}|^2$$

Therefore, the intensity depends only on $|\mathfrak{C}(\boldsymbol{x})|^2$ not the phase S.

4.2.2 Interference of two sources

Consider two sources for which we have a WKB approximation, the total solution is then just the sum of both

$$\boldsymbol{A}(t, \boldsymbol{x}) = \Big(\mathfrak{C}_1(\boldsymbol{x})e^{iS_1(\boldsymbol{x})} + \mathfrak{C}_2(\boldsymbol{x})e^{iS_2(\boldsymbol{x})}\Big)e^{-i\omega t}$$

Computing the intensity of this solution we find

$$I = \frac{\omega}{2\mu_0}\left|\boldsymbol{k}_1|\mathfrak{C}_1|^2 + \boldsymbol{k}_2|\mathfrak{C}_2|^2 + \frac{2\mu_0}{\omega}\boldsymbol{I}_{int}\right|$$

where the interference term $\boldsymbol{I}_{int}$ is

$$\boldsymbol{I}_{int} = \frac{1}{4\mu_0}[\boldsymbol{E}_1^* \times \boldsymbol{B}_2 + \boldsymbol{E}_2^* \times \boldsymbol{B}_1 + \text{c.c.}]$$
$$= \frac{\omega}{2\mu_0}[\mathfrak{C}_1 \times (\boldsymbol{k}_2 \times \mathfrak{C}_2)\cos[S_1(\boldsymbol{x}) - S_2(\boldsymbol{x})] + 1 \leftrightarrow 2]$$

the $1 \leftrightarrow 2$ indicates the same expression as before with subscripts swapped.

Let's consider the situation where both WKB solutions are plane waves propagating almost parallel with the same frequency. In this case $\mathfrak{C}_1, \mathfrak{C}_2,$ and $\boldsymbol{k}_1 \approx \boldsymbol{k}_2$ are constant and $S_1 = \boldsymbol{k}_1 \cdot \boldsymbol{x} + \varphi_1, S_2 = \boldsymbol{k}_2 \cdot \boldsymbol{x} + \varphi_2$ so we have

$$\boldsymbol{I}_{int} = \frac{\omega}{2\mu_0}\mathfrak{C}_1 \cdot \mathfrak{C}_2 2\boldsymbol{k}_1 \cos((\boldsymbol{k}_1 - \boldsymbol{k}_2) \cdot \boldsymbol{x} + \varphi_1 - \varphi_2))$$

Note that the $\mathfrak{C}_1 \cdot \mathfrak{C}_2$ term is picking out the polarizations that are aligned and if they are orthogonal then the interference term vanishes.

The final intensity is then

$$I = \frac{\omega^2}{2\mu_0 c}\Big(|\mathfrak{C}_1|^2 + |\mathfrak{C}_2|^2 + 2\mathfrak{C}_1 \cdot \mathfrak{C}_2 \cos\big((\boldsymbol{k}_1 - \boldsymbol{k}_2) \cdot \boldsymbol{x} + (\varphi_1 - \varphi_2)\big)\Big)$$

which is variable in space.

Interference is the cause of many interesting phenomenon such as beating, in which the interfering waves have slightly different frequencies and the result has an oscillating envelope. Another phenomenon is iridescence, which is due to the constructive interference of certain colors and destructive interference of others, making certain colors appear more vibrant. As the angle of viewing changes the colors being interfered changes, giving the effect of color change.

4.3 Diffraction

Diffraction encompasses the wide array of phenomena for which the geometric optics approximation is insufficient.

4.4 Exercises

1. Verify that the trajectories solving
$$\frac{d^2\boldsymbol{x}}{d\tau^2} = \frac{\omega^2}{c^2} n(\boldsymbol{x}) \nabla n(\boldsymbol{x})$$
are exactly the solutions to the Euler-Lagrange equations obtained by extremizing the action given by
$$\mathcal{S} = \int n(\boldsymbol{x}) \sqrt{\left\| \frac{d\boldsymbol{x}}{d\lambda} \right\|^2} \, d\lambda$$
(up to curve reparameterization).

2. An *optical fiber* is a cylindrical dielectric material that is used to transport light signals. the optical fiber is referred to *graded-index* if the index of refraction decreases gradually away from the axis. If the fiber has sufficiently large diameter compared to inverse wavenumber, the light propagation can be analyzed by the geometric optics approximation, i.e. using light rays, these are called *multi-mode*. Consider a graded-index multi-mode optical fiber of radius R with $\mu = \mu_0$ and dielectric constant ϵ/ϵ_0, (note: $n = \sqrt{\epsilon/\epsilon_0}$), varying for $x^2 + y^2 \leq R$ as
$$\epsilon(\boldsymbol{x})/\epsilon_0 = a - b(x^2 + y^2)$$
where $a, b > 0$ and $a \geq 1 + bR^2$. Write down and solve the ray propagation equation
$$\frac{d^2\boldsymbol{x}}{d\tau^2} = \frac{\omega^2}{c^2} n(\boldsymbol{x}) \nabla n(\boldsymbol{x})$$
Show that rays that initially are sufficiently close to the central axis of the fiber and form a small enough angle with the axis will remain close to the axis for all time.

3. The half-space $z \geq 0$ is filled by a medium with index of refraction n. Consider a point $\boldsymbol{x}_1 = (x_1, y_1, z_1)$ in the vacuum region $z_1 < 0$ and a point $\boldsymbol{x} = (x_2, y_2, z_2)$ in the medium, $z_2 > 0$. Find the path between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ that minimizes the elapsed time in the sense of Fermat's principle. Show that the result agrees with Snell's law.

4. In preparation for the next section, obtain a spacetime version of the geometric optics approximation for a scalar field ψ as follows. Instead of restricting the solutions ψ to oscillate with a definite frequency ω, we could write ψ in the form

$$\psi(t, \boldsymbol{x}) = \mathscr{A}(t, \boldsymbol{x}) e^{i\mathscr{S}(t,\boldsymbol{x})}$$

a) Write the exact wave equation for ψ in terms of $\mathscr{A}, \mathscr{S}$. Then make the approximation that second derivatives of $\mathscr{A}$ can be neglected comapred with squares of first derivatives of $\mathscr{S}$ to obtain analogs of

$$|\nabla S|^2 = \frac{\omega^2}{c^2}$$

$$(\nabla^2 S)\mathfrak{C} + 2\nabla S \cdot \nabla \mathfrak{C} = 0$$

b) Define $k_0 = \frac{1}{c}\frac{\partial \mathscr{S}}{\partial t}$, and $\boldsymbol{k} = \nabla \mathscr{S}$. Show that $\boldsymbol{k}_0^2 = |\boldsymbol{k}|^2$. Define $x^0 = ct$, and define

$$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right)$$

Define $k_\mu = (k_0, k_1, k_2 k_3)$ and $k^\mu = \sum_\nu \eta^{\mu\nu} k_\nu$, where

$$\eta^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Show that

$$\sum_\nu k^\nu \partial_\nu k^\mu = 0$$

This shows that in the geometric optics approximation, light rays move on null straight lines (geodesics) in spacetime.

5 Special Relativity and Lorentzian Manifolds

5.1 Symmetric Bilinear Forms on Vector Spaces

5.1.1 Bilinear forms on a vector space

Definition 5.1.1.
Let V be a real vector space. A bilinear form on V is a bilinear map $\omega : V \times V \to \mathbb{R}$.

Suppose V is a real finite dimensional vector space with basis $v_1, \ldots, v_n$, and suppose ω is a bilinear form on V. Due to the bilinearity of ω, we know ω is completely determined by its values on

$$\{(v_i, v_j) \in V \times V : i, j \in \{1, \ldots, n\}\}.$$

Define the matrix

$$A = [\omega(v_i, v_j)] = \begin{bmatrix} \omega(v_1, v_1) & \omega(v_1, v_2) & \ldots & \omega(v_1, v_n) \\ \omega(v_2, v_1) & \omega(v_2, v_2) & \ldots & \omega(v_2, v_n) \\ \vdots & \vdots & \ddots & \vdots \\ \omega(v_n, v_1) & \omega(v_n, v_2) & \ldots & \omega(v_n, v_n) \end{bmatrix}$$

With respect to the basis $v_1, \ldots, v_n$ on V, we can represent each element $v \in V$ as a column matrix where our correspondence is given by

$$v = \sum_{i=1}^{n} a_i v_i \iff \mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Thus, given $x, y \in V$ where

$$x = \sum_{i=1} a_i v_i \qquad y = \sum_{j=1}^{n} b_j v_j$$

we have

$$\langle x, y \rangle = \langle \sum_{i=1}^{n} a_i v_i, \sum_{j=1}^{n} b_j v_j \rangle = \sum_{i,j=1}^{n} a_i b_j \langle v_i, v_j \rangle = \mathbf{x}^T A \mathbf{y}$$

Similarly, given a matrix A, we can define a bilinear form on V where

$$\langle x, y \rangle = \mathbf{x}^T A \mathbf{y}$$

Thus, for a fixed basis on V, we have a one-to-one correspondence between bilinear forms on V and $\mathbb{R}^{n \times n}$ (the set of n-by-n matrices with real coefficients).

5.1.2 Nondegenerate bilinear forms on a vector space

Suppose ω is a bilinear form on a real vector space V, then we can define a linear map

$$\omega^\flat : V \to V^\vee \qquad \omega^\flat(x) : V \xrightarrow{y \mapsto \omega(x,y)} \mathbb{R}$$

Definition 5.1.2.
Let V be a real vector space. A bilinear form ω on V is non-degenerate if $\omega^\flat$ is injective.

Remark 5.1.3.
For a real finite dimensional vector space V and a fixed basis on V, we have, via the correspondence of bilinear forms and matrices, that nondegenerate bilinear forms are in one-to-one correspondence with matrices A that satisfy $\mathbf{x}^T A\mathbf{y} = 0$ for all $\mathbf{y} \in \mathbb{R}^{\dim(V)}$ implies $\mathbf{x} = 0$.

Proposition 5.1.4.
Let V be a real vector space, and let ω be a bilinear form on V. The following are equivalent:

1. *ω is non-degenerate.*
2. *For all non-zero $v \in V$, there exists $u \in V$ such that $\omega(v, u) \neq 0$.*

Proof.
First suppose ω is non-degenerate. Let $v \in V$ such that $v \neq 0$. Since $\omega^\flat$ is injective and $v \neq 0$, then $\omega^\flat(v) \neq 0$. Thus, there exists $u \in V$ such that

$$0 \neq \omega^\flat(v)u = \omega(v, u)$$

Hence (1) implies (2). Now suppose (2). Let $v \in V$ such that $v \neq 0$, then, by assumption, there exists $u \in V$ such that $\omega(v, u) \neq 0$. Thus $\omega^\flat(v) \neq 0$. Therefore $\ker(\omega^\flat) = \{0\}$ so that $\omega^\flat$ is injective. Hence ω is non-degenerate.

□

In the case V is a real finite dimensional vector space, then $\dim(V) = \dim(V^\vee)$. Thus $\omega^\flat$ is an isomorphism. Denote the inverse of $\omega^\flat$ as $\omega^\sharp$. The pair ω^β and $\omega^\sharp$ are called the musical isomorphisms induced by ω. Using a basis, we can easily write out $\omega^\flat$. Suppose $v_1, \ldots, v_n$ is a basis for V, and let $\alpha^1, \ldots, \alpha^n$ be the dual basis, then

$$\omega^\flat(v) = \sum_{j=1}^{n} g(v, v_j)\alpha^j.$$

To identify $\omega^\sharp$, we will need a notion of an orthonormal basis.

5.1.3 Symmetric bilinear forms on a vector space

Definition 5.1.5.
Let V be a real vector space. A bilinear form ω on V is symmetric if for all $x, y \in V$, $\omega(x,y) = \omega(y,x)$.

Remark 5.1.6.
For a real finite dimensional vector space V and a fixed basis on V, we have, via the correspondence of bilinear forms and matrices, that symmetric bilinear forms are in one-to-one correspondence with symmetric matrices.

Theorem 5.1.7 (Decomposition of Symmetric Bilinear Forms)**.**
Let V be a real finite dimensional vector space, and let ω be a symmetric, bilinear form on V. Let $U = \{u \in V : \forall v \in V, \omega(u,v) = 0\}$. There exists largest vector subspaces (in terms of dimension) P and N of V such that

1. *$v \in P$ if and only if $v = 0$ or $\omega(v,v) > 0$.*
2. *$v \in N$ if and only if $v = 0$ or $\omega(v,v) < 0$.*
3. *$V = U \oplus P \oplus N$.*

Furthermore, there exists a bases $u_1, \dots, u_k \in U$, $e_1, \dots, e_r \in P$, and $f_1, \dots, f_s \in N$ such that

1. *For all $v \in V$ and all $i \in \{1, \dots, k\}$, $\omega(u_i, v) = 0$.*
2. *For all $i, j \in \{1, \dots, r\}$, $\omega(e_i, e_j) = \delta_{ij}$.*
3. *For all $i, j \in \{1, \dots, s\}$, $\omega(f_i, f_j) = -\delta_{ij}$.*
4. *For all $i \in \{1, \dots, r\}$ and all $j \in \{1, \dots, s\}$, $\omega(e_i, f_j) = 0$.*

Proof.
Suppose $\dim(V) = n < \infty$. If $\omega = 0$, then $P = \{0\} = N$ and $U = V$. Additionally, any basis will do for the additional claim. Therefore we assume $\omega \neq 0$. Since ω is bilinear, then U is indeed a vector subspace of V. Let W_0 be a vector subspace of V such that $V = U \oplus W_0$. Note, $w \in W_0$ if and only if $w = 0$ or there exists $v \in V$ such that $\omega(w,v) \neq 0$. Since ω is symmetric, then the $v \in V$ for which $\omega(w,v) \neq 0$ is actually in W_0. Therefore $\omega|_{W_0 \times W_0} \neq 0$.

We now show subspaces P and N in the first claim exists which we do by constructing P and N. Denote $\dim(U) = k$. We claim there exists $w_1 \in W_0$ such that $\omega(w_1, w_1) \neq 0$. Suppose this fails; that is, for each $w \in W_0$, $\omega(w,w) = 0$, then ω is anti-symmetric on W_0. Since the only bilinear form which is both anti-symmetric and symmetric is the the zero bilinear form, then $\omega|_{W_0 \times W_0} = 0$ which is a contradiction.

Thus there indeed exists such a $w_1 \in W_0$. Dividing by $\omega(w_1, w_1)$ if needed, we can assume $\omega(w_1, w_1) = \pm 1$. Define W_1 as

$$W_1 = \{v \in W_0 : \omega(w_1, v) = 0\}$$

This is clearly a vector subspace with $W_1 \cap \text{Span}(w_1) = \{0\}$. Furthermore, given $w \in W_0$, we can write

$$w = (w - \omega(w, w_1)w_1) + \omega(w, w_1)w_1$$

which shows $W_1 \oplus \text{Span}(w_1) = W_0$. If $W_1 = 0$, then $W_0 = \text{Span}(w_1)$. Suppose $W_1 \neq 0$, then for each $v \in W_1$, there exists $u \in W_1$ such that $\omega(v, u) \neq 0$. Indeed, given $v \in W_1$, there exists $w \in W_0$ such that $\omega(v, w) \neq 0$. Since $w = \lambda w_1 + u$, then we have

$$0 \neq \omega(v, w) = \omega(v, \lambda w_1 + u) = \lambda\omega(v, w_1) + \omega(v, u) = \omega(v, u).$$

Therefore $\omega|_{W_1 \times W_1} \neq 0$. By the same argument given on W_0, we know there exists $w_2 \in W_1$ such that $\omega(w_2, w_2) \neq 0$. Thus, we can define W_2 in a similar manner as W_1 and continue this process which must eventually terminate for $\dim(W_0) = n - k$. Therefore, we can find $w_1, \dots, w_{n-k}$ such that

$$W_0 = \text{Span}(w_1) \oplus \ldots \oplus \text{Span}(w_{n-k})$$

and $\omega(w_i, w_i) \neq 0$ for each $i \in \{1, \dots, n-k\}$. Thus $w_1, \dots, w_{n-k}$ is a basis for W_0. Furthermore, by construction, we know $\omega(w_i, w_j) = \pm\delta_{ij}$. Thus, let $e_1, \dots, e_r$ denote the vectors from $w_1, \dots, w_{n-k}$ for which $\omega(e_i, e_i) = 1$. Denote the other vectors as $f_1, \dots, f_s$. We claim that

$$\text{Span}(e_1, \dots, e_r) \qquad \text{Span}(f_1, \dots, f_s)$$

are the desired subspaces for P and N, respectively. Note,

$$V = U \oplus \text{Span}(e_1, \dots, e_r) \oplus \text{Span}(f_1, \dots, f_s)$$

Therefore, if there exists a larger vector space P', then $\dim(P') > r$ which contradicts $\dim(V) = n$ as $P' \cap U = \{0\}$ and $P' \cap \text{Span}(f_1, \dots, f_s) = \{0\}$. Thus $\dim(P) = r$ so that we can take $P = \text{Span}(e_1, \dots, e_r)$. For the same reasons, we can take $N = \text{Span}(f_1, \dots, f_s)$. Thus, in the first claim (1) – (3) holds. For the second claim, let $u_1, \dots, u_k$ be any basis for U, then the basis

$$u_1, \dots, u_k, e_1, \dots, e_r, f_1, \dots, f_s$$

satisfy the properties for the second claim.

□

With respect to the basis $u_1, \dots, u_k, e_1, \dots, e_r, f_1, \dots, f_s$, we know corresponding matrix of ω is a block diagonal matrix of the form

$$\begin{bmatrix} 0_{k\times k} & 0 & 0 \\ 0 & I_{r\times r} & 0 \\ 0 & 0 & -I_{s\times s} \end{bmatrix}.$$

Since the dimension of U, P, and N are invariant of a basis, we know the values of k, r, s are independent of the basis.

Definition 5.1.8.
Let V be a real finite dimensional vector space. We define the signature of a symmetric, bilinear form ω as $(r, s) \in \mathbb{N}_0^2$ where r and s are the values such that there exists a basis

$$u_1, \ldots, u_k, e_1, \ldots, e_r, f_1, \ldots, f_s$$

of V for which the matrix of ω with respect to this basis is given by

$$\eta_{r,s} = \begin{bmatrix} 0_{k\times k} & 0 & 0 \\ 0 & I_{r\times r} & 0 \\ 0 & 0 & -I_{s\times s} \end{bmatrix}.$$

Suppose V is a real finite dimensional vector space, and let ω be a nondegenerate, symmetric, bilinear form on V. Let $e_1, \ldots, e_r, f_1, \ldots, f_s$ be vectors satisfying the decomposition theorem. Denote the dual basis as $\alpha^1, \ldots, \alpha^r$ and $\beta^1, \ldots, \beta^s$. Then, as one can check,

$$\omega^\sharp(\alpha^i) = e_i \qquad \omega^\sharp(\beta^i) = -f_i$$

so that if $\phi = \sum_{i=1}^{r} a_i \alpha^i + \sum_{j=1}^{s} b_j \beta^j \in V^\vee$, then

$$\omega^\sharp(\phi) = \sum_{i=1}^{r} a_i e_i - \sum_{j=1}^{s} b_j f_j.$$

5.1.4 Positive/Negative definite and symmetric bilinear forms on a vector space: inner products

Definition 5.1.9. Let V be a real vector space, and let ω be a bilinear form on V.

- The bilinear form ω is positive definite if for all non-zero $v \in V$, $\omega(v,v) > 0$.
- The bilinear form ω is negative definite if for all non-zero $v \in V$, $\omega(v,v) < 0$.

Remark 5.1.10.

1. For a real finite dimensional vector space V and a fixed basis on V, we have, via the correspondence of bilinear forms and matrices, that symmetric, positive definite bilinear forms are in one-to-one correspondence with symmetric, positive definite matrices. Similarly, symmetric, negative definite bilinear forms are in correspondence with symmetric, negative definite matrices.
2. If ω is positive definite, then the signature of ω is $(\dim(V), 0)$.
3. If ω is negative definite, then the signature of ω is $(0, \dim(V))$.
4. We will refer to symmetric, positive definite bilinear forms as inner products.

5.1.5 Generalized orthogonal group

Definition 5.1.11.
Let V be a real vector space, and let ω be a bilinear form. A linear automorphism T of V is an isometry of ω provided for all $x, y \in V$,

$$\omega(Tx, Ty) = \omega(x, y)$$

In our case, we are interested in ω which are symmetric and nondegenerate. Recall, for bilinear forms ω on a finite dimensional vector space, we can represent the bilinear form via a matrix. Using the decomposition for symmetric, bilinear forms as well as that that linear automorphisms of a vector space correspond to invertible matrices, we can identify the isometries of ω as a subset of $GL_n(\mathbb{R})$ satisfying a simple property.

Proposition 5.1.12.

Let V be a real finite dimensional vector space of dimension n, and let ω be a symmetric, nondegenerate bilinear form with signature (r, s). Then $A \in GL_n(\mathbb{R})$ is an isometry of ω if and only if $A^T \eta_{r,s} A = \eta_{r,s}$.

Proof.

Pick a basis for V such that the matrix of ω is given by $\eta_{r,s}$. Suppose $A \in GL_n(\mathbb{R})$ is an isometry of ω, then for all $x, y \in V$,

$$\begin{aligned} \omega(Ax, Ay) &= (A\mathbf{x})^T \eta_{r,s} (A\mathbf{y}) \\ &= \mathbf{x}^T A^T \eta_{r,s} A\mathbf{y} \\ &= \mathbf{x}^T \eta_{r,s} \mathbf{y} = \omega(x, y) \end{aligned}$$

Thus, for all $x, y \in V$,

$$\mathbf{x}^T (A^T \eta_{r,s} A - \eta_{r,s})\mathbf{y} = 0.$$

This only happens provided $A^T \eta_{r,s} A - \eta_{r,s} = 0$. Hence $A^T \eta_{r,s} A = \eta_{r,s}$. Now suppose the converse, then for all $x, y \in V$,

$$\omega(Ax, Ay) = \mathbf{x}^T A^T \eta_{r,s} A\mathbf{y} = \mathbf{x}^T \eta_{r,s} \mathbf{y} = \omega(x, y)$$

Thus A is an isometry of ω.

□

Definition 5.1.13.

Define the generalized orthogonal group of signature (r, s) as

$$O(r, s) = \{A \in GL_{r+s}(\mathbb{R}) : A^T \eta_{r,s} A = \eta_{r,s}\}$$

Remark 5.1.14.

1. If $A \in O(r, s)$, then $\det(A) = \pm 1$.
2. Since $\eta_{r,s} A^T \eta_{r,s} A = I$, then $A^{-1} = \eta_{r,s} A^T \eta_{r,s}$.
3. Applying A on the left of (2) shows $A^T \in O(r, s)$ whenever $A \in O(r, s)$.
4. When $s = 0$, we obtain the usual orthogonal group: $O(n, 0) = O(n)$.
5. The case $O(3, 1)$ and $O(1, 3)$ are both called the Lorentz group.
6. Consider the map $\phi_{r,s} : GL_{r+s}(\mathbb{R}) \to GL_{r+s}(\mathbb{R})$ where $\phi_{r,s}(A) = A^T \eta_{r,s} A$. Since matrix multiplication and the transpose are continuous maps, then $\phi_{r,s}$ is a continuous map. Thus $O(n, s) = \phi_{r,s}^{-1}(\eta_{r,s})$ is a closed subset of $GL_{r+s}(\mathbb{R})$. Thus, by the Closed Subgroup Theorem for Lie groups, $O(r, s)$ is a Lie group.

5.2 Scalar Product Spaces

5.2.1 Scalar product spaces and the scalar product

Definition 5.2.1.

A real scalar product space is a pair (V, g) where V is a real vector space and g is a symmetric, nondegenerate, bilinear form. The scalar product is finite dimensional if V is finite dimensional.

Remark 5.2.2.

Though g is not necessarily an inner product, we adopt the terminology from inner products such as orthonormal, orthogonal, etc where we define the "norm" as $||v|| = \sqrt{|g(v,v)|}$. Therefore a list of orthonormal vectors may have vectors such that $g(v,v) = -1$.

Proposition 5.2.3.

Let (V, g) be a real finite dimensional scalar product space where g has signature (r, s). Then $A \in O(r, s)$ if and only if A sends an orthonormal basis to an orthonormal basis.

Consider a real finite dimensional scalar product space (V, g). Since g is non-degenerate and V is finite dimensional, then we have the musical isomorphisms:

$$g^\flat : V \to V^\vee \qquad g^\sharp : V^\vee \to V$$

Using $g^\sharp$, we can define a map

$$(\cdot,\cdot) : V^\vee \times V^\vee \to \mathbb{R} \qquad (\alpha, \beta) = g(\alpha^\sharp, \beta^\sharp)$$

Note, that $(\cdot,\cdot)$ is a nondegenerate, symmetric, bilinear map on $V^\vee$. Furthermore, note that if $e_1, \ldots, e_n$ is an orthonormal basis on V, then the dual is orthonormal with respect to $(\cdot,\cdot)$. We wish to extend this notion to the exterior power on $V^\vee$. Note the musical isomorphisms induce isomorphisms

$$\tilde{g}^\flat : \bigwedge^k V \to \bigwedge^k V^\vee \qquad \tilde{g}^\sharp : \bigwedge^k V^\vee \to \bigwedge^k V$$

which, on simple tensors, are given by

$$\tilde{g}^\flat(v_1 \wedge \ldots \wedge v_k) = g^\flat(v_1) \wedge \ldots \wedge g^\flat(v_k) \qquad \tilde{g}^\sharp(\alpha_1 \wedge \ldots \wedge \alpha_k) = g^\sharp(\alpha_1) \wedge \ldots \wedge g^\sharp(\alpha_k).$$

Thus, we can define a map

$$(\cdot,\cdot) : \bigwedge^k V^\vee \times \bigwedge^k V^\vee \to \mathbb{R}$$

which, on simple tensors, is given by

$$
\begin{aligned}
(\alpha^1 \wedge \ldots \wedge \alpha^k, \beta^1 \wedge \ldots \wedge \beta^k) &= \det\left([g(g^\sharp(\alpha^i), g^\sharp(\beta^j))]\right) \\
&= \det \begin{bmatrix} g(g^\sharp(\alpha^1), g^\sharp(\beta^1)) & g(g^\sharp(\alpha^1), g^\sharp(\beta^2)) & \ldots & g(g^\sharp(\alpha^1), g^\sharp(\beta^k)) \\ g(g^\sharp(\alpha^2), g^\sharp(\beta^1)) & g(g^\sharp(\alpha^2), g^\sharp(\beta^2)) & \ldots & g(g^\sharp(\alpha^2), g^\sharp(\beta^k)) \\ \vdots & \vdots & \ddots & \vdots \\ g(g^\sharp(\alpha^k), g^\sharp(\beta^1)) & g(g^\sharp(\alpha^k), g^\sharp(\beta^2)) & \ldots & g(g^\sharp(\alpha^k), g^\sharp(\beta^k)) \end{bmatrix}
\end{aligned}
$$

The map $(\cdot, \cdot)$ is called the scalar product on (V, g). Since g is symmetric, then $(\cdot, \cdot)$ is symmetric. By fixing each component and using the Universal Property for the Exterior Product due to the alternating multilinearity of the determinant, one is able to see $(\cdot, \cdot)$ is bilinear. Furthermore if $e_1, \ldots, e_n$ is an orthonormal basis with dual basis $\alpha^1, \ldots, \alpha^n$ on V, then

$$\{\alpha^{i_1} \wedge \ldots \wedge \alpha^{i_k} : 1 \leq i_1 < \ldots < i_k \leq n\}$$

is an orthonormal basis of $\bigwedge^k V^\vee$ with respect to $(\cdot, \cdot)$. This also shows that $(\cdot, \cdot)$ is nondegenerate on $\bigwedge^k V^\vee$. Note from this basis, we can see that the signature of $(\cdot, \cdot)$ is the same as that of g. Therefore the isometry group of $(\cdot, \cdot)$ is precisely the isometry group of g. In the case $k = 0$, we have $\bigwedge^0 V^\vee = \mathbb{R}$. Thus we can extend $(\cdot, \cdot)$ to $k = 0$ where $(a, b) = ab$.

5.2.2 Volume form for scalar product spaces

Let V be a real finite dimensional vector space of dimension n. Recall an orientation for V can be described via an equivalence class of basis elements $[(v_1, \ldots, v_n)]$ where

$$(v_1, \ldots, v_n) \sim (u_1, \ldots, u_n) \iff \exists A \in GL_n^+(\mathbb{R}),\ \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} = A \begin{bmatrix} u_1 & \ldots & u_n \end{bmatrix}$$

Equivalently, we can specify an orientation for V by an equivalence class of non-zero top covectors $[\mu]$ (elements of $\bigwedge^n V^\vee$) where

$$\mu \sim \tau \iff \exists \lambda > 0,\ \mu = \lambda \tau$$

Note, these two notions of orientation agree (as one should check). Thus, suppose $[\mu]$ is an orientation on a real finite dimensional scalar product space (V, g). Using g and the existence of an orthonormal basis on V via the Decomposition Theorem for Symmetric Bilinear Forms, we can identify a form $\omega_g \in [\mu]$. Note, given an orientation $[\mu]$ on V, a basis $v_1, \ldots, v_n$ of V is orientated provided $\mu(v_1, \ldots, v_n) > 0$.

Proposition 5.2.4.

Let (V, g) be a real finite dimensional scalar product space. Orientate V by an equivalence class of non-zero top covectors $[\mu]$. Then there exists a unique $\omega_g \in [\mu]$ such that for any orientated orthonormal basis $v_1, \ldots, v_n$ of V,

$$\omega_g(v_1, \ldots, v_n) = 1$$

Proof.

Let $e_1, \ldots, e_n$ be any orthonormal basis for V. Swapping e_1 and e_2 if needed, we assume $\mu(e_1, \ldots, e_n) > 0$. Let $\alpha^1, \ldots, \alpha^n$ denote the dual basis. We claim

$$\omega_g = \alpha^1 \wedge \ldots \wedge \alpha^n.$$

Note, $\mu \sim \omega_g$ as $\mu = \mu(e_1, \ldots, e_n)\omega_g$. Thus $\omega_g \in [\mu]$. Suppose $v_1, \ldots, v_n$ is any other orientated orthonormal basis for V. Let $A \in GL_n(\mathbb{R})$ such that

$$\begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} = A \begin{bmatrix} e_1 & \ldots & e_n \end{bmatrix}$$

Then

$$\begin{aligned} 0 < \omega_g(v_1, \ldots, v_n) &= \omega_g(Ae_1, \ldots, Ae_n) \\ &= \det(A)\omega_g(e_1, \ldots, e_n) \\ &= \det(A) \end{aligned}$$

Therefore $\det(A) > 0$. Furthermore, since A sends an orthonormal basis to an orthonormal basis, then $A \in O(r, s)$ where (r, s) is the signature of g; Therefore $\det(A) = 1$ so that

$$\omega_g(v_1, \ldots, v_n) = 1.$$

Therefore ω_g has the desired property. For uniqueness, note that any top form is uniquely determined on a basis for V.

□

Using properties for change of basis on top covectors, we can describe ω_g in terms of any oriented basis.

Corollary 5.2.5.

Let (V, g) be a real finite dimensional scalar product space. Orientate V by an equivalence class of non-zero top covectors $[\mu]$. For any orientated basis $v_1, \ldots, v_n$,

$$\omega_g = \sqrt{|\det([g_{ij}])|}\alpha^1 \wedge \ldots \wedge \alpha^n$$

where $\alpha^1, \ldots, \alpha^n$ is the dual basis of $v_1, \ldots, v_n$ and

$$g = \sum_{i,j=1}^{n} g_{i,j}\alpha^i \otimes \alpha^j.$$

Proof.

Let (r, s) denote the signature of g. Let $e_1, \ldots, e_n$ be an orientated orthonormal basis with dual basis $\beta^1, \ldots, \beta^n$. We know $\omega_g = \lambda\alpha^1 \wedge \ldots \wedge \alpha^n$ where $\lambda > 0$. Write

$$v_j = \sum_{i=1}^{n} a^i_j e_i \qquad A = [a^i_j]$$

then

$$\lambda = \omega_g(v_1, \ldots, v_j) = \omega_g\left(\sum_{i=1}^{n} a^i_1 e_i, \ldots, \sum_{i=1}^{n} a^i_n e_i\right) = \det(A)$$

Since we also have $g = \sum_{i,j=1}^{n} g_{i,j}\alpha^i \otimes \alpha^j$, where

$$\begin{aligned} g_{i,j} = g(v_i, v_j) &= g(\sum_{k=1}^{n} a^k_i e_k, \sum_{l=1}^{n} a^l_j e_l) \\ &= \sum_{k,l=1}^{n} a^k_i a^l_j g(e_k, e_l) = \sum_{k=1}^{n} a^k_i a^k_j g(e_k, e_k) = \left(A^T \eta_{r,s} A\right)_{i,j} \end{aligned}$$

then $\det([g_{i,j}]) = \det(A)^2 \det(\eta_{r,s}) = (-1)^s \det(A)^2$. Since $\lambda > 0$, then we conclude

$$\lambda = \sqrt{|\det([g_{i,j}])|}$$

so that

$$\omega_g = \sqrt{|\det([g_{ij}])|}\alpha^1 \wedge \ldots \wedge \alpha^n$$

as claimed.

□

5.2.3 Hodge star on scalar product spaces

Using the scalar product and the canonical top covector, we can define a pairing of covectors for an oriented vector space.

Theorem 5.2.6 (Existence of the Hodge Star)**.**
Let (V, g) be a real finite dimensional scalar product space with $\dim(V) = n$. Equip V with an orientation. Then for each $k \in \{1, \ldots, n\}$, there exists a unique linear isomorphism

$$* : \bigwedge^k V^\vee \to \bigwedge^{n-k} V^\vee$$

satisfying the property that for all $\alpha, \beta \in \bigwedge^k V^\vee$,

$$\alpha \wedge *\beta = (\alpha, \beta)\omega_g$$

Proof.
For each $\tau \in \bigwedge^{n-k}(V^\vee)$, define

$$\phi_\tau : \bigwedge^k V^\vee \to \mathbb{R} \qquad \phi_\tau(\alpha) \in \mathbb{R} \text{ such that } \phi_\tau(\alpha)\omega_g = \alpha \wedge \tau$$

Clearly ϕ_τ is a linear map. Furthermore, $\alpha \wedge \tau = 0$ for all $\alpha \in \bigwedge^k V^\vee$ if and only if $\tau = 0$. Therefore we have an injective linear map

$$\bigwedge^{n-k} V^\vee \to \left(\bigwedge^k V^\vee\right)^\vee \qquad \tau \mapsto \phi_\tau$$

Due to dimensions, we know this in fact an isomorphism. Therefore, for each $\beta \in \bigwedge^k V^\vee$, there exists a unique $*\beta \in \bigwedge^{n-k} V^\vee$ such that $\phi_{\star\beta}(\alpha) = (\alpha, \beta)$ for all $\alpha \in \bigwedge^k V^\vee$. Hence we have an isomorphism

$$* : \bigwedge^k V^\vee \to \bigwedge^{n-k} V^\vee$$

satisfying the property that for all $\alpha, \beta \in \bigwedge^k V^\vee$,

$$\alpha \wedge *\beta = (\alpha, \beta)\omega_g$$

Clearly $*$ is unique.

□

Remark 5.2.7.
For $k = 0$, we have $(a, b) = ab$. Thus we can extend $*$ for k zero where

$$a \wedge *b = ab\omega_g$$

Definition 5.2.8.
Let (V, g) be a finite dimensional scalar product space with $\dim(V) = n$. Equip V with an orientation, and let $k \in \{0, \ldots, n\}$. The Hodge star operation on $\bigwedge^k V^\vee$ is the unique linear isomorphism

$$* : \bigwedge^k V^\vee \to \bigwedge^{n-k} V^\vee$$

satisfying the property that for all $\alpha, \beta \in \bigwedge^k V^\vee$, $\alpha \wedge *\beta = (\alpha, \beta)\omega_g$

By using the uniqueness of the Hodge Star, one can explicitly write $*\alpha$. The proof of the following theorem can be found in [Lee09].

Theorem 5.2.9 (Computing the Hodge Star)**.**
Let (V, g) be a finite dimensional scalar product space with $\dim(V) = n$. Equip V with an orientation. Let $v_1, \ldots, v_n$ be an orientated basis, and let $\alpha^1, \ldots, \alpha^n$ be the dual basis. Then for any $1 \le i_1 < \ldots < i_k \le n$,

$$*(\alpha^{i_1} \wedge \ldots \wedge \alpha^{i_k}) = \sqrt{|\det([g_{i,j}])|}\epsilon_{i_1 \ldots i_k i_{k+1} \ldots i_n} \alpha^{i_{k+1}} \wedge \ldots \wedge \alpha^{i_n}$$

In particular, if $v_1, \ldots, v_n$ is an orientated orthonormal basis, then for any $1 \le i_1 < \ldots < i_k \le n$,

$$*(\alpha^{i_1} \wedge \ldots \wedge \alpha^{i_k}) = g_{i_1 i_1} \ldots g_{i_k i_k} \epsilon_{i_1 \ldots i_k i_{k+1} \ldots i_n} \alpha^{i_{k+1}} \wedge \ldots \wedge \alpha^{i_n}$$

Corollary 5.2.10.
Let (V, g) be a finite dimensional scalar product space, and denote the signature of g as (r, s). Equip V with an orientation.

1. *$*1 = \omega_g$. Thus, for each $a \in \mathbb{R}$, $*a = a\omega_g$.*
2. *$*\omega_g = (-1)^s$*
3. *For all $\omega \in \bigwedge^k V^\vee$, $**\omega = (-1)^s(-1)^{k(n-k)}\omega$.*

Proof.
We begin by proving (1). Since $b \wedge *1 = (b, 1)\omega_g = b\omega_g$, then, by uniqueness of $*1$, we know $*1 = \omega_g$. Thus, by linearity of $*$ we have $*a = a * 1 = a\omega_g$. For (2) and (3) follow from writing the forms in terms of an orthonormal basis.

□

Using isometries map orthonormal basis to orthonormal basis as well as the computation of the Hodge Star in terms of the orthonormal basis, we know the Hodge Star commutes with isometries of g.

Corollary 5.2.11 (Hodge Star commutes with isometries)**.**
Let (V, g) be a finite dimensional scalar product space, and denote the signature of g as (r, s). Let $\alpha^1, \ldots, \alpha^n$ be an orthonormal basis of $V^\vee$ with respect to $(\cdot, \cdot)$. For each $A \in O(r, s)$ and each $1 \leq i_1 < \ldots < i_k \leq n$,

$$A\left(*\alpha^{i_1} \wedge \ldots \wedge \alpha^{i_k}\right) = *\left(A\left(\alpha^{i_1} \wedge \ldots \wedge \alpha^{i_k}\right)\right)$$

Proof.
Let $A \in O(r, s)$ be with respect to the orthonormal basis, then, defining $\beta^i = A\alpha^i$, we have an orthonormal basis $\beta^1, \ldots, \beta^n$. Since

$$\begin{aligned}
A(*\alpha^{i_1} \wedge \ldots \alpha^{i_k}) &= A(g_{i_1 i_1} \ldots g_{i_k i_k} \epsilon_{i_1 \ldots i_k i_{k+1} \ldots i_n} \alpha^{i_{k+1}} \wedge \ldots \wedge \alpha^{i_n}) \\
&= g_{i_1 i_1} \ldots g_{i_k i_k} \epsilon_{i_1 \ldots i_k i_{k+1} \ldots i_n} A(\alpha^{i_{k+1}} \wedge \ldots \wedge \alpha^{i_n}) \\
&= g_{i_1 i_1} \ldots g_{i_k i_k} \epsilon_{i_1 \ldots i_k i_{k+1} \ldots i_n} A\alpha^{i_{k+1}} \wedge \ldots \wedge A\alpha^{i_n} \\
&= g_{i_1 i_1} \ldots g_{i_k i_k} \epsilon_{i_1 \ldots i_k i_{k+1} \ldots i_n} \beta^{i_{k+1}} \wedge \ldots \wedge \beta^{i_n} \\
&= *(\beta^{i_1} \wedge \ldots \wedge \beta^{i_k}) \\
&= *(A(\alpha^{i_1} \wedge \ldots \wedge \alpha^{i_k}))
\end{aligned}$$

then we conclude, by linearity of A, $A* = *A$.

□

5.3 Riemannian and Pseudo-Riemannian Metrics

5.3.1 Riemannian and Pseudo-Riemannian metrics

Since the tangent bundle of a C^∞ manifold has real vector space structures for each fiber, then it makes sense to define a fiber-wise symmetric, nondegenerate bilinear forms. To ensure the bilinear forms do not vary too wildly from fiber to fiber, we will want to require them to vary smoothly across the fibers. To codify these two items, we utilizes that a bilinear map $\omega : V \times V \to \mathbb{R}$ is equivalent to a linear map $\tilde{\omega} : V \otimes_{\mathbb{R}} V \to \mathbb{R}$.

Definition 5.3.1.
Let M be a C^∞ manifold.

- A Riemannian metric is a C^∞ section $g : M \to T^*M \otimes_{\mathbb{R}} T^*M$ such that for each $p \in M$, g_p is an inner product.
- A pseudo-Riemannian metric is a C^∞ section $g : M \to T^*M \otimes_{\mathbb{R}} T^*M$ such that for each $p \in M$, g_p is symmetric and nondegenerate. Furthermore, the signature of g_p is the same as the signature of g_q for all $p, q \in M$. Define the signature of g as the signature of g_p where $p \in M$.

Suppose g is a section of $T^*M \otimes_{\mathbb{R}} T^*M \to M$. Let $(U, x^1, \ldots, x^m)$ be a coordinate chart on M. Since $dx^1, \ldots, dx^m$ is a local frame on U of the cotangent bundle, then there exists unique $a_{i,j} \in C^\infty(U)$ such that on U,

$$g = \sum_{i,j=1}^{n} a_{i,j} dx^i \otimes dx^j$$

Using that dx^i is dual to $\frac{\partial}{\partial x^i}$, we can identify $a_{i,j}$:

$$\begin{aligned} g\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) &= \sum_{i,j=1}^{n} a_{i,j} \left(dx^i \otimes dx^j\right)\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) \\ &= \sum_{i,j=1}^{n} a_{i,j} dx^i \left(\frac{\partial}{\partial x^k}\right) dx^j \left(\frac{\partial}{\partial x^l}\right) \\ &= \sum_{i,j=1}^{n} a_{i,j} \delta_{i,k} \delta_{j,l} = a_{k,l} \end{aligned}$$

Consider the matrix

$$A = [a_{i,j}] = \begin{bmatrix} g\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}\right) & g\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right) & \cdots & g\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^n}\right) \\ g\left(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^1}\right) & g\left(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2}\right) & \cdots & g\left(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^n}\right) \\ \vdots & \vdots & \ddots & \vdots \\ g\left(\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}\right) & g\left(\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^2}\right) & \cdots & g\left(\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^n}\right) \end{bmatrix}$$

Since each $a_{i,j} \in C^\infty(U)$, then the map

$$\phi_g : U \to GL_n(\mathbb{R}) \qquad \phi_g(p) = A(p)$$

is a C^∞ map. Using this matrix and our correspondence of bilinear forms and matrices, we have the following for each $p \in U$:

- g_p is symmetric if and only if A_p is symmetric.
- g_p is positive definite if and only if A_p is positive definite.

Thus

- A Riemannian metric g defines a C^∞ map $\phi_g : U \to GL_n(\mathbb{R})$ which takes values in matrices which are symmetric and positive definite.
- A pseudo-Riemannian metric g defines a C^∞ map $\phi_g : U \to GL_n(\mathbb{R})$ which takes values in matrices which are symmetric and nondegenerate.

Now suppose we have a C^∞ map

$$\phi : U \to GL_n(\mathbb{R}) \qquad \phi = \begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \dots * \phi_{1,n} & \\ \phi_{2,1} & \phi_{2,2} & \dots * \phi_{2,n} & \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n,1} & \phi_{n,2} & \dots * \phi_{n,n} & \end{bmatrix}$$

where $\phi_{i,j} \in C^\infty(U)$, then we can define

$$g_\phi : U \to T^*M \otimes_{\mathbb{R}} T^*M \qquad g_\phi = \sum_{i,j=1}^{n} \phi_{i,j} dx^i \otimes dx^j$$

Since each $\phi_{i,j} \in C^\infty(U)$, then g_ϕ is a C^∞ section on U. Therefore, we have the following:

- g_ϕ is a Riemannian metric if and only if ϕ takes value in symmetric and positive definite matrices.
- g_ϕ is a pseudo-Riemannian metric if and only if ϕ takes value in symmetric matrices.

Therefore, on any coordinates chart $(U, x^1, \dots, x^n)$ for M we have

- Riemannian metrics on U are in one-to-one correspondence with C^∞ maps $U \to GL_n(\mathbb{R})$ which take value in symmetric and positive definite matrices.
- Pseudo-Riemannian metrics on U are in one-to-one correspondence with C^∞ maps $U \to GL_n(\mathbb{R})$ which take value in symmetric matrices.

Using these characterizations, we can easily come up with examples of local (pseudo) Riemannian metrics.

Example 5.3.2.

1. Consider $\mathbb{R}^n$, then $g = \sum_{i=1}^{n} dx^i \otimes dx^i$ is a Riemannian metric called the standard metric on $\mathbb{R}^n$.

2. Consider the upper half plane $\mathbb{H}^2$. We can define $g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$ which gives a hyperbolic metric on $\mathbb{H}^2$.

Definition 5.3.3.
An $n+1$ dimensional C^∞ manifold M with pseudo Riemannian metric g is a Lorentzian manifold provided the signature of g is either $(1, n)$ or $(n, 1)$.

Though every C^∞ manifold admits a Riemannian metric, it is not the case that every C^∞ manifold admits a Lorentzian metric. In particular

1. Every non-compact, connected C^∞ manifold admits a Lorentzian metric.
2. A compact, connected C^∞ manifold admits a Lorentzian metric if and only if the Euler characteristic is zero.

Thus, for example, $\mathbb{S}^n$ does not admit a Lorentzian metric for any $n \in \mathbb{N}_0$ as $\chi(\mathbb{S}^n) = 2$.

5.3.2 Hodge star and Maxwell's Equations

The theory for scalar product spaces extends nicely to C^∞ manifolds with (pseudo) Riemannian metrics. In fact, the local picture of these scalar product spaces and the Hodge star appears essentially the same as the vector space counter part beside the fact our bases are now local frames and the coefficients are smooth functions.

Using the Hodge star and the exterior derivative on $\mathbb{R}^4$, we can encode Maxwell's equations in a coordinate free way. To see this, we first write Maxwell's equations using natural units so that Maxwell's equations become

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}$$
$$\nabla \cdot \vec{E} = \rho \qquad \nabla \times \vec{E} - \frac{\partial \vec{B}}{\partial t} = 0$$

Let us consider the time independent case of these equations so that we have

$$\nabla \times \vec{E} = 0 \qquad \nabla \cdot \vec{B} = 0.$$

In this regime, we can encode the electric field as a differential one form

$$\mathcal{E} = E_x dx + E_y dy + E_z dz$$

and the magnetic field as a differential two form

$$\mathcal{B} = B_x dy \wedge dz - B_y dx \wedge dz + B_z dx \wedge dy$$

As one can check, we have

$$\nabla \times \vec{E} = 0 \iff d\mathcal{E} = 0 \qquad \nabla \cdot \vec{B} = 0 \iff d\mathcal{B} = 0$$

If we return to the time dependence, then we could write $\mathcal{E}$ and $\mathcal{B}$ as one differential two form

$$\mathcal{F} = \mathcal{B} + \mathcal{E} \wedge dt.$$

To make some of the computation simple, we will write the exterior derivative on $\mathbb{R}^4$ as $d = d_S + dt \wedge \partial_t$ where d_S is the exterior derivative for the spatial coordinates x, y, and z. Using this, we can easily compute $d\mathcal{F}$:

$$\begin{aligned} d\mathcal{F} &= d_S\mathcal{B} + d_S\mathcal{E} \wedge dt + dt \wedge \partial_t \mathcal{B} \\ &= d_S\mathcal{B} + (d_S\mathcal{E} - \partial_t\mathcal{B}) \wedge dt \end{aligned}$$

Recall, from Maxwell's equation, we have

$$d_S\mathcal{B} = \nabla \cdot \vec{B} = 0 \qquad d_S\mathcal{E} - \partial_t\mathcal{B} = \nabla \times \vec{E} - \frac{\partial \vec{B}}{\partial t} = 0$$

Thus we can encode Faraday's Law and the divergence of magnetic fields in the single equation $d\mathcal{F} = 0$. The important thing to note about writing these two equations in this way is that we automatically know these are preserved under *any* self diffeomorphism of $\mathbb{R}^4$since the pullback commutes with the exterior derivative. In particular, these are invariant under any change of coordinates.

To encode Gauss's Law and Ampere's Law, we utilize the Hodge star. Endow $\mathbb{R}^4$ with the Lorentz metric given by

$$\phi : \mathbb{R}^4 \to GL_4(\mathbb{R}) \qquad \phi(x) = \eta_{1,3}$$

where the coordinates are in the order of t, x, y, z. We are going to define the four current as the differential form

$$\mathcal{J} = \rho dt - J_x dx - J_y dy - J_z dz$$

where $\vec{J} = (J_x, J_y, J_z)$ and ρ is the charge density. We claim Gauss Law and Ampere's Law is equivalent to

$$*d * \mathcal{F} = \mathcal{J}$$

This verification is left as an exercise. Note, since the Hodge star commutes with isometries, then we know Ampere's Law and Gauss's Law are invariant under any isometry of the Lorentz metric. In particular, we know these two laws are invariant under the Lorentz group $O(1, 3)$.

5.4 Special Relativity

5.4.1 Galilean transformations

To construct the Galilean group for classical mechanics, we follow the same approach as [Wil22]. Suppose we have two observers $\mathcal{O}$ and $\mathcal{O}'$. Denote the coordinate of $\mathcal{O}$ as x_1, x_2, x_3, t, and denote the coordinates of

$\mathcal{O}'$ as x_1', x_2', x_3', t. We will denote any time derivatives with respect to t using dots (e.g. $\dot{\vec{x}}(t)$). Furthermore, suppose the observers measure the same lengths; that is,

$$t_2' - t_1' = t_2 - t_1 \qquad ||\vec{x}_2'(t') - \vec{x}_1'(t')|| = ||\vec{x}_2(t) - \vec{x}_1(t)||.$$

Then we know the coordinates vary as

$$t' = t + s \qquad \vec{x}'(t') = R(t)\vec{x}(t) + \vec{b}(t)$$

where $R(t)$ is orthogonal matrices for each t. Now, suppose the observers are both inertial which is to say that the trajectory of a free particle in $\mathcal{O}$ and $\mathcal{O}'$ is given by

$$\vec{x}(t) = \vec{x}_0 + \vec{u}t \qquad \vec{x}'(t') = \vec{x}_0' + \vec{u}'t'$$

Then we know $R(t)$ is constant and $\vec{b}(t) = \vec{v}t + \vec{a}$. Indeed, since $t' = t + s$ and $\vec{x}'(t') = R(t)\vec{x}(t) + \vec{b}(t)$, then

$$\begin{aligned} \vec{x'}(t+s) &= \vec{x}_0' + \vec{u}'(t+s) \\ &= R(t)\vec{x}(t) + \vec{b}(t) \\ &= R(t)(\vec{x}_0 + \vec{u}t) + \vec{b}(t) \end{aligned}$$

Differentiating with respect to t yields

$$\vec{u}' = \dot{R}(t)(\vec{x}_0 + \vec{u}t) + R(t)\vec{u} + \dot{\vec{b}}(t) \tag{1}$$

so that differentiating again yields

$$0 = \ddot{R}(t)(\vec{x}_0 + \vec{u}t) + 2\dot{R}(t)\vec{u} + \ddot{\vec{b}}(t)$$

This last equation must be true for all trajectories. In particular, taking $\vec{u} = 0 = \vec{x}_0$ shows $\ddot{\vec{b}} = 0$ so that $\vec{b}(t) = \vec{v}t + \vec{a}$. Taking $\vec{u} = 0$ shows $\ddot{R}(t) = 0$. Therefore $\dot{R}(t) = 0$ implying $R(t)$ is constant. Hence we have the transformation

$$t' = t + s \qquad \vec{x}'(t') = R\vec{x}(t) + \vec{v}t + \vec{a}$$

as claimed. We can express this transformation as

$$\begin{bmatrix} R & \mathbf{v} & \mathbf{a} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} \vec{x}'(t') \\ t' \\ 1 \end{bmatrix} = \begin{bmatrix} R & \mathbf{v} & \mathbf{a} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} R\mathbf{x}(t) + t\mathbf{v} + \mathbf{a} \\ t + s \\ 1 \end{bmatrix}$$

The set of all such matrices is called the Galilean group on $\mathbb{R}^3$ which is denoted by Gal(3). Note, under such transformations, we obtain an addition law for velocities:

$$\dot{\vec{x}}'(t') = R\dot{\vec{x}}(t) + \vec{v}$$

Furthermore, under such transformations, Newton's Second Law is preserved. It it due to these two items, that it was assumed that all inertial frames where related via a Galilean transformation. However, an issue arises with Maxwell's equations. Recall that we obtained the speed of light is constant from Maxwell's equations. Thus, under a Galilean transformation for which $\vec{v} \neq 0$, we would have $c' = c - \vec{v}$ where c' is the speed of light measured by observer $\mathcal{O}'$ and c is the speed of light measured by observer $\mathcal{O}$. To account for this issue, an ether model was put forth, but the model soon became moot with the Michelson-Morley experiment as well as with Einstein's postulates for special relativity.

5.4.2 Postulates for special relativity and Lorentz transformations

To over come the issues of preserving Maxwell's Equation under transformations, Einstein put forth two postulates:

1. The laws of physics are the same in all inertial reference frames.

2. The speed of light is constant and the same for all inertial reference frames.

To see what transformations preserve these postulates, suppose we have two inertial observers $\mathcal{O}$ and $\mathcal{O}'$. Consider a wave front which travels a distance $\Delta\vec{x}$ over a time of Δt, then

$$||\Delta\vec{x}|| = c\Delta t$$

so that

$$0 = c^2\Delta t^2 - ||\Delta\vec{x}||^2 = c^2\Delta t^2 - \Delta x_1^2 - \Delta x_2^2 - \Delta x_3^2$$

Since $\mathcal{O}'$ would agree the wave front travelled at speed c, then we have

$$||\Delta\vec{x}'|| = c\Delta t'$$

so that

$$0 = c^2\Delta t'^2 - ||\Delta\vec{x}'||^2 = c^2\Delta t'^2 - \Delta x_1'^2 - \Delta x_2'^2 - \Delta x_3'^2$$

Therefore, any two inertial observers would agree that

$$c^2\Delta t^2 - \Delta x_1^2 - \Delta x_2^2 - \Delta x_3^2 = c^2\Delta t'^2 - \Delta x_1'^2 - \Delta x_2'^2 - \Delta x_3'^2$$

which is called the space-time interval. Thus, using coordinates ct, x_1, x_2, x_3 and ct', x_1', x_2', x_3', we are interested in linear transformations A such that

$$A^T\eta_{1,3}A = \eta_{1,3}$$

which is precisely the Lorentz group $O(1,3)$.

5.4.3 Finding the Lorentz transformation between two observers

Suppose we have two observers $\mathcal{O}$ and $\mathcal{O}'$ with observer $\mathcal{O}'$ moving with some constant velocity $\vec{v}$ with respect to $\mathcal{O}$. What is the change of coordinates from $\mathcal{O}$ to $\mathcal{O}'$? Denote the coordinates of $\mathcal{O}$ as ct, x, y, z and the coordinates of $\mathcal{O}'$ as ct', x', y', z'. Furthermore, let's first consider the easy case that x is in the direction of travel with $y = y'$ and $z' = z$. Then our change of coordinates is a matrix of the form

$$\Lambda = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $\Lambda^T \eta_{1,3} \Lambda = \eta_{1,3}$, then we obtain

$$a^2 - c^2 = 1 \quad ab = cd \quad d^2 - b^2 = 1$$

Note, $a \neq 0$ and $d \neq 0$. Thus we can define

$$\beta = \frac{c}{a} = \frac{b}{d}$$

Using this in the first and third equation, we obtain

$$a^2 = \frac{1}{1-\beta^2} \qquad d^2 = \frac{1}{1-\beta^2}$$

which implies

$$c = \frac{\beta}{\sqrt{1-\beta^2}} \qquad b^2 = \frac{\beta}{\sqrt{1-\beta^2}}$$

Let $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, then $c = \gamma\beta = b$. Therefore our matrix Λ becomes

$$\Lambda = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We now use the velocity to determine β. Suppose we have some mass in $\mathcal{O}$ that is stationary over some time Δt. Using Λ, we know $\mathcal{O}'$ sees

$$\begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta t \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma\Delta t \\ \gamma\beta\Delta t \\ 0 \\ 0 \end{bmatrix}$$

Therefore, observer $\mathcal{O}'$ see the particle travelled $\Delta x' = \gamma\beta\Delta t$ over the time $c\Delta t' = \gamma\Delta t$. Thus

$$v = \frac{\Delta x'}{\Delta t'} = c\beta$$

so that

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

For the general case, we can always rotate x, y, z so that x is in the direction of travel with $y = y'$ and $z = z'$, apply Λ with $v = ||\vec{v}||$, and then rotate back to our original coordinates.

5.5 Exercises

Exercises (1) – (3) provided another way to show there exists an orthonormal basis for a finite dimensional vector space equipped with a nondegenerate, symmetric bilinear form.

1. Let V be a finite dimensional vector space with symmetric, nondegenerate bilinear form ω on V. For a vector subspace $S \subset V$, define $S^\perp = \{u \in V : \forall v \in S, \omega(u,v) = 0\}$.

   (a) Show $\dim(S^\perp) + \dim(S) = \dim(V)$.

   (b) Show $(S^\perp)^\perp = S$.

2. Let V be a finite dimensional vector space with symmetric, nondegenerate bilinear form ω on V. A vector subspace $S \subset V$ is nondegenerate if $\omega|_{S\times S}$ is nondegenerate. Show the following are equivalent:

   - S is nondegenerate.
   - $S^\perp$ is nondegenerate.
   - $S^\perp \cap S = \{0\}$.
   - $V = S \oplus S^\perp$.

3. Let V be a finite dimensional vector space with symmetric, nondegenerate bilinear form ω on V.

   (a) Show that if S is nondegenerate and $S \neq 0$, then there exists a $v \in S$ for which $\omega(v,v) \neq 0$.

   (b) A linearly independent list of vectors $v_1, \ldots, v_k \in V$ are nondegenerate provided $\omega(v_j, v_j) \neq 0$ for each $j \in \{1, \ldots, k\}$. Suppose $\dim(V) = n$ and $k < n$. Show there exists $v_{k+1}, \ldots, v_n \in V$ such that $v_1, \ldots, v_n$ is a nondegenerate basis for V.

   (c) Show that V has a nondegenerate basis.

   (d) Show that V has a basis $e_1, \ldots, e_n$ for such that $\omega(e_i, e_j) = \delta_{ij}$ or $\omega(e_i, e_j) = -\delta_{ij}$.

4. The following exercise investigates some of the topological features of $O(r,s)$.

   (a) Assume $r, s > 0$. Show that $O(r,s)$ has at least four connected components. *Hint: Write any matrix $A \in O(r,s)$ in terms of four blocks B, C, D, E where the sizes depend on r and s. Use $A^T \eta_{r,s} A = \eta_{r,s}$ to come up with a continuous function $f : O(r,s) \to \mathbb{R}^2 \backslash \{(x,0),(0,x) : x \in \mathbb{R}\}$.*

   (b) Assume $r, s > 0$. Show that $O(r,s)$ is not compact.

   (c) Show for all $r, s \in \mathbb{N}_0$, $O(r,s)$ and $O(s,r)$ are isomorphic as Lie groups (recall that bijective Lie group homomorphisms are automatically Lie group isomorphisms)

5. Verify $*d*\mathcal{F} = \mathcal{J}$.

6. Though every C^∞ manifold M admits a Riemannian metric, it is not the case that all C^∞ manifolds admit a Lorentzian metric. The goal of this exercise is to construct the obstruction of the existence for a Lorentzian metric. Suppose M is a connected manifold.

(a) Prove that if there exists a rank one subbundle $S \subset TM$ over M, then M admits a Lorentzian metric.

(b) Prove that if M admits a Lorentzian metric, then there exists a rank one subbundle $S \subset TM$ over M.

(c) Prove a compact, connected C^∞ manifold admits a Lorentzian metric if and only if the Euler characteristic is zero (see section 11 in [BT13]).

6 E&M as a Gauge Theory

In this section, we uncover the limitations of our current description of electromagnetic fields as vector potentials, and reinterpret electromagnetic fields as connections. We will follow [Wal22, Chapter 9] closely, while also drawing some ideas from Naber. **TODO: insert citation for Naber**

6.1 Klein-Gordon Fields

6.1.1 The Uncoupled Case

Definition 6.1.1. A **Klein-Gordon charged scalar field** is a function $\Phi : \mathbb{R}^4 \to \mathbb{C}$ whose Lagrangian density is given by

$$\mathcal{L}_{\text{KG}} = -\frac{1}{2}\left[\eta^{\mu\nu}\partial_\mu \Phi^* \partial_\eta \Phi + \left(\frac{mc}{\hbar}\right)^2 |\Phi|^2\right].$$

where Φ^* denotes the complex conjugate of Φ and where m is some constant. We call m the **mass** of the field, and we call $\frac{\hbar}{mc}$ the **Compton wavelength** of the field. Then $\frac{mc}{\hbar}$ is called the **inverse Compton wavelength** of the field.

The equation of motion which arises as from the Euler-Lagrange equations and a Klein-Gordon field Φ is given as

$$\Box\Phi - \left(\frac{mc}{\hbar}\right)^2 \Phi = 0. \tag{2}$$

where we recall that the $\Box$ operator is the d'Lambertian operator

$$\Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2.$$

If the space derivatives of Φ are much smaller than the inverse Compton wavelength (i.e. if we are in the non-relativistic case), then we make the approximation $\nabla^2\Phi \approx 0$ and Equation 2 gives rise to the equation

$$-\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \left(\frac{mc}{\hbar}\right)^2 \Phi = 0.$$

This in turn gives that $\Phi \sim e^{\pm imc^2 t/\hbar}$. If we write

$$\Phi(t,x) = e^{-imc^2 t/\hbar}\,\Upsilon(t,x)$$

then Equation 2 yields

$$\begin{aligned}0 &= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(e^{-imc^2t/\hbar}\,\Upsilon(t,x)\right) + \nabla^2\left(e^{-imc^2t/\hbar}\,\Upsilon(t,x)\right) - \frac{m^2c^2}{\hbar^2}\Phi \\ &= -\frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{imc^2}{\hbar}e^{imc^2t/\hbar}\Upsilon(t,x) + e^{-imc^2t/\hbar}\frac{\partial\Upsilon}{\partial t}\right) + e^{-imc^2t/\hbar}\,\nabla^2\Upsilon - \frac{m^2c^2}{\hbar^2}\Phi \\ &= -\frac{1}{c^2}\left(\frac{-2imc^2}{\hbar}e^{imc^2t/\hbar}\frac{\partial\Upsilon}{\partial t} + e^{imc^2t/\hbar}\frac{\partial^2\Upsilon}{\partial^2 t}\right) + e^{imc^2t/\hbar}\,\nabla^2\Upsilon.\end{aligned}$$

If we assume that $\frac{\partial^2\Upsilon}{\partial^2 t} \approx 0$, then we recover the equation

$$0 = 2i\frac{m}{\hbar}\frac{\partial\Upsilon}{\partial t} + \nabla^2\Upsilon,$$

which is the Schrödinger equation for a free particle.

6.1.2 Gauge Transformations in the Coupled Case

We now consider the case where our Klein-Gordon field is coupled to a electromagnetic field. In (non-relativistic) quantum mechanics, we have the Schrodinger equation

$$-\left(\frac{\partial}{\partial t} + i\frac{q}{\hbar}\phi\right)\Psi = \frac{1}{2m}\left(\nabla - i\frac{q}{\hbar}\mathbf{A}\right)\cdot\left(\nabla - i\frac{q}{\hbar}\mathbf{A}\right)\Psi \tag{3}$$

When we couple our Klein-Gordon field to an electromagnetic field, we would like the coupling to be compatible with the spacetime structure of special relativity while also satisfying the Schrodinger equation in the non-relativistic approximation/limit. To do so, we replace $\partial_\mu\Phi$ with $\partial_\mu\Phi - i\frac{q}{\hbar}A_\mu\Phi$ such that the coupled Klein-Gordon Lagrangian density is

$$\mathcal{L}_{\text{KG}}^{\text{EM}} = -\frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\Phi^* + i\frac{q}{\hbar}A_\mu\Phi^*\right)\left(\partial_\nu\Phi - i\frac{q}{\hbar}A_\nu\Phi\right) - \frac{1}{2}\left(\frac{mc}{h}\right)^2|\Phi|^2. \tag{4}$$

The equation of motion implied by this Lagrangian density is

$$\eta^{\mu\nu}\left(\partial_\mu - i\frac{q}{\hbar}A_\mu\right)\left(\partial_\nu - i\frac{q}{\hbar}A_\nu\right)\Phi - \left(\frac{mc}{\hbar}\right)^2\Phi = 0. \tag{5}$$

However, notice that Equation 5 is not invariant under our current notion of "gauge transformation" (this is Exercise 1 of this week). This is to say that if make the transformation

$$A_\mu \mapsto A_\mu + \partial_\mu\chi,$$

then this does not preserve Equation 5. However, if we make the simultaneous transformations

$$\begin{aligned}\Phi &\to e^{i\frac{q}{\hbar}\chi}\Phi; \\ A_\mu &\to A_\mu + \partial_\mu\chi\end{aligned}$$

then it will be shown shortly that Equation 5 is preserved. We will view these simultaneous transformations as being *the* gauge transformations for the coupled Klein-Gordon field, and say that (Φ, A_μ) and (Φ', A'_μ) are physically equivalent if there is some $\chi(t,x)$ where $\Phi' = e^{i\frac{q}{\hbar}\chi}\,\Phi$ and $A'_\mu = A_\mu + \partial_\mu\chi$.

6.1.3 The Derivative Operator

The quantity $\partial\Phi - i\frac{q}{\hbar}A_\mu\Phi$ which we placed into the Lagrangian density given in Equation 4 can be interpreted as giving a notion of differentiation of Φ. Define an operator

$$\mathcal{D}_\mu := \partial_\mu - i\frac{q}{\hbar}A_\mu.$$

We notice that for any Klein-Gordon field Φ, under a gauge transformation corresponding to χ,

$$\begin{aligned}
\mathcal{D}_\mu(\Phi') &= (\partial_\mu - i\frac{q}{\hbar}(A_\mu - \partial_\mu\chi))e^{i\frac{q}{\hbar}\chi}\Phi \\
&= \partial_\mu(e^{i\frac{q}{\hbar}\chi}\Phi) - i\frac{q}{\hbar}A_\mu e^{i\frac{q}{\hbar}\chi}\Phi - i\frac{q}{\hbar}\partial_\mu\chi e^{i\frac{q}{\hbar}\chi}\Phi \\
&= e^{i\frac{q}{\hbar}}(\partial_\mu\Phi - i\frac{q}{\hbar}A_\mu\Phi) \\
&= e^{i\frac{q}{\hbar}\chi}\mathcal{D}_\mu(\Phi),
\end{aligned}$$

i.e. the operator itself is invariant under gauge transformations. If we rewrite Equation 4 as

$$\mathcal{L}_{\mathrm{KG}}^{\mathrm{EM}} = -\frac{1}{2}\eta^{\mu\nu}(\mathcal{D}_\mu\Phi)^*\mathcal{D}_\nu\Phi - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2|\Phi|^2,$$

then it becomes clear that $\mathcal{L}_{\mathrm{KG}}^{\mathrm{EM}}$ is invariant under gauge transformations since under a gauge transformation $(\Phi, A_\mu) \to (\Phi', A'_\mu)$ corresponding to χ, we have

$$\begin{aligned}
-\frac{1}{2}\eta^{\mu\nu}(D_\mu\Phi')^*D_\nu\Phi' - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2|\Phi'|^2 &= -\frac{1}{2}\eta^{\mu\nu}(e^{i\frac{q}{\hbar}\chi}D_\mu\Phi)^*e^{i\frac{q}{\hbar}\chi}D_\nu\Phi - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2|e^{i\frac{q}{\hbar}\chi}\Phi|^2 \\
&= -\frac{1}{2}\eta^{\mu\nu}e^{-i\frac{q}{\hbar}\chi}(D_\mu\Phi)^*e^{i\frac{q}{\hbar}\chi}D_\nu\Phi - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2|\Phi|^2 \\
&= -\frac{1}{2}\eta^{\mu\nu}\left(D_\mu\Phi\right)^*\left(D_\nu\Phi\right) - \frac{1}{2}\left(\frac{mc}{h}\right)^2|\Phi|^2.
\end{aligned}$$

Definition 6.1.2. Given a Klein-Gordon field (Φ, A_μ), we define a the **curvature** of the electromagnetic field to be $C_{\mu\nu}$ where

$$C_{\mu\nu}\Phi = [\mathcal{D}_\mu, \mathcal{D}_\nu] = D_\mu D_\nu\Phi - D_\nu D_\mu\Phi.$$

Simplifying the expression, we have

$$\begin{aligned}
C_{\mu\nu} &= (\partial_\mu - i\frac{q}{\hbar}A_\mu)(\partial_\nu - i\frac{q}{\hbar}A_\nu) - (\partial_\nu - i\frac{q}{\hbar}A_\nu)(\partial_\mu - i\frac{q}{\hbar}A_\mu) \\
&= \partial_\mu\partial_\nu - i\frac{q}{\hbar}\partial_\mu A_\nu - \frac{q^2}{\hbar^2}A_\mu A_\nu - \left(\partial_\nu\partial_\mu - i\frac{q}{\hbar}\partial_\nu A_\mu - \frac{q^2}{\hbar^2}A_\nu A_\mu\right) \\
&= -i\frac{q}{\hbar}(\partial_\mu A_\nu - \partial_\nu A_\mu) \\
&= i\frac{q}{\hbar}F_{\mu\nu}.
\end{aligned}$$

This simplification gives an interpretation of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ as expressing the curvature of the electromagnetic field with respect to the operator D_μ.

6.2 The Electromagnetic Field as a Connection

Up to this point we have treated a electromagnetic field as an equivalence class of vector potentials where two vector potentials are equivalent if they only differ by a gauge transformation. We will now transition to the perspective that an electromagnetic field is a connection on a $U(1)$-bundle.

Definition 6.2.1. Suppose that X is a smooth manifold and that G is a Lie group. A **G-bundle over X** is a smooth manifold P, a smooth map $\mathcal{P} : P \to X$ and a smooth right action $\sigma : P \times G \to P$ which satisfy:

- $\mathcal{P}(\sigma(p, g)) = \mathcal{P}(p)$ for any $p \in P$ and $g \in G$. In other words, σ preserves the fibers of $\mathcal{P}$.
- For every $x_0 \in X$ there exists an open set $V \subseteq X$ where $x_0 \in V$ and where there exists a diffeomorphism $\Psi : \mathcal{P}^{-1}(V) \to V \times G$ where we write $\Psi(p) = (\mathcal{P}(p), \psi(p))$ and $\psi(\sigma(p, g)) = \sigma(\psi(p), g)$. The pair (V, Ψ) is called a **local trivialization** of $\mathcal{P}$ at x_0. In this language, this condition can be stated as "for every $x_0 \in X$, $\mathcal{P}$ admits a local trivialization at x_0".

We shall sometimes refer to $\mathcal{P}$ or even just P as a G-bundle over X if the remaining structure is clear from context.

Definition 6.2.2. A **connection** on a G-bundle $\mathcal{P}$ over X with action σ is a dual vector field on $\mathcal{P}$ which is valued in the Lie algebra of G, along with some additional technical conditions (which we will only specify in special cases for the purposes of this section).

For the purposes of this section, we will only consider the case where $\mathcal{P} = M \times U(1)$. The Lie algebra of $U(1)$ is $\mathbb{R}$, so a connection is a dual vector field on $\mathcal{P}$ in the ordinary sense. Say we have a connection (dual vector field on $\mathcal{P}$) $\mathcal{A}_\Lambda$ ($\Lambda = 0, 1, 2, 3, 4$). Then the technical conditions not specified in the above definition yields that $\mathcal{A}_\Lambda(x^\mu, s)$ is independent of s and $\mathcal{A}_4 = 1$. Given that $\mathcal{A}_\Lambda(x^\mu, s)$ is independent of s, we can automatically recover a dual vector field $A_\nu(x^\mu)$ on M. Furthermore, when we perform the transformation $s \mapsto s - \chi(x^\mu)$ for some arbitrary function $\chi(x^\mu)$, the action of $U(1)$ on $\mathcal{P}$ is preserved and the structure of $\mathcal{P}$ is preserved also. Under this coordinate transformation, for a dual vector field $\mathcal{A}_\lambda$ on $\mathcal{P}$, $\mathcal{A}_4 \mapsto \mathcal{A}'_4 = 1 = \mathcal{A}_4$ and $\mathcal{A}_\lambda \mapsto \mathcal{A}_\lambda + \partial_\lambda \chi$ for $\lambda = 0, 1, 2, 3$.

6.3 Dirac Magnetic Monopoles

To give an example of working with a nontrivial G-bundle, we will conclude this section by investing whether a magnetic monopole could exist, i.e. whether it could be the case that $\nabla \cdot \mathbf{B} \neq 0$ in some circumstance.

Definition 6.3.1. We say that a *magnetic charge* is present in an electromagnetic field if said field has the property that on some 2-sphere S, we have

$$0 \neq g \equiv \frac{1}{\mu_0} \int_S \mathbf{B} \cdot \hat{\mathbf{n}} dS.$$

If the field potential $\mathbf{A}$ is non-singular, then using the relation $\mathbf{B} = \nabla \times \mathbf{A}$,

$$g = \frac{1}{\mu_0} \int_S (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{n}} dS = \frac{1}{\mu_0} \int_{\partial S} \mathbf{A} \cdot d\mathbf{l} = 0,$$

where the final equality is given by $\partial S = \emptyset$.

However, we can relax the condition that $\mathbf{A}$ is non-singular to the condition that $\mathbf{A}$ is defined locally in such a way that it is smooth on each patch it is defined on and where if two patches overlap, then the local expressions of $\mathbf{A}$ are physically equivalent (i.e. they are related by a gauge transformation). In this situation, we can for example consider vector potentials on S^2 of the forms

$$\mathbf{A}(r, \varphi, \theta) = \frac{\mu_0 g}{4\pi} \frac{1}{r \sin\theta} (1 - \cos\theta) \hat{\varphi}$$
$$\mathbf{A}'(r, \varphi, \theta) = -\frac{\mu_0 g}{4\pi} \frac{1}{r \sin\theta} (1 + \cos\theta) \hat{\varphi}.$$

$\hat{\varphi}$ is the unit vector in the φ direction, where φ is the (multi-valued) spherical coordinate function. The potential $\mathbf{A}$ is singular at $\theta = \pi$ and the potential $\mathbf{A}'$ is singular at $\theta = 0$. On the region where $\theta \neq 0$ and $\theta \neq \pi$ (i.e. S^2 with the north and south poles removed), in some sense the potentials $\mathbf{A}$ and $\mathbf{A}'$ are related by a gauge transformation which corresponds to the function

$$\chi = -\frac{\mu_0 g}{2\pi} \varphi.$$

However, this is not actually a gauge transformation since φ is a multi-valued function and if we make some particular choices for each value, then φ is necessarily discontinuous. Nevertheless, we could relax our definition of a gauge transformation to allow for such things provided that the additional transformation

$$\Phi \mapsto e^{i \frac{q}{\hbar} \chi} \Phi$$

is well-defined mathematically. In general, this will not be the case for a gauge transformation which corresponds to a multivalued function. In the particular case of the gauge transformation relating $\mathbf{A}$ and $\mathbf{A}'$, we have

$$\Phi \mapsto e^{i \frac{\mu_0 g q}{2\pi\hbar} \chi}.$$

This is not generally well-defined, but in the particular case where we have

$$g = n \frac{2\pi\hbar}{\mu_0 q} \tag{6}$$

for some integer n, then the transformation becomes

$$\Phi \mapsto e^{in\varphi} \Phi,$$

which is well-defined even though φ is multivalued since $e^{in\varphi} = e^{in(\varphi + 2\pi)}$. The condition given in Equation 6 is called the **Dirac quantization condition**. Assuming that a magnetic charge g satisfies the Dirac quantization condition, then we can discuss $\mathbf{A}$ and $\mathbf{A}'$ as giving a coherent electromagnetic field on S^2 which has the property that

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 g}{4\pi r^2} \hat{\mathbf{r}}.$$

6.4 Exercises

1. Show that if we make the transformation $A_\mu \mapsto A_\mu + \partial_\mu \chi$, then Equation 5 is not necessarily preserved.

2. Show that if we assume that the inverse Compton wavelength is much larger than the space derivatives of Φ, then we can recover Equation 3 from Equation 5.

3. Given G-bundles $\mathcal{P}_1 : P_1 \to X$ and $\mathcal{P}_2 : P_2 \to X$, a morphism of G-bundles is a map $P_1 \to P_2$ which is equivariant. Show that a G-bundle over X is isomorphic to the trivial bundle $X \times G$ if and only if it admits a global section. If a G-bundle is isomorphic to the trivial bundle, then we say that it is **trivializable**.

Remark 6.4.1. The property that trivializability is equivalent to admitting a global section is fairly unique to G-bundles; this is not true about other fiber bundles (vector bundles, sphere bundles, etc.).

7 Spin-1/2 Electrodynamics as Gauge theory

7.1 Intro

The goal of this section is to formulate a spin$\frac{1}{2}$ particle's wavefunction ψ in a way that accounts for all symmetries of the system. These symmetries are internal symmetries of the system i.e. local gauge symmetry $\psi \mapsto e^{i\theta(x,y,z,t)}\psi$ and external symmetries, such as invariance under Lorentz transformations. This can be accomplished by pasting together locally defined wavefunctions into a globally defined matter field ϕ.

7.1.1 Ingredients for a gauge theory

- A smooth, oriented, (pseudo-)Riemannian manifold, M in which the particle reside. In the physics context, generally this will be space $\mathbb{R}^3$, spacetime $\mathbb{R}^{1,3}$ or some open subset of either, i.e. contractible. But this does not have to be the case, in fact things are in a sense more interesting when we consider manifolds with non trivial topology.

- A finite dimensional inner product space V. This is considered the inner space of the particle. Think $\mathbb{C}, \mathbb{C}^2, \mathbb{C}^4$. This is the space we want the wavefunction to take values in. The inner product allows for computation of squared norms (probability densities).

- A matrix Lie group G with a representation

$$\rho : G \to GL(V) \quad g \mapsto \rho(g)$$

which preserves the inner product of V

$$\langle v, w\rangle = \langle \rho(g)v, \rho(g)w\rangle$$

This is the group of symmetries of our system, e.g. rotational symmetry of $\mathbb{R}^3$ is $SO(3)$ and the group of rotational symmetries of $\mathbb{R}^{1,3}$ is the restricted Lorentz group $SO(1,3)_+ = \mathcal{L}_+^\uparrow$. Having a symmetry of our systems means that G acts both on the frames of the system and on the state. For example, rotational invariance of $\mathbb{R}^3$ means that if we rotate our reference frame of $\mathbb{R}^3$ using $R \in SO(3)$ and rotate the corresponding state vector in V we should expect no change, that is

$$\psi(x, y, z) = \rho(g)R\,\psi(R(x, y, z))$$

- A smooth principal G-bundle P over M

$$G \hookrightarrow P \xrightarrow{\pi} M$$

Recall, that P admits a global section iff it is trivializable. But P does have local sections $s : U \to P$ for $U \subset M$, regardless of its trivializability. These local sections correspond to the local gauge (frame of V) with respect to which the wavefunction ψ can be described. A change of gauge corresponds to an action of G on the fibers of P, that is, $s(m) \mapsto s(m)g$, and changes the wavefunction by $\psi \mapsto \rho(g)^{-1}\psi$.

- A connection $\boldsymbol{\omega}$ on P with curvature $\boldsymbol{\Omega}$. Given a local section $s : U \to P$ we can pullback each to give the local gauge potential $s^*\boldsymbol{\omega} = \boldsymbol{\mathcal{A}}$ on M, and the local field strength $s^*\boldsymbol{\Omega} = \boldsymbol{\mathfrak{F}}$.
- In order to account for the gauge symmetries in a way that gives a globally defined wavefunction (matter field) we will consider global sections of the associated bundle

  $$V \hookrightarrow P \times_\rho V \xrightarrow{\hat{\pi}} M$$

  which are in one-to-one correspondence with maps $\phi : P \to V$ which are G-equivariant, that is

  $$\phi(pg) = g^{-1}\phi(p)$$

  This ϕ is the globally defined matter field over M.
- A potential function

  $$\mathcal{U} : V \to \mathbb{R}$$

  which is smooth, non-negative, real-valued and is G-invariant, $\mathcal{U}(gv) = \mathcal{U}(v)$. The self interaction energy of a matter field ϕ is then given by $\mathcal{U} \circ \phi$.
- Finally, we need an action functional $S(\boldsymbol{\omega}, \phi)$, from which we can determine the stationary points, i.e., the field configurations satisfying the Euler-Lagrange equations.

All of these building blocks combine to create our gauge theory. We formulate a gauge theory for a spin-$\frac{1}{2}$ particle without interactions, and then with interactions.

7.2 Spin

7.2.1 Some historical background

The first experimental evidence of spin is due to Stern and Gerlach, which in 1922, showed that the spin of silver atoms is quantized, there are only two possible states. Initially thought to be a magnetic moment created by the orbit of the electron around the nucleus, it was later proposed by Pauli that spin is a fundamentally quantum mechanical internal degree of freedom, which can only take on one of two possible values. Puzzling over this opaque description led Uhlenbeck and Goudsmit to describe this degree of freedom as a form of angular momentum, as if the electron were spinning about an axis. It is not, but the notion of intrinsic angular momentum provided a nice explanation of the two valuedness: the electron can either spin clockwise or counterclockwise. While the physicality of spin is still undetermined, a complete description of the sate of the particle nonetheless needs to incorporate this degree of freedom.
The first attempt to account for this was by Pauli, who eventually developed of a linear algebraic approach in which the spin state of a particle is represented by a unit vector in $\mathbb{C}^2$. This theory, while quite successful, is unfortunately non-relativistic and was later superseded by Dirac's relativistic formulation in which the state of the particle lives in $\mathbb{C}^4$.

7.2.2 Details of QM spin treatment

Spin observables and Pauli matrices....

7.2.3 Non-relativistic treatment

Consider a particle described by a wave function $\psi(x, y, z, t)$. As discussed above, a compelte description of this particle will depend on the spin state. That is, if we fix a preferred axis to measure along e.g. z-axis, then ψ will also depend on the value of the operator σ_z. As $\sigma_z = \pm 1$ we really have a two wavefunction

$$\psi(x, y, z, t, \sigma_z) = \begin{bmatrix} \psi(x, y, z, t, +1) \\ \psi(x, y, z, t, -1) \end{bmatrix} = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} \in \mathbb{C}^2$$

Our basic quantum mechanics tells us that for fixed time t and a region of space $R \subset \mathbb{R}^3$, the probabilities of finding the particle in R with spin up or spin down are given by

$$\int_R \|\psi_+\|^2 \quad \text{and} \quad \int_R \|\psi_-\|^2$$

respectively. While the probability of finding it in R at all is

$$\int_R \|\psi_+\|^2 + \|\psi_-\|^2$$

7.2.4 Rotational invariance and representations

Let's consider the spatial part of a stationary state, i.e. the object

$$\begin{bmatrix} \psi_+(x, y, z) \\ \psi_-(x, y, z) \end{bmatrix} \in \mathbb{C}^2$$

where $x, y, z = v$ are a certain oriented, orthonormal frame of $\mathbb{R}^3$. This object should transform accordingly when a rotation of the frame is applied. That is, for $R_1 \in SO(3)$

$$\rho(R_1)\psi(v) = \psi(R_1^{-1}v)$$

where $\rho(R_1) \in GL(\mathbb{C}^2)$. Moreover, each $\rho(R)$ should have an inverse $\rho(R^{-1}) = \rho(R)^{-1}$, and given another $R_2 \in SO(3)$ these operations should compose

$$\rho(R_2R_1) = \rho(R_2)\rho(R_1)$$

In other words we are looking for a representation

$$SO(3) \xrightarrow{\tilde{\rho}} GL(\mathbb{C}^2) \quad R \mapsto \rho(R)$$

These representations are well known. There is one for each positive integer l, the dimension of which is $2l + 1$. This puts a kink in our plan as the dimension of the representation we want it 2. This indicates that perhaps $SO(3)$ is not able to resolve all the information we need.

In order to proceed we need to pass to the universal cover of $SO(3)$ given by the double covering map

$$SU(2) \xrightarrow{\varphi} SO(3) \quad g \mapsto \pm g$$

and investigate representations of the form

$$SU(2) \xrightarrow{\rho} GL(\mathbb{C}^n)$$

A fact of note is that all representations of $SO(3)$ can be turned into representations of $SU(2)$ by precomposing with φ

$$SU(2) \xrightarrow{\varphi} SO(3) \xrightarrow{\tilde{\rho}} GL(\mathbb{C}^n)$$

but representation of $SU(2)$ will only induce a representation of $SO(3)$ if $\rho(-g) = \rho(g)$ for all $g \in SU(2)$. This is because ρ must factor through $SO(3)$ via the double covering map φ.

The irreducible representations of $SU(2)$ are easily described. Consider the vector space of polynomials $\mathbb{C}[x, y]$, then for each $k = 0, 1, \dots$ we have a corresponding subspace V_k spanned by polynomials of the form $x^{k-n}y^n$ for $n = 0, 1, \dots, k$. The action of a $g \in SU(2)$ is given by $x^{k-n}y^n \mapsto (x')^{k-n}(y')^n$ where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

As $V_k \cong \mathbb{C}^{k+1}$ we have representations

$$D^{\frac{k}{2}} : SU(2) \to GL(\mathbb{C}^{k+1})$$

These are refered to as *spin-j representations* where $j = 0, 1/2, 1, 3/2, \dots.$ When $k = 0$ we have only the trivial representation

$$D^0 : SU(2) \to GL(\mathbb{C}) \quad g \mapsto 1$$

When $k = 1$ we have

$$D^{\frac{1}{2}} : SU(2) \to GL(\mathbb{C}^2) \quad g \mapsto g$$

giving us an irreducible 2-dimensional representation.
We could also build a 2-dimensional representation from the 1-dimensional D^0 by

$$D^0 \oplus D^0 : SU(2) \to GL(\mathbb{C}^2) \quad g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The issue here is that this trivial action does not transform our state in the way we expect e.g. if we rotate the frame in a way that interchanges the z-axis then the components of our vector should be interchanged

$$\begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} \to \begin{bmatrix} \psi_- \\ \psi_+ \end{bmatrix}$$

which $D^0 \oplus D^0$ certainly does not do. This means the representation that we are in need of is in fact $D^{\frac{1}{2}}$.

Explain more why the half integer reps are different

Here is where we leave the non relativistic treatment.

7.3 Relativistic theory

7.3.1 Dirac Equation and Bispinors

Since Pauli's theory was non-relativistic we move now to Dirac's treatment which is relativistic, i.e. invariant under Lorentz transformations.

The Dirac equation, which will be described in detail later I think...is

$$(\gamma^\mu \partial_\mu \phi + im)\phi = 0$$

here m is the mass, γ^μ are 4×4 matrix representations of elements of a Clifford algebra, to be explained later, and ϕ is an element of $\mathbb{C}^4$, this be the crucial detail for our current purposes. This element ϕ can be written

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$$

where $\phi_i \in \mathbb{C}$ and $\psi_L, \psi_R \in \mathbb{C}^2$ are called left and right-handed spinors, respectively. This object ϕ is known as a bispinor (or four-component spinor), and is what led to the prediction of antimatter.

In order for Dirac's theory to be Lorentz invariant a change to the coordinates x^0, x^1, x^2, x^3 of $\mathbb{R}^{1,3}$ of the form

$$y^\mu = \Lambda^\mu{}_\nu x^\nu$$

with $\Lambda^\mu{}_\nu \in \mathcal{L}_+^\uparrow = SO(1,3)_+$ should leave the Dirac equation in the same form.

7.3.2 Representations of $SL(2,\mathbb{C})$

As was done for the non-relativistic theory we require a representation

$$\mathcal{L}_+^\uparrow \to GL(\mathbb{C}^4)$$

Note: we now are looking for a four dimensional representation not a two dimensional representation.

The group $SO(3)$ is contained in $\mathcal{L}_+^\uparrow$ as a closed subgroup and the group $SU(2)$ is a closed subgroup of $SL(2,\mathbb{C})$. One would hope that representations of $\mathcal{L}_+^\uparrow$ will be analogous to the situation with $SO(3)$ and maybe even restrict to those representations on the closed subgroups. This is indeed the case.
There is a double covering map

$$SL(2,\mathbb{C}) \xrightarrow{\Phi} \mathcal{L}_+^\uparrow$$

constructed in detail in Naber. We then have

$$\begin{array}{ccc} SL(2,\mathbb{C}) & & \\ \Phi\downarrow & \searrow^{\rho} & \\ \mathcal{L}_+^\uparrow & \underset{\tilde{\rho}}{\dashrightarrow} & GL(\mathbb{C}^4) \end{array}$$

where any representation of $\mathcal{L}_+^\uparrow$ can be pulled back to a representation on $SL(2,\mathbb{C})$ but not every representation of $SL(2,\mathbb{C})$ can be pushed forward to one on $\mathcal{L}_+^\uparrow$. This is the case iff $\rho(g) = \rho(-g)$ for all $g \in SL(2,\mathbb{C})$. Like the situation with $SU(2)$ and $SO(3)$ the representations of $SL(2,\mathbb{C})$ are the fundamental objects of interest.

The n-dimensional irreducible representations of $SL(2,\mathbb{C})$ are well described in the literature so we will only look at the ones relevant to this discussion. For the $n = 2$ case we have the *left-handed spinor representation*

$$D^{(\frac{1}{2},0)} : SL(2,\mathbb{C}) \to GL(\mathbb{C}^2) \quad g \mapsto g$$

and the *right-handed spinor representation*

$$D^{(0,\frac{1}{2})} : SL(2,\mathbb{C}) \to GL(\mathbb{C}^2) \quad g \mapsto (g^*)^{-1}$$

where g^* is the conjugate transpose of g. As we are looking for a four-dimensional representation we can combine these two to get

$$D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})} : SL(2,\mathbb{C}) \to GL(\mathbb{C}^4)$$

$$g \mapsto \begin{bmatrix} g & 0 \\ 0 & (g^*)^{-1} \end{bmatrix}$$

Transformations of the bispinor under this representation should leave the Dirac equation invariant. That is, for given coordinates $x = (x^0, x^1, x^2, x^3)$ $\mathbb{R}^{1,3}$ and a Lorentz transformation $\Lambda = \Lambda_\mu^\nu \in \mathcal{L}_+^\uparrow$ we have new coordinates $y = (y^0, y^1, y^2, y^3)$ related by

$$y^\nu = \Lambda_\mu^\nu x^\mu$$

and $\hat{\partial}_\nu = \frac{\partial}{\partial y^\nu}$ related by

$$\partial_\mu = \Lambda_\mu^\nu x^\mu \hat{\partial}_\nu$$

the transformed bispinor is

$$\hat{\phi}(y) = D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}(g)\phi(\Lambda^{-1}x)$$

Invariance of the Dirac equation means that if

$$(\gamma^\mu \partial_\mu \phi + im)\phi = 0$$

then

$$(\gamma^\nu \partial_\nu \hat{\phi} + im)\hat{\phi} = 0$$

This is indeed the case.

7.3.3 Bundle Formalism for Free Dirac Electron

Let's build our gauge theory of a free electron, i.e. no electromagnetic interactions.

- Base manifold is $\mathbb{R}^{1,3}$
- The internal space is $\mathbb{C}^4$ with the inner product

$$\langle v, w\rangle = \frac{1}{2}\big(h(v,w) + h(w,v)\big)$$

  where h is the twisted Hermitian form

$$h(v,w) = v_1\overline{w}_3 + v_2\overline{w}_4 + v_3\overline{w}_1 + v_4\overline{w}_2$$

- Lie group $SL(2,\mathbb{C})$ with representation

$$D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})} : SL(2,\mathbb{C}) \to GL(\mathbb{C}^4)$$

  which encapsulates the external symmetry of Lorentz invariance. This representation is unitary with respect to $\langle\cdot,\cdot\rangle$, as can be verified.

- Naively Lorentz invariance should be encapsulated using the oriented, time-oriented, orthonormal frame bundle

$$\mathcal{L}_+^\uparrow \hookrightarrow L \xrightarrow{\pi} \mathbb{R}^{1,3}$$

  on which $\mathcal{L}_+^\uparrow$ acts. But as was seen earlier the representation on $\mathbb{C}^4$ actually arises from $SL(2,\mathbb{C})$. So in fact we need to have each fiber $\mathcal{L}_+^\uparrow$ of L double covered by $SL(2,\mathbb{C})$, that is the bundle

$$SL(2,\mathbb{C}) \hookrightarrow S \xrightarrow{\Pi} \mathbb{R}^{1,3}$$

  with a map $S \xrightarrow{\tilde{\Phi}} L$, which is the double cover map on each fiber

$$\Pi^{-1}(x) \cong SL(2,\mathbb{C}) \xrightarrow{\Phi} \mathcal{L}_+^\uparrow \cong \pi^{-1}(x)$$

  such that the diagram

$$\begin{array}{ccccc} S \times SL(2,\mathbb{C}) & \xrightarrow{\;\cdot\;} & S & \xrightarrow{\Pi} & \mathbb{R}^{1,3} \\ \downarrow{\scriptstyle \tilde{\Phi}\times\Phi} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ L \times \mathcal{L}_+^\uparrow & \xrightarrow[\cdot]{} & L & \xrightarrow[\pi]{} & \mathbb{R}^{1,3} \end{array}$$

  commutes. This is the spin structure on $\mathbb{R}^{1,3}$.

  In general a spin structure on a manifold M need not exist. This will be explored more later.

The free Dirac electron is then an $SL(2,\mathbb{C})$-equivariant map $\phi : S \to \mathbb{C}^4$. Explicitly ϕ satisfies

$$\phi(pg) = g^{-1}\phi(p) = \begin{bmatrix} g^{-1} & 0 \\ 0 & g^* \end{bmatrix}$$

Equivalently the free Dirac electron is a section of the associated bundle

$$\mathbb{C}^4 \hookrightarrow S \times_\rho \mathbb{C}^4 \to \mathbb{R}^{1,3}$$

where the representation $\rho = D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$. That is, a 4-component spinor field on $\mathbb{R}^{1,3}$.

We know that if

$$(\gamma^\mu \partial_\mu \phi + im)\phi = 0$$

then

$$(\gamma^\mu \partial_\mu \phi + im)e^{i\theta}\phi = 0$$

that is, ϕ has internal $U(1)$ symmetry in addition to the external Lorentz invariance. If we want this $U(1)$ symmetry to be local not just global we need the presence of an electromagnetic potential which is the data of a connection on a $U(1)$-principal bundle over $\mathbb{R}^{1,3}$. In order to do this we will need to couple the $SL(2,\mathbb{C})$-bundle to a $U(1)$-bundle into a $SL(2,\mathbb{C}) \times U(1)$-bundle over $\mathbb{R}^{1,3}$.

7.4 Coupling matter field to EM

7.4.1 Spliced bundles

Consider two principal bundles over M

$$G \hookrightarrow P \xrightarrow{\pi_P} M$$

$$H \hookrightarrow Q \xrightarrow{\pi_Q} M$$

Define

$$P \circ Q = \{(p,q) \in P \times Q \mid \pi_P(p) = \pi_Q(q)\}$$

as the total space which is a submanifold of $P \times Q$. The map

$$\Pi : P \circ Q \quad \Pi(p,q) = \pi_P(p) = \pi_Q(q)$$

is then the bundle projection and we have a smooth right action defined by

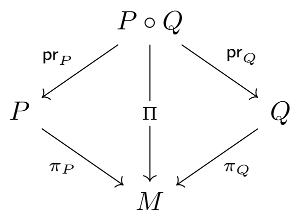
$$P \circ Q \times (G \times H) : P \circ Q \quad ((p,q),(g,h)) \mapsto (pg, qh)$$

This makes $P \circ Q$ into a $G \times H$-principal bundle over M.
By identifying $G \cong G \times \{e_H\}$ and $H \cong \{e_G\} \times H$ we also have the principal bundles

$$H \hookrightarrow P \circ Q \xrightarrow{\mathsf{pr}_P} P$$

$$G \hookrightarrow P \circ Q \xrightarrow{\mathsf{pr}_Q} Q$$

where $\mathsf{pr}_P, \mathsf{pr}_Q$ are projection on to P and Q, respectively. hence we have the commutative diagram



Now suppose we have connections $\boldsymbol{\omega}_P, \boldsymbol{\omega}_Q$ on P, Q, respectively. Note

$$\mathfrak{g} \cong \mathfrak{g} \oplus \{0\} \subseteq \mathfrak{g} \oplus \mathfrak{h}$$

$$\mathfrak{h} \cong \{0\} \oplus \mathfrak{h} \subseteq \mathfrak{g} \oplus \mathfrak{h}$$

so that $\mathsf{pr}_P^* \boldsymbol{\omega}_P$ is a connection on $P \circ Q \xrightarrow{\mathsf{pr}_Q} Q$ and $\mathsf{pr}_Q^* \boldsymbol{\omega}_Q$ is a connection on $P \circ Q \xrightarrow{\mathsf{pr}_P} P$. Then their direct sum

$$\mathsf{pr}_P^* \boldsymbol{\omega}_P \oplus \mathsf{pr}_Q^* \boldsymbol{\omega}_Q$$

is a connection on $P \circ Q \xrightarrow{\Pi} M$.

The last piece of data we need is a representation of $G \times H$ on a vector space V. If we have representations

$$\rho_P : G \to GL(V)$$

$$\rho_Q : H \to GL(V)$$

that are commutative,i.e. $\rho_P(g)\rho_Q(h) = \rho_Q(h)\rho_P(g)$ then we can form the product representation

$$\rho_P \times \rho_Q : G \times H \to GL(V)$$

$$\rho_P \times \rho_Q(g, h) = \rho_P(g)\rho_Q(h) = \rho_Q(h)\rho_P(g)$$

giving the associated action on V.

Now the matter field coupled to an electromagnetic potential is given by a map $\phi : P \circ Q \to V$ satisfying

$$\phi((p, q)(g, h)) = \rho_P(g)^{-1}\rho_Q(h)^{-1}\phi(p, q)$$

7.4.2 Electron coupled to EM potential

Let's look at the example of an electron coupled to an electromagnetic potential.

For the electron we have the bundle

$$SL(2, \mathbb{C}) \hookrightarrow S \to \mathbb{R}^{1,3}$$

with the spinor connection $\boldsymbol{\omega}_S$ on S (details alluded to but not found in Naber), and the representation

$$\rho_S = D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})} : SL(2,\mathbb{C}) \to GL(\mathbb{C}^4)$$

For the electromagnetic potential we need some $U(1)$-principal bundle

$$U(1) \hookrightarrow P \to \mathbb{R}^{1,3}$$

with connection $\boldsymbol{\omega}_E$. Define the representation

$$\rho_E : U(1) \to GL(\mathbb{C}^4) \quad z \mapsto z\mathbb{I}_{4\times 4}$$

where here $\mathbb{I}_{4\times 4}$ is the 4×4 identity matrix.
The Dirac electron coupled to this EM potential is then a smooth map

$$\phi : S \circ P \to \mathbb{C}^4$$

that is $SL(2,\mathbb{C}) \times U(1)$ equivariant under the representation $\rho_S \times \rho_E$.

7.5 Gauge transformations and connections

Let $P \xrightarrow{\pi} M$ be a smooth principal G-bundle. A global choice of gauge is given by a global section

$$M \xrightarrow{s} P$$

A global gauge transformation is a bundle diffeomorphism

$$f : P \to P$$

that is G-equivariant, i.e.

A local gauge transformation is a diffeomorphism of the bundle

$$\pi^{-1}(U) \xrightarrow{\pi} U$$

Given a local gauge $s : U \to \pi^{-1}(U)$ and a connection $\boldsymbol{\omega}$ we can pullback the connection to get the local gauge potential $s^*\boldsymbol{\omega} = \boldsymbol{A}$ which is a lg-valued one form on M. There is also a corresponding curvature two-form $\boldsymbol{\Omega}$ on P which we can pullback to get $s^*\boldsymbol{\Omega}$, the local field strength.

In the case of electromagnetism on spacetime, the bundle

$$U(1) \hookrightarrow P \to \mathbb{R}^{1,3}$$

will have a global gauge $s : \mathbb{R}^{1,3} \to P$. The connection $\boldsymbol{\omega}$ can be pulled back to $s^*\boldsymbol{\omega}$ on all of $\mathbb{R}^{1,3}$. Then we define the gauge potential as

$$A_\mu = s^*\boldsymbol{\omega}(\partial_\mu) \in \mathfrak{u}(1) \cong \mathbb{R}$$

and the field strength as

$$F_{\mu\nu} = s^*\mathbf{\Omega}(\partial_\mu, \partial_\nu) = \partial_\mu A_\nu - \partial_\nu A_\mu$$

from which we can get the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.

Since $F = dA$, we know that A is unique only up to a closed 0-form, i.e. for any two potentials A, A' for F they are related by

$$A' = A + df$$

for f a continuous real-valued function on $\mathbb{R}^{1,3}$. This f is what constitutes a gauge transformation of the potential.

In order to recover this notion from that of a bundle automorphism consider a new section $s' : \mathbb{R}^{1,3} \to P$ related to the first choice by $s' = sg$ where $g : \mathbb{R}^{1,3} \to U(1)$. Then the local gauge potential transforms as

$$A'_\mu = g^{-1}A_\mu g + g^{-1}\partial_\mu g$$

Since $U(1)$ is abelian

$$A'_\mu = A_\mu + g^{-1}\partial_\mu g$$

Writing $g = e^{-if}$ then the transform potential becomes

$$A'_\mu = A_\mu + g^{-1}\partial_\mu g = A_\mu + e^{if}e^{-if}(-i\partial_\mu f)$$

recovering

$$A'_\mu = A_\mu + \partial_\mu f$$

7.6 Exercises

1. Let H be a normal subgroup of G and $\mathbb{K}$ a field. Prove that every $\mathbb{K}$-representation of G/H gives rise to a representation of G. Prove that a $\mathbb{K}$-representation ρ of G gives rise to a representation of G/H iff $H \subseteq \mathsf{ker}\rho$.

2. Show that $SL(2, \mathbb{C})$ has an action on $\mathbb{R}^{1,3}$, identified as the pace of 2×2 complex Hermitian matrices

$$x = (x^0, x^1, x^2, x^3) \leftrightarrow \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix}$$

   via conjugation. Also show that $SL(2, \mathbb{C})$ is the double cover of the restricted Lorentz group $\mathcal{L}_+^\uparrow$.

3.

8 Spin-Structures

8.1 Clifford Algebra

The construction of Spin group arises naturally from Clifford algebra.

Definition. Let V be a finite-dimensional vector space over $k = \mathbb{R}$, and

$$q : V \times V \to k$$

a symmetric bilinear form. We often call the pair (V, q) a quadratic vector space. The Clifford algebra $\mathrm{Cl}(V, q)$ is the associative k-algebra with unit, generated by V, and subject to the relations

$$\{u, v\} = uv + vu = -2q(u, v) \cdot \mathbf{1} \quad \forall u, v \in V$$

In other words, if we denote by I_q the two-sided ideal in $T(V)$ generated by the set $\{v \otimes v + q(v, v)1, v \in V\} \subset T(V)$, the Clifford algebra $\mathrm{Cl}(V, g)$ is the quotient

$$\mathrm{Cl}(V, g) := T(V)/I_q$$

Proposition. The Clifford algebra $\mathrm{Cl}(V, q)$ exists, and is uniquely defined by its universality property: for every linear map $\jmath : V \to \mathcal{A}$ such that $\mathcal{A}$ is an associative unital k-algebra, and $\{\jmath(u), \jmath(v)\} = -2q(u, v) \cdot \mathbf{1}$, there exists a unique morphism of algebras $\Phi : \mathrm{Cl}(V, q) \to \mathcal{A}$ such that the diagram below is commutative.

$$\begin{array}{ccc} V & \xrightarrow{\imath} & \mathrm{Cl}(V, q) \\ & \searrow^{\jmath} & \downarrow \Phi \\ & & \mathcal{A} \end{array}$$

$\imath$ denotes the natural inclusion $V \hookrightarrow \mathrm{Cl}(V, q)$.

The algebra $\mathrm{Cl}(V, q)$ depends only on the isomorphism class of the pair (V, q) = vector space + quadratic form. It is known from linear algebra that some simple invariants classify the isomorphism classes of such pairs: $(\dim V, \operatorname{rank} q, \operatorname{sign} q)$. We will be interested in the special case when $\dim V = \operatorname{rank} q = \operatorname{sign} q = n$, i.e., when q is an Euclidean metric on the n-dimensional space V. In this case, the Clifford algebra $\mathrm{Cl}(V, q)$ is usually denoted by $\mathrm{Cl}(V)$, or Cl_n. If (e_i) is an orthonormal basis of V, then we can alternatively describe Cl_n as the associative $\mathbb{R}$-algebra with 1 generated by (e_i), and subject to the relations

$$e_i e_j + e_j e_i = -2\delta_{ij}$$

Using the universality property of Cl_n, we deduce that the map

$$V \to \mathrm{Cl}(V), \quad v \mapsto -v \in \mathrm{Cl}(V)$$

extends to an automorphism of algebras $\alpha : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$. Note that α is involutive, i.e., $\alpha^2 = \mathbf{1}$. Set

$$\mathrm{Cl}^0(V) := \ker(\alpha - \mathbf{1}), \quad \mathrm{Cl}^1(V) = \ker(\alpha + \mathbf{1})$$

Note that $\mathrm{Cl}(V) = \mathrm{Cl}^0(V) \oplus \mathrm{Cl}^1(V)$, and moreover

$$\mathrm{Cl}^{\varepsilon}(V) \cdot \mathrm{Cl}^{\eta}(V) \subset \mathrm{Cl}^{(\varepsilon+\eta) \bmod 2}(V)$$

i.e., the automorphism α naturally defines a $\mathbb{Z}_2$-grading of $\mathrm{Cl}(V)$. In other words, the Clifford algebra $\mathrm{Cl}(V)$ is naturally a super-algebra.

Note: Let A be superalgebra ($\mathbb{Z}_2$-graded). Then the supercommutator is the bilinear map $[\cdot,\cdot]_s : A \times A \to A$ defined on homogeneous elements by $[a,b]_s := ab - (-1)^{|a||b|} ba$. A superalgebra A is supercommutative if the supercommutator vanishes identically, equivalently, for all homogeneous $a, b \in A$ we have

$$ab = (-1)^{|a||b|} ba$$

8.2 The Spin Group

Definition. The Clifford group is defined to be

$$\Gamma(V) = \left\{ x \in \mathrm{Cl}^\star(V); \quad \alpha(x) \cdot V \cdot x^{-1} \subset V \right\}$$

where $\mathrm{Cl}(V)^\star$ denotes the group of invertible elements.

Definition. Define

$$\mathrm{Pin}(V) := \{ x \in \Gamma(V); |N(x)| = 1 \}$$

where N is the spinorial norm, a map $N : \mathrm{Cl}(V) \to \mathrm{Cl}(V), \quad N(x) = x^b x$. ^{b} is the anti-automorphism map of $\mathrm{Cl}(V)$.

Define

$$\mathrm{Spin}(V) := \{ x \in \Gamma(V); \quad N(x) = 1 \} = \mathrm{Pin}(V) \cap \Gamma^0(V)$$

Note that $\cap \Gamma^0(V)$ represents the even part that is fixed by α. Alternatively, $\mathrm{Spin}(V)$ can be described by the following equality,

$$\mathrm{Spin}(V) = \{ v_1 \cdots v_{2k}; k \geq 0, v_i \in V, |v_i| = 1, \forall i = 1 \ldots 2k \}$$

In particular, this shows that $\mathrm{Spin}(V)$ is a compact topological group. Observe that $\mathrm{Spin}(V)$ is a closed subgroup of the Lie group $\mathrm{GL}(\mathrm{Cl}(V))$. This implies that $\mathrm{Spin}(V)$ is in fact a Lie group, and the map $\mathrm{Spin}(V) \hookrightarrow \mathrm{Cl}(V)$ is a smooth embedding.

The *orthogonal group* of a vector space V, denoted $\mathrm{O}(V)$, is the group of all linear transformations $T : V \to V$ that preserve the quadratic form q, i.e.,

$$\mathrm{O}(V) = \{ T : V \to V \mid q(T(v)) = q(v) \text{ for all } v \in V \},$$

where $q(v)$ is the quadratic form associated with the vector space V.

Proposition. There exist short exact sequences

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}(V) \to O(V) \to 1$$

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(V) \to SO(V) \to 1$$

Proposition. The morphism $\rho : \mathrm{Spin}(V) \to SO(V)$ is a covering map. Moreover, the group $\mathrm{Spin}(V)$ is connected if $\dim V \geq 2$, and simply connected if $\dim V \geq 3$. In particular, $\mathrm{Spin}(V)$ is the universal cover of $SO(V)$, when $\dim V \geq 3$.

8.2.1 Low-dimensional examples

(The case $n = 1$). The Clifford algebra Cl_1 is isomorphic with the field of complex numbers $\mathbb{C}$. The $\mathbb{Z}_2$-grading is $\mathbf{Re}\mathbb{C} \oplus \mathbf{Im}\mathbb{C}$. The group Spin(1) is isomorphic with $\mathbb{Z}_2$.

(The case $n = 2$). The Clifford algebra Cl_2 is isomorphic with the algebra of quaternions $\mathbb{H}$. This can be seen by choosing an orthonormal basis $\{e_1, e_2\}$ in $\mathbb{R}^2$. The isomorphism is given by

$$1 \mapsto 1, \quad e_1 \mapsto \boldsymbol{i}, \quad e_2 \mapsto \boldsymbol{j}, \quad e_1 e_2 \mapsto \boldsymbol{k},$$

where $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ are the imaginary units in $\mathbb{H}$. Note that

$$\mathrm{Spin}(2) = \left\{a + b\boldsymbol{k}; a, b \in \mathbb{R}, a^2 + b^2 = 1\right\} \cong S^1$$

The natural map $\mathrm{Spin}(1) \to SO(2) \cong S^1$ takes the form $e^{i\theta} \mapsto e^{2i\theta}$.

(The case $n = 3$). The Clifford algebra Cl_3 is isomorphic, as an ungraded algebra, to the direct sum $\mathbb{H} \oplus \mathbb{H}$. More relevant is the isomorphism $\mathrm{Cl}_3^{\text{even}} \cong \mathrm{Cl}_2 \cong \mathbb{H}$ given by

$$1 \mapsto 1, \quad e_1 e_2 \mapsto \boldsymbol{i}, \quad e_2 e_3 \mapsto \boldsymbol{j}, \quad e_3 e_1 \mapsto \boldsymbol{k}$$

where $\{e_1, e_2, e_3\}$ is an orthonormal basis in $\mathbb{R}^3$. Under this identification the operation $x \mapsto x^b$ coincides with the conjugation in $\mathbb{H}$

$$x = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \mapsto \bar{x} = a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k}$$

In particular, the spinorial norm coincides with the usual norm on $\mathbb{H}$

$$N(a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}) = a^2 + b^2 + c^2 + d^2$$

Thus, any $x \in \mathrm{Cl}_3^{\text{even}} \backslash\{0\}$ is invertible, and

$$x^{-1} = \frac{1}{N(x)} x^b$$

Moreover, a simple computation shows that $x\mathbb{R}^3 x^{-1} \subset \mathbb{R}^3, \forall x \in \mathrm{Cl}_3^{\text{even}} \backslash\{0\}$, so that

$$\Gamma^0\left(\mathbb{R}^3\right) \cong \mathbb{H}\backslash\{0\}$$

Hence

$$\mathrm{Spin}(3) \cong \{x \in \mathbb{H}; \quad |x| = 1\} \cong SU(2)$$

8.3 Whitehead Towers

Definition. If we take X to be an arbitrary CW complex with the subspace A a point, then the resulting tower of n-connected CW models amounts to a sequence of maps

$$\cdots \to Z_2 \to Z_1 \to Z_0 \to X$$

with Z_n n-connected and the map $Z_n \to X$ inducing an isomorphism on all homotopy groups π_i with $i > n$. The space Z_0 is path-connected and homotopy equivalent to the component of X containing A, so one may as well assume Z_0 equals this component. The next space Z_1 is simply-connected, and the map $Z_1 \to X$ has the homotopy properties of the universal cover of the component Z_0 of X. For larger values of n one can by analogy view the map $Z_n \to X$ as an ' n-connected cover' of X.

8.3.1 Whitehead Towers of Spin(n), $n \geq 3$

This section discusses the Whitehead towers related to Spin groups for $n \geq 3$.

The **orthogonal group** $O(n)$ is defined as the set of all $n \times n$ orthogonal matrices:

$$O(n) = \{A \in GL(n, \mathbb{R}) \mid A^T A = I\}$$

The determinant of an orthogonal matrix is either $+1$, representing rotations, or -1, representing reflections. The zeroth homotopy group of the orthogonal group $O(n)$ is given by:

$$\pi_0(O(n)) = \mathbb{Z}_2$$

The orthogonal group can be divided into two disconnected components, with the component of determinant $+1$ being the special orthogonal group, $SO(n)$.
Moving to $SO(n)$ in the Whitehead tower, we have

$$\pi_0(SO(n)) = 0$$

since $SO(n)$ is connected. However, $SO(n)$ is not simply connected, and has its fundamental group

$$\pi_1(SO(n)) = \mathbb{Z}_2$$

Spin(n) being the double cover and the universal cover of $SO(n)$ is simply connected and therefore "kills" the fundamental group and has

$$\pi_1(\text{Spin}(n)) = 0$$

The next non-trivial homotopy group is π_3, and the space that kills it is String(n), an infinite dimensional group.

8.4 Tangential Structure

Consider a closed, connected, smooth manifold M of dimension n. The *tangent bundle* TM is a vector bundle over M, where the fiber at each point $p \in M$ is the tangent space T_pM, consisting of the tangent vectors to M at p. Locally, in a coordinate chart $U \subset M$, the tangent space at each point can be represented as an n-dimensional vector space.

The tangent space T_pM at a point p is spanned by the partial derivatives $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ in local coordinates $(x^1, \ldots, x^n)$ near p.

Take two overlapping coordinate charts U and V with coordinates $(x^1, \ldots, x^n)$ on U and $(y^1, \ldots, y^n)$ on V, there is a smooth change of coordinates function $\phi : U \cap V \to \mathbb{R}^n$ that relates the coordinates x and y.

When you move from one coordinate chart U to another overlapping chart V, the basis of the tangent space changes according to the change of coordinates. Specifically, if the coordinates in U are $(x^1, \ldots, x^n)$ and in V are $(y^1, \ldots, y^n)$, then the relationship between the tangent vectors in the two charts is given by the *Jacobian matrix* of the coordinate transformation: $\frac{\partial y^i}{\partial x^j}$. This matrix describes how the components of a tangent vector in chart U transform when you switch to chart V. The Jacobian matrix $\frac{\partial y^i}{\partial x^j}$ of a smooth change of coordinates is an element of $\mathrm{GL}(n, \mathbb{R})$.

8.4.1 Deformation Retraction

A *deformation retraction* from $GL(n)$ to $O(n)$ is a continuous map $H : \mathrm{GL}(n, \mathbb{R}) \times [0, 1] \to \mathrm{GL}(n, \mathbb{R})$ such that:

- $H(A, 0) = A$ for all $A \in \mathrm{GL}(n, \mathbb{R})$,
- $H(A, 1) \in \mathrm{O}(n)$ for all $A \in \mathrm{GL}(n, \mathbb{R})$,
- $H(Q, t) = Q$ for all $Q \in \mathrm{O}(n)$ and all $t \in [0, 1]$.

We use the *Gram-Schmidt process* to construct the deformation retraction.

For each invertible matrix $A \in \mathrm{GL}(n, \mathbb{R})$, decompose A as:

$$A = QR$$

where $Q \in \mathrm{O}(n)$ is the orthogonal matrix obtained from the Gram-Schmidt process and R is an upper triangular matrix with positive diagonal entries (from the QR-decomposition).

Define the deformation retraction by interpolating between A and Q as:

$$H(A, t) = Q\left((1 - t)I + tR\right)$$

where I is the identity matrix and R is the upper triangular matrix from the QR-decomposition of A.

- At $t = 0$, we have $H(A, 0) = A$.
- At $t = 1$, we have $H(A, 1) = Q \in \mathrm{O}(n)$.

A vector bundle $V \to M$ is associated with a principal $GL_n(\mathbb{R})$-bundle $\mathcal{B}(V) \to M$ of bases, often called the frame bundle of $V \to M$. If we endow $V \to M$ with a metric, then we can take orthonormal frames and so construct a principal $O(n)$-bundle of frames $\mathcal{B}_O(V) \to M$ by doing the Gram-Schmidt process in each trivialization.

8.5 Orientation

Let V be a real vector space of dimension $n > 0$. A basis of V is a linear isomorphism $b : \mathbb{R}^n \to V$. Let $\mathcal{B}(V)$ denote the set of all bases of V. The group $GL_n(\mathbb{R})$ of linear isomorphisms of $\mathbb{R}^n$ acts simply transitively on the right of $\mathcal{B}(V)$ by composition: if $b : \mathbb{R}^n \to V$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are isomorphisms, then so too is $b \circ g : \mathbb{R}^n \to V$. We say that $\mathcal{B}(V)$ is a right $GL_n(\mathbb{R})$-torsor. For any $b \in \mathcal{B}(V)$ the map $g \mapsto b \circ g$ is a bijection from $GL_n(\mathbb{R})$ to $\mathcal{B}(V)$, and we use it to topologize $\mathcal{B}(V)$. Since $GL_n(\mathbb{R})$ has two components, so does $\mathcal{B}(V)$. An orientation of V is a choice of component of $\mathcal{B}(V)$.
Recall that the components of $GL_n(\mathbb{R})$ are distinguished by the determinant homomorphism

$$\det : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^{\neq 0}$$

the identity component consists of $g \in GL_n(\mathbb{R})$ with $\det(g) > 0$, and the other component consists of g with $\det(g) < 0$. On the other hand, an isomorphism $b : \mathbb{R}^n \to V$ does not have a numerical determinant. Rather, its determinant lives in the determinant line Det V of V. Namely, define

$$\operatorname{Det} V = \{\epsilon : \mathcal{B}(V) \to \mathbb{R} : \epsilon(b \circ g) = \det(g)^{-1}\epsilon(b) \text{ for all } b \in \mathcal{B}(V), g \in GL_n(\mathbb{R})\}$$

$\mathfrak{o}(V) = \{\epsilon : \mathcal{B}(V) \to \{\pm 1\} : \epsilon(b \circ g) = \operatorname{sign}\det(g)^{-1}\epsilon(b) \text{ for all } b \in \mathcal{B}(V), g \in GL_n(\mathbb{R})\}$. An orientation of V is a point of $\mathfrak{o}(V)$.

The 2:1 map $\mathfrak{o}(V) \to X$ is called the orientation double cover associated to $V \to X$, and equivalently a principle $\mathbb{Z}_2$ bundle. In case $V = TX$ is the tangent bundle, it is called the orientation double cover of X.

(i) An orientation of a real vector bundle $V \to X$ is a section of $\mathfrak{o}(V) \to X$. (ii) If $o : X \to \mathfrak{o}(V)$ is an orientation, then the opposite orientation is the section $-o : X \to \mathfrak{o}(V)$. (iii) An orientation of a manifold X is an orientation of its tangent bundle $TX \to X$.

The orientation may or may not exist. The obstruction to existence is the isomorphism class of the orientation double cover: orientations exists if and only if $\mathfrak{o}(V) \to M$ is trivializable. This isomorphism class is an element of $H^1(M; \mathbb{Z}/2\mathbb{Z})$ which is isomorphic to $\operatorname{Hom}(\pi_1(M), \mathbb{Z}/2\mathbb{Z})$.

The isomorphism class of $\mathfrak{o}(V) \to M$ is the first Stiefel-Whitney class $w_1(V) \in H^1(M; \mathbb{Z}/2\mathbb{Z})$. The Stiefel-Whitney classes are characteristic classes of real vector bundles. They live in the cohomology algebra $H^\bullet(BO; \mathbb{Z}/2\mathbb{Z})$.

8.5.1 An example

Any n-dimensional manifold M has a double cover

$$p : M_O \to M$$

where M_O is an oriented manifold.

Consider $\mathbb{RP}^n$ for n even. The orientation double cover is S^n; the deck transformation reverses orientation. For $\mathbb{RP}^n$ for n odd, the orientation double cover is a disjoint union of two copies of $\mathbb{RP}^n$, oriented with opposite orientations.

If M is a connected manifold, define the orientation character or the first Stiefel–Whitney class

$$w : \pi_1(M) \to \{\pm 1\}$$

by $w[\gamma] = 1$ if γ lifts to a loop in the orientation double cover and $w[\gamma] = -1$ if γ lifts to a path which is not a loop. Intuitively, $w[\gamma] = -1$ if going around the loop γ reverses the orientation. M is orientable if and only if w is trivial.

8.6 Classifying Space

Theorem. Let M be a paracompact Hausdorff manifold and G a Lie group. Then there is a bijection between the set of homotopy classes of continuous maps $M \to BG$ and the set of isomorphism classes of principle G-bundles over M. We call BG the classifying space of G.

We now work with an arbitrary homomorphism $\rho : H \to G$, where H and G are compact. Let $EH \to BH$ be the universal H-bundle. The associated G-bundle has a classifying map

$$\begin{array}{ccc} EH \times_\rho G & \longrightarrow & EG \\ \downarrow & & \downarrow \\ BH & \xrightarrow{B\rho} & BG \end{array}$$

which we denote $B\rho$. The top horizontal arrow in induces an isomorphism $\theta^{\text{univ}} : EH \times_\rho G \cong (B\rho)^*(EG)$. The pair $(EG \times_\rho G, \theta^{\text{univ}})$ is the universal reduction of a G-bundle to an H-bundle.

Proposition. Let $P \to M$ be a principal G-bundle and $f : M \to BG$ a classifying map. Then a lift $\tilde{f}$ in the diagram

$$\begin{array}{ccc} & & BH \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \\ M & \xrightarrow{f} & BG \end{array}$$

induces a reduction to H, and conversely, a reduction to H induces a lift $\tilde{f}$. Isomorphism classes of reductions are in 1:1 correspondence with homotopy classes of lifts.

Example. $BU(1) \cong S^\infty / U(1) \cong \mathbb{CP}^\infty$

8.7 Reducing Structure Group

Let H, G be Lie groups and $\rho : H \to G$ a homomorphism. (For the discussion of orientations this is the inclusion $GL_n(\mathbb{R}) \hookrightarrow GL_n(\mathbb{R})$.)

Definitions.

(i) Let $Q \to M$ be a principal H-bundle. The associated principal G-bundle $Q_\rho \to M$ is the quotient

$$Q_\rho = (Q \times G)/H,$$

where H acts freely on the right of $Q \times G$ by

$$(q, g) \cdot h = (q \cdot h, \rho(h)^{-1} g), \quad q \in Q,\ g \in G,\ h \in H.$$

(ii) Let $P \to M$ be a principal G-bundle. Then a *reduction to* H is a pair (Q, θ) consisting of a principal H-bundle $Q \to M$ and an isomorphism

$$\begin{array}{ccc} Q_\rho & \xrightarrow{\theta} & P \\ & \searrow & \downarrow \\ & & M \end{array}$$

of principal G-bundles.

8.8 Spin Structure

Let $V \to M$ be a real vector bundle of rank n with a metric. A spin structure on V is a reduction of the structure group of the orthonormal frame bundle $\mathcal{B}_O(V) \to M$ along $\rho : \mathrm{Spin}(n) \to O(n)$.
Here ρ is the projection $\mathrm{Spin}(n) \to SO(n)$ followed by the inclusion $SO(n) \to O(n)$. So the reduction can be thought of in two steps: an orientation followed by a lift to the double cover.

Let $\rho : H \to G$ be a double cover of the Lie group G. We have in mind $G = SO(n)$ and $H = \mathrm{Spin}(n)$. Let $P \to M$ be a principal G-bundle and (Q, θ) a reduction along ρ to a principal H-bundle.

Given a tangent bundle TM over M, there is an associated principle $\mathrm{GL}(n, \mathbb{R})$-bundle, which is the frame bundle.

$$\begin{array}{ccc} \mathrm{GL}(n\mathbb{R}) & \hookrightarrow & \mathrm{Fr}(M) \\ & & \downarrow \\ & & M \end{array}$$

Again we can get a $O(n)$-bundle from the frame bundle using the deformation retraction.

$$\begin{array}{ccc} O(n) & \hookrightarrow & \mathrm{Fr}_o(M) \\ & & \downarrow \\ & & M \end{array}$$

Hence by

$$[M, BO(n)] \overset{1:1}{\longleftrightarrow} \{\text{isomorphism classes of } O(n)\text{-bundles on } M\}$$

the orthogonal frame bundle $\mathrm{Fr}_o(M)$ determines a map $M \to BO(n)$ up to homotopy.

The inclusion map

$$SO(n) \hookrightarrow O(n)$$

induces a map between the classifying spaces

$$BSO(n) \to BO(n)$$

and an orientation is a lift

$$\begin{array}{ccc} & & BSO(n) \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \\ M & \xrightarrow{f} & BO(n) \end{array}$$

so the diagram commutes.

The existence of such a lift is obstructed by $w_1(M)$.

The covering map

$$P : \mathrm{Spin}(n) \to SO(n)$$

induces a map

$$B\mathrm{Spin}(n) \to BSO(n)$$

and a spin structure is a lift

$$\begin{array}{ccc} & & B\mathrm{Spin}(n) \\ & \overset{\tilde{\tilde{f}}}{\nearrow} & \downarrow \\ M & \xrightarrow{\tilde{f}} & BSO(n) \\ & \underset{f}{\searrow} & \downarrow \\ & & BO(n) \end{array}$$

making all diagrams commute. The existence such a lift is obstructed by $w_2(M)$, and the following session will explain why.

8.8.1 Cech Cocycle

Let (M^n, g) be an n-dimensional, oriented Riemannian manifold. In other words, the tangent bundle TM admits an $\mathrm{SO}(n)$ structure so that it can be defined by an open cover (U_α), and transition maps

$$g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{SO}(n)$$

satisfying the cocycle condition.

Note: by the cocycle condition we mean:

(a) $g_{\alpha\alpha} = 1_F$

(b) $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1_F \quad \text{over} \quad U_\alpha \cap U_\beta \cap U_\gamma.$

The manifold is said to be *spinnable* if there exist smooth maps

$$\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Spin}(n)$$

satisfying the cocycle condition, and such that

$$\rho(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta} \quad \forall \alpha, \beta,$$

where $\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ denotes the canonical double cover. The collection $\tilde{g}_{\alpha\beta}$ as above is called a *spin structure*. A pair (manifold, spin structure) is called a *spin manifold*.

Not all manifolds are spinnable. To understand what can go wrong, let us start with a trivializing cover $U = (U_\alpha)$ for TM, with transition maps $g_{\alpha\beta}$, and such that all the multiple intersections $U_{\alpha\beta\cdots\gamma}$ are contractible.

Since each of the overlaps $U_{\alpha\beta}$ is contractible, each map $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{SO}(n)$ admits at least one lift

$$\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Spin}(n).$$

From the equality $\rho(\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}) = g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$, we deduce

$$\epsilon_{\alpha\beta\gamma} = \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} \in \ker \rho = \mathbb{Z}_2.$$

Thus, any lift of the gluing data $g_{\alpha\beta}$ to $\mathrm{Spin}(n)$ produces a degree 2 Čech cochain of the trivial sheaf $\mathbb{Z}_2$, namely the 2-cochain

$$(\epsilon_\bullet) : U_{\alpha\beta\gamma} \mapsto \epsilon_{\alpha\beta\gamma}.$$

Note that for any $\alpha, \beta, \gamma, \delta$ such that $U_{\alpha\beta\gamma\delta} \neq \emptyset$, we have

$$\epsilon_{\beta\gamma\delta} - \epsilon_{\alpha\gamma\delta} + \epsilon_{\alpha\beta\delta} - \epsilon_{\alpha\beta\gamma} = 0 \in \mathbb{Z}_2.$$

In other words, $\epsilon_\bullet$ defines a Čech 2-cocycle, and thus defines an element in the Čech cohomology group $\check{H}^2(M, \mathbb{Z}_2)$.

It is not difficult to see that this element is independent of the various choices: the cover U, the gluing data $g_{\alpha\beta}$, and the lifts $\tilde{g}_{\alpha\beta}$. This element is intrinsic to the tangent bundle TM. It is called the second Stiefel-Whitney class of M, and it is denoted by $w_2(M)$. An oriented Riemannian manifold M admits a spin structure if and only if $w_2(M) = 0$.

8.9 Exercises

1. Show Spin(n) is the double cover of $SO(n)$ by writing out the covering map.

2. Verify Spin(4) $\cong SU(2) \times SU(2)$.

3. Find the representative for $w_1(M) = [M_{\text{orientation}}]$ in $H^1(M, \mathbb{Z}/2)$ directly using Čech cocycles built from transition functions.

9 Dirac Operators and Indices

n preceding talks we have seen spin-$\frac{1}{2}$ systems and their non-relativistic formulation due to Pauli. Moreover, we have seen the symmetry group, $\mathrm{Spin}(n)$, relevant to these systems. In the last lecture, we began a discussion of how to pass from a local (coordinate) description of spinors to a more global description on a spacetime with (potentially) non-trivial topology.

In this lecture, we complete our description of spinors on smooth spacetimes. Moreover, we sketch some mathematical consequences/constructions based on spinors and the Dirac equation. Of note, index theory is useful in the analysis of moduli spaces of solutions to the Yang–Mills equations; this topic will be introduced in a later lecture.

9.1 The Dirac Equation

In 1928 Paul Dirac found a relativistic extension of Pauli's work on spinors. Dirac began by pondering the Klein–Gordon equation $(\partial^2 + m^2)\phi = 0$ and asking if there was a relativistic wave equation which was first order in the spacetime derivatives.[1] Inspired by factoring the sum of squares over the complex numbers we may consider an equation of the form

$$(i\gamma^\mu \partial_\mu - m)\phi = 0, \tag{7}$$

where γ^μ are some constants. But what kind of constants? If the γ^μ were scalars, then the equation would fail to be Lorentz invariant.

Multiplying the Dirac equation by $(i\gamma^\mu\partial_\mu + m)$, i.e., the other half of our putative factorization, we obtain

$$-(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\phi = 0.$$

As partial derivatives commute, we can rewrite this equation as

$$\left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu + m^2\right)\phi = 0,$$

where $\{-,-\}$ is the anti-commutator bracket. If we want this equation (and hence equation 7) to describe a free particle of mass m, we should recover the Klein–Gordon equation. Indeed, we do, provided that

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{8}$$

where $\eta^{\mu\nu}$ is the matrix corresponding to the standard metric on Minkowski space. Expanding equation 8, we have

$$(\gamma^0)^2 = 1;\ (\gamma^j)^2 = -1,\ j \neq 0;\ \text{ and }\ \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu, \mu \neq \nu.$$

We can recognize that what $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ generate is a four dimensional Clifford algebra: it's the Clifford algebra $\mathrm{Cl}_{1,3}$. (It is standard that the γ-matrices can be expressed in terms of Pauli's spin matrices.)

[1] Here we follow the derivation from Zee's *QFT in a Nutshell*, [Zee03].

Now, let V be any complex vector space equipped with a representation of $\mathrm{Cl}_{1,3}$, we can consider the Dirac equation for V valued functions, i.e., $\phi \in C^\infty(\mathbb{R}^{1,3}, V)$. Moreover, we can add various interactions or potential terms. As an example, if we fix a electromagnetic potential, then we have the corresponding Dirac equation:

$$(i\gamma^\mu(\partial_\mu - ieA_\mu) - m)\,\phi = 0. \tag{9}$$

9.1.1 Summary of (Local) Structure

Let us briefly summarize the structures we utilized in the preceding section.

1. A pseudo-Riemannian manifold: $\mathbb{R}^{1,3} = (\mathbb{R}^4, \eta)$.
2. The Clifford algebra built from our pseudo-Riemannian manifold: $\mathrm{Cl}_{1,3}$ (or its complexification $\mathrm{Cl}_{1,3} \otimes \mathbb{C}$).
3. A vector space V equipped with a representation of $\mathrm{Cl}_{1,3}$.
4. The partial/covariant derivatives of V-valued functions on our manifold.

9.2 Globalizing Our Work: Dirac Operators and Spinor Bundles

In this section, we will extend the constructions of the preceding section to general (pseudo-)Riemannian manifolds. We will restrict to the Riemannian case, though the Lorentzian and general psuedo-Riemmanian case can also be considered. A standard reference is Lawson and Michelson's *Spin Geometry* [LM89], see also Chapter 11 of [Nic21] or [BGV04]. (For the psuedo-Riemannian manifolds a reference is Helga Baum's text [Bau81].)

Another note, we will work with spin structures and the group $\mathrm{Spin}(n)$. In many physical applications, the relevant structure is that of a Spin^c structure (or even $\mathrm{Pin}^\pm$ structure), the case of Spin^c can be treated similarly and is discussed in Appendix D of [LM89]. It is a theorem that any bundle equipped with a spin structure has a canonical Spin^c structure. There is a slight difference, however, as any almost complex manifold admits a Spin^c structure while not necessarily admitting a spin structure; the classic example of such a manifold is $\mathbb{C}P^2$.

9.2.1 Dirac Bundles

Let (M, g) be a closed (compact without boundary), connected, oriented Riemannian manifold of dimension n. As in the previous lecture, let $P_{SO}(X)$ denote the principal $SO(n)$-bundle of orthonormal tangent frames.

Consider $\mathbb{R}^n$ with its standard inner product and the corresponding Clifford algebra $\mathrm{Cl}_n = \mathrm{Cl}(\mathbb{R}^n)$. The defining action of $SO(n)$ on $\mathbb{R}^n$ naturally extends to a representation $\rho\colon SO(n) \to \mathrm{Cl}_n$. We can then form the associated bundle, which we call the *Clifford bundle of* M,

$$\mathrm{Cl}(M) := P_{SO}(M) \times_\rho \mathrm{Cl}_n.$$

The bundle $\mathrm{Cl}(M)$ is a bundle of Clifford algebras, i.e., each fiber has the structure of a Clifford algebra. One can check that another construction of $\mathrm{Cl}(M)$ is to apply the Clifford algebra construction to each tangent space using the Riemannian metric as the quadratic form.

Next, let $S \to M$ be a vector bundle of left modules over $\mathrm{Cl}(M)$, i.e., for each $p \in M$, the fiber S_p is a left module for the Clifford algebra $\mathrm{Cl}(M)_p$, we will denote this action, "Clifford multiplication", by the symbol $\cdot$, e.g., $v \cdot \sigma$. Further, assume that S has a Riemannian metric and is equipped with a metric connection ∇, e.g., the Levi-Civita connection. We then define the *Dirac operator* to be

$$D\colon \Gamma(M, S) \to \Gamma(M, S), \quad D\sigma := \sum_{j=1}^{n} e_j \cdot \nabla_{e_j} \sigma,$$

at $p \in M$, for $\{e_1, \ldots, e_n\}$ an orthonormal basis of T_pM. Let us note a few properties of D, which we recall from Section III.5 of [LM89].

Proposition 9.2.1. *Let $D\colon \Gamma(M, S) \to \Gamma(M, S)$ be a Dirac operator.*

1. *The operator D is a first-order differential operator which is elliptic.*
2. *The* Dirac Laplacian *D^2 is a generalized Laplacian.*

While we won't recall the symbol calculus, nor develop the theory of elliptic operators, we will use some results as necessary.

Note that there is an inner product on sections of S given by integrating the fiberwise inner product $\langle -, - \rangle$:

$$(\sigma_1, \sigma_2) := \int_M \langle \sigma_1, \sigma_2 \rangle, \quad \sigma_1, \sigma_2 \in \Gamma(M, S).$$

It is desirable that the Dirac operator D is formally self-adjoint, i.e., for all $\sigma_1, \sigma_2 \in \Gamma(M, S)$,

$$(D\sigma_1, \sigma_2) = (\sigma_1, D\sigma_2).$$

This will follow (Proposition 5.3 of Chapter III [LM89]) provided that

1. Clifford multiplication is orthogonal, i.e., at each $p \in M$, and for each unit vector $e \in T_pM$,

   $$\langle e \cdot \sigma_1, e \cdot \sigma_2 \rangle = \langle \sigma_1, \sigma_2 \rangle,$$

   for all $\sigma_1, \sigma_2 \in S_p$, and

2. The covariant derivative on S is a module derivation, i.e., for all $\sigma \in \Gamma(M, S)$ and $\psi \in \Gamma(M, \mathrm{Cl}(M))$,

   $$\nabla(\psi \cdot \sigma) = (\nabla' \psi) \cdot \sigma + \psi \cdot (\nabla \sigma),$$

   where ∇' is the covariant derivative on the Clifford bundle associated to the Levi-Civita connection corresponding to the Riemannian metric on M.

Definition 9.2.2. A *Dirac bundle* on a Riemannian manifold (M, g) is a bundle S of left modules over the Clifford bundle $\mathrm{Cl}(M)$ which is equipped with a metric and metric connection satisfying properties (1) and (2).

For both physics and geometry, the most relevant Dirac bundles are the so-called *spinor bundles*, so we describe these next.

9.2.2 Spinor Bundles in General

We now add to our manifold (M, g) the assumption that it is equipped with a spin structure: $P_{\mathrm{Spin}}(M) \to P_{SO}(M)$. Let V be a left module for Cl_n equipped with an inner product, such that Clifford multiplication is orthogonal. As $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^0$, there is an (oriented) orthogonal representation $\mu\colon \mathrm{Spin}(n) \to SO(V)$. The (real) *spinor bundle* determined by V is the associated bundle

$$S := P_{\mathrm{Spin}}(M) \times_\mu V.$$

We use the same letter as in preceding subsection as this spinor bundle is naturally a Dirac bundle (and the word spinor starts with the letter 's' in English (and French)). Indeed, the (metric) connection on TM determines a connection on $P_{SO}(M)$ and this lifts to a connection on $P_{\mathrm{Spin}}(M)$ which in turns induces a connection on the spinor bundle S. That S is a module bundle for the Clifford bundle, $\mathrm{Cl}(M)$, is a standard check (Proposition III.3.8 of [LM89]). Moreover, the connection is a derivation with respect to Clifford multiplication (Proposition III.4.11 of [LM89]).

We also have complex spinor bundles, where V is a module for $\mathrm{Cl}_n \otimes \mathbb{C}$ and graded versions of both. Summarizing, we have the following.

Theorem 9.2.3. *Let $S \to X$ be a spinor bundle (real, complex, graded or ungraded). Then, S is a Dirac bundle, and in particular is equipped with a canonical Dirac operator.*

The canonical Dirac operator on a spinor bundle was first constructed by Atiyah and Singer and is sometimes called the *Atiyah–Singer operator*.

9.2.3 Examples of Spinor Bundles

As a first example of a spinor bundle, we can consider Cl_n as a module over itself. Let ℓ denote the module structure by left multiplication. We then form the *Clifford linear spinor bundle*

$$\mathrm{Cl}_{\mathrm{Spin}}(M) := P_{\mathrm{Spin}}(M) \times_\ell \mathrm{Cl}_n.$$

This bundle is naturally $\mathbb{Z}/2$-graded and is contains a lot of geometric information about the spin manifold M.

The representation theory Clifford algebras is tame and every spinor bundle is completely reducible, i.e., any spinor bundle (real, complex, graded, or ungraded) can be decomposed into a sum of irreducible ones.

The classification of irreducible representations is discussed in Section I.5 of [LM89], later (Section II.3) the same authors summarize the number of irreducible representations as follows.

Dimension $n \pmod 8$	Real Ungraded	Complex Ungraded	Real Graded	Complex Graded
1	1	2	1	1
2	1	1	1	2
3	2	2	1	1
4	1	1	2	2
5	1	2	1	1
6	1	1	1	2
7	2	2	1	1
8	1	1	2	2

A few comments are in order. First, the number of real irreducible representations is 8-fold periodic; while the number of complex representations has periodicity 2. This periodicity is an incarnation of *Bott Periodicity*; a beautiful topic for another time. Secondly, spin/Clifford representations are fundamental for the description of *supersymmetric* field theories. While we won't address supersymmetry in these notes, we note that the capital 'N' notation, e.g., $N = 2$ supersymmetry, approximately corresponds (depending on conventions) to the number of copies of irreducible representations we sum.

9.3 Fredholm Operators and Indices

In preparation for the next section, let us recall some elementary results from functional analysis/operator theory.

Definition 9.3.1. Let $T\colon X \to Y$ be a bounded linear operator between Banach spaces. The operator T is *Fredholm* if

1. The kernel of T, $\ker T$, is finite dimensional, and
2. The cokernel of T, $\operatorname{coker} T$, is finite dimensional, and
3. The image/range of T is closed.

One characterization of Fredholm operators is that they are exactly those bounded linear operators which are invertible up to compact operators (see Chapter III Lemma 5.1 of [LM89]).

Definition 9.3.2. Let $T\colon X \to Y$ be a Fredholm operator. The *index* of T is the integer given by

$$\operatorname{Ind} T = \dim \ker T - \dim \operatorname{coker} T.$$

The notion of (Fredholm) index is really only interesting in infinite dimensions. Indeed, if $T\colon X \to Y$ is Fredholm between finite dimensional spaces, then $\operatorname{Ind} T = \dim X - \dim Y$ by the rank-nullity theorem.

Example 9.3.3. Consider the "shift right" operator $R\colon \ell^2 \to \ell^2$, where

$$R(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

Clearly, R has trivial kernel and its cokernel is one-dimensional, hence $\operatorname{Ind} R = -1$.

The following is simple, but very useful in practice.

Lemma 9.3.4. *Let $T\colon X \to Y$ be Fredholm and $T^*\colon Y \to X$ a (formal) adjoint, then*

$$\operatorname{Ind} T = \dim\ker T - \dim\ker T^*.$$

Example 9.3.5. Consider the circle, $S^1 \cong \mathbb{R}/\mathbb{Z}$, and the operator $\frac{d}{dx}\colon L^2_1(S^1) \to L^2(S^1)$ between Sobolev spaces of complex valued functions. One can compute that the kernel and cokernel are both one dimensional, so the index is zero. Moreover, one could consider the modified operator $\frac{d}{dx} - \lambda$ for $\lambda \in \mathbb{C}$ of unit norm; the adjoint of this operator is simply $\frac{d}{dx} - \overline{\lambda}$ and it follows that the index is still zero.

The following is known as the *topological invariance of the index*, see Chapter III Section 7 of [LM89] for a proof in the setting of Hilbert Spaces.

Theorem 9.3.6. *Let $\mathcal{B}(X,Y)$ be the space of bounded linear operators. The index map*

$$\operatorname{Ind}\colon \mathcal{B}(X,Y) \to \mathbb{Z}$$

is locally constant. Moreover, if X and Y are separable Hilbert spaces, then the index induces a bijection

$$\pi_0\mathcal{B}(X,Y) \cong \mathbb{Z}.$$

9.4 The Atiyah–Singer Index Theorem

In this section we want to recall the Atiyah–Singer Index Theorem in its cohomological form and work through some elementary examples. Atiyah and Singer proved their theorem(s) in the early 1960's building on critical work of Borel, Bott, Chern, Gelfand, Grothendieck, Hirzebruch, Thom, and others; it was one of the mathematical high points of the twentieth century. Our overview will be very brief; for an elegant and more elaborative account see [Fre21].

To begin, in order to compute an (analytic) index we need a Fredholm operator. Fortunately, any elliptic operator on a manifold admits a *Fredholm* extension. The idea is to impose regularity on the space of sections we consider, so for a Dirac bundle $S \to M$ with Dirac operator D, we take the Sobolev space[2] of sections $W^{1,2}(M,S)$. Next, there is an extension of the operator D to a Fredholm operator

[2] Sobolev spaces are a central tool in much of functional and geometric analysis. Sobolev spaces on Riemannian manifolds are discussed in Taylor's three volume set [Tay97] or Section III.2 of [LM89].

$D\colon W^{1,2}(M,S) \to L^2(M,S)$. For the precise statements and proofs, see Theorems II.5.5 and II.5.7 of [LM89].

With the help of its Fredholm extension, our Dirac operator D—or any elliptic operator on M—now has a well defined index $\operatorname{Ind} D \in \mathbb{Z}$. The Index Theorem now relates this analytic index to the integral of a local quantity over our (compact, oriented) manifold M.

Theorem 9.4.1 (Atiyah–Singer (Cohomological Version))**.** *Let P be an elliptic operator on the compact oriented manifold M of dimension n. Then*

$$\operatorname{Ind} P = (-1)^{\frac{n(n+1)}{2}} \int_M \operatorname{ch} P \cdot \widehat{A}(M)^2.$$

The integrand appearing in the Index Theorem is a product of two (inhomogeneous) cohomology classes on our manifold M: the *Chern class of the symbol bundle* associated to the operator P and the $\widehat{A}$-class of the manifold itself. An introduction to characteristic classes such as these will take place in the next lecture.

The Index Theorem is useful in at least three ways:

1. As the right hand side is a topological quantity it is invariant under homotopy, thus providing an answer to the question "Why is the analytic index locally constant?"

2. It turns out that any elliptic operator can be deformed to a "twisted" Dirac operator and for such operators the right hand side of the Index Theorem is often computable.

3. The left hand side of the Index Theorem is an integer, so so is the right hand side. Thus, integrability and divisibility results for certain topological invariants of manifolds can be proven using the Index Theorem, e.g., Hirzebruch's Signature Theorem.

9.4.1 Fundamental Examples

We now illustrate two specific examples of indices of Dirac operators. In both cases we will consider a graded bundle $S = S^0 \oplus S^1$ and a Dirac operator of the form

$$D = \begin{pmatrix} 0 & D^1 \\ D^0 & 0 \end{pmatrix} : S^0 \oplus S^1 \to S^0 \oplus S^1.$$

If D is self-adjoint, then D^0 and D^1 are adjoints of each other. Therefore,

$$\operatorname{Ind} D^0 = \dim\ker D^0 - \dim\ker D^1.$$

Example 9.4.2. Let (M,g) be a compact, oriented Riemannian manifold and consider the Clifford bundle itself $\operatorname{Cl}(M)$. This bundle is graded, $\operatorname{Cl}(M) = \operatorname{Cl}^0(M) \oplus \operatorname{Cl}^1(M)$. Moreover, in the previous lecture we saw that for a (quadratic) vector space (V,q), $\operatorname{Cl}(V,q) \cong \Lambda^*(V)$ as vector spaces. This fiberwise identification assembles to a global identification at the level of sections: $\Gamma(M,\operatorname{Cl}(M)) \cong \Omega^*(M)$. We then consider half of the "off diagonal" part of the (graded) Dirac operator and have the identification

$$D^0\colon \Gamma(M,\operatorname{Cl}^0(M)) \to \Gamma(M,\operatorname{Cl}^1(M)) \quad \text{is} \quad d+d^*\colon \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M),$$

where d is the de Rham derivative and $d^* = *d*$ its adjoint. So then

$$\operatorname{Ind} D^0 = \dim H^{\text{even}}_{dR}(M) - \dim H^{\text{odd}}_{dR}(M) = \chi(M)$$

the Euler characteristic of the manifold M.

Example 9.4.3. Let M be compact Riemannian spin manifold of dimension $4k$ and let $\$_{\mathbb{C}}$ be the unique irreducible (ungraded) complex spinor bundle on M. We now induce a grading on $\$_{\mathbb{C}}$ following Section II.6 of [LM89]. By choosing a positively oriented orthonormal tangent frame $(e_1, \ldots, e_{4k})$ define the *complex volume element*, $\omega_{\mathbb{C}} \in \Gamma(M, \mathrm{Cl}(M) \otimes \mathbb{C})$ by

$$\omega_{\mathbb{C}} := i^{2k} e_1 \cdot e_2 \cdots e_{4k}.$$

Multiplication by $\omega_{\mathbb{C}}$ splits our spinor bundle into $+1$ and -1 eigenbundles:

$$\$_{\mathbb{C}} = \$^+_{\mathbb{C}} \oplus \$^-_{\mathbb{C}} \quad \text{and} \quad D^+ \colon \Gamma(M, \$^+_{\mathbb{C}}) \to \Gamma(M, \$^-_{\mathbb{C}}).$$

The Index Theorem then computes the index of D^+: $\operatorname{Ind} D^+ = \widehat{A}(M)$. An immediate consequence is that $\widehat{A}(M) \in \mathbb{Z}$. While that conclusion might not seem surprising note that $\widehat{A}(M)$ is given in terms of Pontryagin classes (more characteristic classes!) with the first few terms

$$\widehat{A} = 1 - \frac{1}{24} p_1 + \frac{1}{5760}(-4p_2 + 7p_1^2) + \cdots,$$

so there is no reason to expect $\widehat{A}(M)$ is an integer. Yet, with the help of the Index Theorem, we have shown that for a spin manifold $\widehat{A}(M) \in \mathbb{Z}$. For our favorite non-spin manifold $\mathbb{C}P^2$, one can compute that $\widehat{A}(\mathbb{C}P^2) = -\frac{1}{8}$.

9.5 Exercises

1. Check that if the γ^μ in Dirac's equation (7) were simply scalars, then the equation fails to be Lorentz invariant.

2. (This exercise follows Section II.1 of [Zee03].) Let us consider complex conjugating the EM Dirac equation 9.

   (a) To begin, show that the matrices $\{-(\gamma^\mu)^*\}$ also generate the Clifford algebra $\mathrm{Cl}_{1,3}$.

   (b) Hence, there is a matrix C, called the *charge conjugation matrix*, such that

   $$-(\gamma^\mu)^* = (C\gamma^0)^{-1} \gamma^\mu (C\gamma^0).$$

   Factoring out a copy of γ^0 from the change of basis matrix is a convention, and far be it from us to break this convention. Find C.

(c) Define $\phi_c := C\gamma^0\phi^*$. Show that if ϕ satisfies equation 9, then

$$(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\,\phi_c = 0.$$

Hence, corresponding to every solution of the massive, charged Dirac equation, there is a corresponding solution with the same mass but opposite charge. For the electron, the corresponding particle is called the *positron*. One should perhaps note that the (early) historical understanding of the positron was through a dialogue between Dirac, Oppenheimer, Weyl, and others.

3. Let $L\colon \ell^2 \to \ell^2$ be the left shift operator. Compute $\operatorname{Ind} L$.

4. Let $T\colon X \to Y$ and $S\colon Y \to Z$ be Fredholm operators. Prove that $S \circ T$ is Fredholm and moreover

$$\operatorname{Ind} S \circ T = \operatorname{Ind} S + \operatorname{Ind} T.$$

Note that you now know how to realize any integer as the index of a Fredholm operator.

5. (This exercise is Theorem 3.50 of [BGV04].) In this exercise you will prove the McKean–Singer Formula. Let Δ be a generalized Laplacian acting on a $\mathbb{Z}/2$-graded Banach space. For $\lambda \in \mathbb{R}$, let $n_\lambda^\pm$ be the dimension of the λ-eigenspace $\mathcal{H}^\pm$. We can consider the corresponding heat operator, $e^{-t\Delta}$, and define its *supertrace* as

$$\operatorname{Str}\left(e^{-t\Delta}\right) = \sum_{\lambda \geq 0} \left(n_\lambda^+ - n_\lambda^-\right) e^{-t\lambda}.$$

That this operation is well defined uses the spectral theorem/calculus for the operator Δ.

Now, let $(E^\pm \to M, D)$ be a $\mathbb{Z}/2$-graded Dirac bundle over a compact Riemannian manifold M. Then D^2 is a generalized Laplacian acting on a Sobolev extension of $\Gamma(M, E^\pm)$. Using the fact that D commutes with D^2 prove that

$$\operatorname{Ind} D = \operatorname{Str} e^{-tD^2}.$$

10 Lie Algebras and Characteristic Classes

10.1 Some Representation Theory for Lie Algebras

Given a Lie group G, we can consider representations of the group to study the structure of G. In particular, a representation of G is a Lie group homomorphism

$$\rho : G \to GL(V)$$

where V is a complex or real vector space. Our goal in this section is to study the representation of a Lie algebra and what that structure provides for us.

10.1.1 Definitions for representation of a Lie algebra

We first begin with some definitions of a Lie algebra representation.

Definition 10.1.1.
Let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$. Let $(W, [\cdot,\cdot])$ be a Lie algebra where W is a $\mathbb{F}$ vector space.

- A representation of W on a $\mathbb{F}$ vector space V is a Lie algebra homomorphism

$$\phi : W \to \mathrm{End}_{\mathbb{F}-\mathrm{linear}}(V) := \mathrm{End}(V)$$

- A representation ϕ of W is faithful if ϕ is injective.
- A representation ϕ on V is irreducible if there are no proper or trivial vector subspaces $U \subset V$ such that $W \cdot U \subset U$.
- A representation ϕ on V is reducible if it is not irreducible.

For a Lie algebra $\mathfrak{g}$ of a Lie group G, we can come up with representations using the representations of G. In particular, if $\rho : G \to GL(V)$ is a representation of G, then the differential $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$ is a Lie algebra homomorphism as the differential of any Lie group homomorphism is a Lie algebra homomorphism.

10.1.2 Adjoint representation

If G is a Lie group with Lie algebra $\mathfrak{g}$, then we have a diffeomorphism

$$r_g \circ l_{g^{-1}} : G \to G \qquad h \mapsto ghg^{-1}$$

which gives rise to a vector space automorphism of $\mathfrak{g}$:

$$\mathrm{Ad}_g := (r_g \circ l_{g^{-1}})_{*,e} : \mathfrak{g} \to \mathfrak{g} \qquad \mathrm{Ad}_g(X) = gXg^{-1}$$

As we vary over G, we obtain a Lie group homomorphism

$$\text{Ad} : G \to GL(\mathfrak{g}) \qquad g \mapsto \text{Ad}_g$$

Thus, we have a representation of G on $\mathfrak{g}$ given by Ad called the adjoint representation. Taking the differential $\text{ad} := \text{Ad}_*$, we obtain a representation of $\mathfrak{g}$ also called the adjoint representation.

10.1.3 Killing forms

Suppose $(V, [\cdot, \cdot])$ is a Lie algebra. Since V is a vector space, then we can consider bilinear forms on V. We can define an adjoint representation on V given by

$$\text{ad} : V \to \text{End}(V) \qquad X \mapsto (Y \mapsto [X, Y])$$

Since ad_X is a linear map on V, we can write it down as a matrix in the case V is finite dimensional. Hence it makes sense to consider the trace of such maps which will give us a number in the ground field. Therefore, we can obtain a map

$$K_V : V \times V \to \mathbb{F} \qquad K_V(X, Y) = \text{tr}(\text{ad}_X \circ \text{ad}_Y)$$

called the Killing form.

Proposition 10.1.2.
Let $(V, [\cdot, \cdot])$ be a finite dimensional Lie algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let K_V be the killing form on V.

1. *K_V is bilinear and symmetric.*

2. *For any automorphism $\sigma : V \to V$, $K_V(\sigma(X), \sigma(Y)) = K_V(X, Y)$.*

10.1.4 Simple and semi-simple Lie algebras

We now consider the condition in the Killing form of a Lie algebra is non-degenerate. It turns out non-degeneracy is linked with the notion of semi-simple.

Definition 10.1.3. Let $(V, [\cdot, \cdot])$ be a Lie algebra.

- Given $U, W \subset V$, define
$$[U, W] = \{\sum_{i=1}^{k} [X_i, Y_i] : X_i \in U, Y_i \in W\}$$

- An ideal in V is a vector subspace $U \subset V$ such that $[V, U] \subset U$.

- An ideal is commutative if the ideal is a commutative subalgebra of V.

- The Lie algebra V is simple if V is a non-commutative algebra and the only ideals in V are 0 and V.
- The Lie algebra V is semi-simple if V if every non-trivial ideal in V is non-commutative abelian.

Theorem 10.1.4 (Cartan's Criterion)**.**
A Lie algebra is semi-simple if and only if the Killing form is non-degenerate.

Theorem 10.1.5.
Every semi-simple Lie algebra can be written as the direct sum of simple Lie algebras which are pairwise orthogonal with respect to the Killing form.

For proofs of these two theorems, see chapter 2 in [Ham17].

10.1.5 Computing the Killing form for a matrix Lie group

Theorem 10.1.6.
Let $n \in \mathbb{N}$ with $n \geq 2$.

1. *The Killing form for $\mathfrak{gl}_n(\mathbb{R})$ is given by $K(X,Y) = 2ntr(XY) - 2tr(X)tr(Y)$.*
2. *The Killing form for $\mathfrak{sl}_n(\mathbb{R})$ is given by $K(X,Y) = 2ntr(XY)$.*
3. *The killing form for $\mathfrak{su}(u)$ is given by $K(X,Y) = 2ntr(XY)$.*
4. *The killing form for $\mathfrak{so}(n)$ is given by $K(X,Y) = (n-2)tr(XY)$.*

10.2 Chern-Weil Homomorphism for Principal Bundles

10.2.1 Overview

Consider the torus $\mathbb{T}^2$ with its standard topology. We can embed $\mathbb{T}^2$ into $\mathbb{R}^3$ via a surface revolution where one chart is given by

$$\psi : (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^3 \qquad \psi(x, \theta) = (f(x)\cos(\theta), f(x)\sin(\theta), g(x))$$

where $f, g : (0, 2\pi) \to \mathbb{R}^3$ are given by $f(x) = R + r\cos(x)$ and $g(x) = r\sin(x)$. Equipping $\mathbb{R}^3$ with its standard metric, we can pullback the metric to $\mathbb{T}^2$ and equipped $\mathbb{T}^2$ with the corresponding Levi-Civita connection. Using the affine connection and metric, one can compute the Gaussian curvature K of $\mathbb{T}^2$ to obtain

$$K = \frac{\cos(x)}{r(R + r\cos(x))}.$$

Due to Gauss' Theorem Ergegium, we know K is invariant under isometries of $\mathbb{T}^2$. Furthermore, note K takes on positive and negative values.

Since K was dependent on the choice of C^∞ structure and metric, it makes sense to consider the following question: Does there exists a C^∞ structure on $\mathbb{T}^2$ and Riemannian metric such that, when equipped with the Levi-Civita connection, K is strictly non-positive or non-negative? The following Theorem will help answer this question.

Theorem 10.2.1 (Gauss-Bonnet)**.**
Let M be a compact, oriented two dimensional Riemannian manifold. Then

$$\int_M K\,vol = 2\pi\chi(M)$$

where $\chi(M)$ is the Euler characteristic.

Since $\chi(\mathbb{T}^2) = 0$, then, from the Gauss-Bonnet Theorem, we know either $K = 0$ or K takes on both positive and negative values. Thus the answer to the question is no. Note that we were able to answer this question (which is rather challenging on its face) by simply computing a number that depended only on the topology. The goal of characteristic classes is to imitate this perspective: answer differential topology and differential geometry questions via topological obstructions that can be computed using cohomology.

10.2.2 Polynomials on a vector space

Let k be a field with characteristic zero. Let V be a k-vector space with basis $v_1, \ldots, v_n$ and dual basis $\alpha^1, \ldots, \alpha^n$. For notation, denote degree l homogeneous polynomials in n-variables over k as $k[x_1, \ldots, x_n]^l$. In the case of $l = 0$, $k[x_1, \ldots, x_n]^0 := k$. We can define for each $l \in \mathbb{N}_0$ a function

$$\mathrm{Sym}_l(V^\vee) \to k[x_1, \ldots, x_n]^l \qquad f \mapsto \sum_{I \in \{1,\ldots,n\}^{\times l}} f(v_{i_1}, \ldots, v_{i_l}) x^{i_1} \ldots x^{i_l} \tag{10}$$

As one can check in exercise 1, this map is a k-algebra isomorphism. We also have the injective k-algebra homomorphism

$$k[x_1, \ldots, x_n] \to \mathrm{Fun}(V, k) \qquad x_i \mapsto \alpha^i$$

Thus, we have an injective k-algebra homomorphism

$$\mathrm{Sym}_l(V^\vee) \to \mathrm{Fun}(V, k) \qquad f \mapsto (v \mapsto f(v, \ldots, v)) \tag{11}$$

Elements in the image of this k-algebra homomorphism are called homogeneous degree l polynomials on V.

10.2.3 Invariant polynomials on a Lie algebra

Recall, if G is a Lie group with Lie algebra $\mathfrak{g}$, then we have a diffeomorphism

$$r_g \circ l_{g^{-1}} : G \to G \qquad h \mapsto ghg^{-1}$$

which gives rise to a vector space automorphism of $\mathfrak{g}$:

$$\mathrm{Ad}_g := (r_g \circ l_{g^{-1}})_{*,e} : \mathfrak{g} \to \mathfrak{g} \qquad \mathrm{Ad}_g(X) = gXg^{-1}$$

As we vary over G, we obtain a Lie group homomorphism

$$\mathrm{Ad} : G \to \mathrm{End}_{Vect}(\mathfrak{g}) \qquad g \mapsto \mathrm{Ad}_g$$

Thus, we have a group action of G on $\mathfrak{g}$ given by Ad called the adjoint action. This adjoint action ascends to an action on $T(\mathfrak{g}^\vee)$ where we have the trivial action in degree zero, and we have in degree $k > 0$

$$\widetilde{\mathrm{Ad}} : G \times T^k(\mathfrak{g}^\vee) \to T^k(\mathfrak{g}^\vee) \qquad (g, f) \mapsto ((u_1, \dots, u_k) \mapsto f(\mathrm{Ad}_g u_1, \dots, \mathrm{Ad}_g u_k).$$

The adjoint action on the tensor algebra descends to an action on the symmetric algebra of $\mathfrak{g}^\vee$. The fixed points of this action are called Ad-G invariant polynomials.

We can also define an Ad-G action on $\mathrm{Fun}(\mathfrak{g}, \mathbb{R})$ given by

$$G \times \mathrm{Fun}(\mathfrak{g}, \mathbb{R}) \to \mathrm{Fun}(\mathfrak{g}, \mathbb{R}) \qquad (g, f) \mapsto (X \mapsto f(gXg^{-1}))$$

The fixed points of this action are also called Ad-G invariant polynomials. As one can check in Exercise 3, the k-algebra homomorphism in (11) is in fact a G-equivariant isomorphism. Thus a homogeneous polynomial is a fixed point if and only if the corresponding element in the symmetric algebra is a fixed point.

Example 10.2.2 (Examples of Ad-invariant polynomials)**.**
Let $k = \mathbb{R}$ or $\mathbb{C}$. Consider the Lie group $G = GL_n(k)$ whose Lie algebra is $\mathfrak{gl}_n(k) = k^{n\times n}$. Consider the function

$$\det : k \times k^{n\times n} \to k \qquad (\lambda, X) \mapsto \det(\lambda I + X) = \sum_{k=0}^{n} f_k(X)\lambda^{n-k}$$

where $f_k(X)$ is a homogeneous polynomial in X valued in k with integer coefficients. Since the determinant is invariant under conjugation of elements in $GL_n(k)$, then $f_0, \dots, f_n$ are all Ad-$GL_n(k)$ invariant homogeneous polynomials. It turns out that these polynomials generate the set of Ad-$GL_n(k)$ homogeneous polynomials (see appendix B in [Tu17]).

10.2.4 The Chern-Weil homomorphism for principal bundles

To see how one constructs a characteristic class for a principal bundles, we give a non-rigorous overview of the process. The interested reader can find more details in the last section of [Tu17] . Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $\pi : P \to M$ is a principal G-bundle. Recall, we are able to pick an Erhesmann connection ω for P which has a curvature form $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$. If $e_1, \ldots, e_n$ is a basis for $\mathfrak{g}$ with dual basis $\alpha^1, \ldots, \alpha^n$, then we can write

$$\Omega = \sum_{i=1}^{n} \Omega^i \otimes e_i$$

where $\Omega^1, \ldots, \Omega^n \in \Omega^2(P)$. Given a homogeneous Ad-G invariant polynomial $f : \mathfrak{g} \to \mathbb{R}$ of degree k, we can write $f = \sum_{I \in \{1,\ldots,n\}^{\times k}} a_I \alpha^{i_1} \ldots \alpha^{i_k}$. Now consider the differential $2k$-form

$$f(\Omega) := \sum_{I \in \{1,\ldots,n\}^{\times k}} a_I \Omega^{i_1} \wedge \ldots \wedge \Omega^{i_k}$$

We have the following facts about $f(\Omega)$:

1. There exists $\zeta \in \Omega^{2k}(M)$ such that $\pi^*(\zeta) = f(\Omega)$.
2. ζ is a closed form.
3. $[\zeta] \in H^{2k}(M)$, called the characteristic class of P associated to f, is independent of ω.

Thus, we obtain a ring homomorphism from homogeneous Ad-G invariant polynomials on $\mathfrak{g}$ and $H^*(M)$ where $f \mapsto [\zeta]$. This homomorphism is called the Chern-Weil homomorphism.

10.2.5 Pontrjagin and Chern classes

Let $k = \mathbb{R}$ or $\mathbb{C}$. Consider a principal $GL_n(k)$ bundle. Then the Ad-$GL_n(k)$ invariant polynomials are generated by the the coefficients for the determinant function $f_0, \ldots, f_n$. However, we will consider a slight adjustment to these where we will take f_j to be $f_j(\frac{i}{2\pi}X)$ instead of $f_j(X)$. Since these polynomials generate the invariant polynomials, it make sense to consider the characteristic classes associated to these polynomials. In the case $k = \mathbb{R}$, the corresponding classes are called the Pontrjagin classes and are denoted as $p_0, \ldots, p_n$ (which correspond to $f_0, \ldots, f_n$, respectively). In the case $k = \mathbb{C}$, the corresponding classes are called the Chern classes and are denotes as $c_0, \ldots, c_n$ (which correspond to $f_0, \ldots, f_n$, respectively).

10.3 Exercises

1. Verify (10) is a k-algebra isomorphism.
2. Verify the Ad-G action on $T(\mathfrak{g}^\vee)$ is well defined and that this action does in fact descend to an action on the symmetric algebra.
3. Verify (11) is a G-equivariant isomorphism onto its image.
4. Prove proposition 10.1.2

11 A Smattering of Gauge Theories

This lecture will proceed in two parts: first, we will introduce a number of different gauge theories and some of their applications, we will then begin to describe Yang–Mills Theory in a bit greater detail. We will revisit Yang–Mills Theory in a later lecture when we discuss some of the its striking results to 4-manifold topology as described by Donaldson.

Throughout we will comment little, if at all, on the physical origins of various theories. Further, we won't discuss the details of quantization for our theories, only noting relevant obstructions when we discuss any consequences of the existence of a quantization.

11.1 A Bit More on Bundles and Connections

Except for the first paragraph where we fix come conventions/notation, the advanced reader can skip to the next section.

Throughout, let G be a Lie group with Lie algebra $\mathfrak{g}$, M a smooth manifold (typically closed) and $P \to M$ a principal G-bundle. We will also fix a connection A on P.

Let $\rho\colon G \to GL(V)$ be a representation. We form the *vector bundle associated to* ρ, $P \times_\rho V \to M$, with total space given by

$$P \times_\rho V = (P \times V)/G, \quad \text{where} \quad (p, v) \cdot g = (p \cdot g, \rho(g^{-1})v).$$

Note that $[p \cdot g, v] = [p, \rho(g)v]$. The obvious projection map is well-defined. Though this construction has already appeared (briefly) above, we will use it extensively in several of the lectures that follow.

Of particular relevance in gauge theory is the *adjoint bundle* of P corresponding to the adjoint representation of G on its Lie algebra $\mathfrak{g}$ (as defined in the previous lecture). We will denote the adjoint bundle as $\mathfrak{g}_P$.

To get a feel for the adjoint bundle, let us prove the following "sanity check."

Lemma 11.1.1. *Let $P = M \times G$ be the trivial G-bundle, then the adjoint bundle $\mathfrak{g}_P$ is trivializable.*

Proof. The result naturally extends to the case of the bundle associated to any representation $\rho\colon G \to GL(V)$ and it is somewhat cleaner to prove it in this generality, so we will proceed by doing so. Define the map

$$\Phi\colon P \times_\rho V \to M \times V, \quad [(x, g), v] \mapsto (x, \rho(g)v).$$

Provided this map is well-defined, it will define a map of vector bundles since it preserves the fibers and is linear on them. Let $h \in G$, then $[(x, gh), \rho(h^{-1})v] \sim [(x, g), v]$, so we consider

$$\begin{aligned}
\Phi([(x, gh), \rho(h^{-1})v]) &= (x, \rho(gh)\rho(h^{-1})v) \\
&= (x, \rho(g)\rho(h)\rho(h)^{-1}v) \\
&= (x, \rho(g)v).
\end{aligned}$$

The map

$$\Psi\colon M \times V \to P \times_\rho V, \quad (x, v) \mapsto [(x, e), v]$$

is clearly linear on fibers and hence defines a bundle map. Finally,

$$\begin{aligned}
\Phi\left(\Psi(x, v)\right) &= \Phi([(x, e), v]) & \qquad \Psi\left(\Phi([(x, g), v])\right) &= \Psi(x, \rho(g)v) \\
&= (x, \rho(e)v) & &= [(x, e), \rho(g)v] \\
&= (x, v) & &= [(x, g), v]
\end{aligned}$$

□

It is possible for the adjoint bundle to be trivializable even if the original principal bundle is not trivial. The bundle associated to the trivial representation is always trivializable as one can find enough linearly independent sections. For instance, if the Lie group is Abelian, then the adjoint representation is trivial. This is one thing that makes $U(1)$-gauge theory, so E&M, more approachable than general gauge theory for spacetimes with non-trivial topology. (For sufficiently nice Lie groups, e.g., compact, connected, and the restriction to unitary representations, the associated bundle to a non-trivial principal bundle is trivial if and only if the representation is trivial.)

If the G-bundle $P \to M$ is equipped with a connection, there is an induced connection (covariant derivative) on any associated bundle $P \times_\rho V \to M$. We won't give the general theory, but only the specific case of the adjoint bundle inspired by [Coh98]; the general construction can be found in Section 11.5 of [Tau11].

To begin, let us define $\mathfrak{g}_P$-valued differential forms, $\Omega^*(M, \mathfrak{g}_P)$:

$$\Omega^k(M, \mathfrak{g}_P) := \Gamma(M, \Lambda^k T^\vee M \otimes \mathfrak{g}_P) \cong \Omega^k(M) \otimes_{C^\infty(M)} \Gamma(M, \mathfrak{g}_P).$$

We will now describe the relationship between connections on $P \to M$ and elements of $\Omega^1(M, \mathfrak{g}_P)$. To this end, begin by defining the *Maurer–Cartan* form on the Lie group G

$$\omega_{MC} \in \Omega^1(G, \mathfrak{g}), \quad (\omega_{MC})_g\,(v) = \left(L_{g^{-1}}\right)_* v, \ v \in T_g G,$$

where L_h is the left translation homomorphism. For matrix groups, one can write

$$(\omega_{MC})_g = g^{-1} dg.$$

More intuitively, we can describe the Maurer–Cartan form in terms of natural frames. Indeed, if $\{E_i\}$ is a frame for TG consisting of left-invariant vector fields, let $\{\theta^i\}$ be the associated co-frame (dual basis). With this data, we have $\omega_{MC} = E_i(e) \otimes \theta^i$.

Next, given a trivializing cover $\{U_\alpha\}$ and transition functions $\{g_{\alpha\beta}\}$ for our bundle $P \to M$, there is a bijection between connections on P and families of 1-forms

$$\{A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})\} \quad \text{such that} \quad A_\beta = \mathrm{A}d(g_{\alpha\beta}^{-1}) \circ A_\alpha + g_{\alpha\beta}^* \omega_{MC}.$$

It then follows that the difference of any two connections lies in $\Omega^1(M, \mathfrak{g}_P)$, i.e., if A and A' are connections on P, then $A - A' \in \Omega^1(M, \mathfrak{g}_P)$. Indeed more is true, connections on P actually form an affine space modeled on $\Omega^1(M, \mathfrak{g}_P)$. Only if the bundle P is trivial do we have a natural base point (the trivial connection), in which case the space of connections is identified with the vector space $\Omega^1(M, \mathfrak{g}_P)$.

It follows from the preceding paragraph that if A is a connection on P, then its curvature 2-form defines an element $F_A \in \Omega^2(M, \mathfrak{g}_P)$. One write this explicitly using "horizontal lifts" of vector fields. Recall that our connection defines a horizontal subbundle $\mathcal{H} \subset TP$ and the notion of horizontal lift of a vector field X on M to one $\widetilde{X}$ on P.

We finish our preparatory work by stating the Bianchi identity: the curvature form F_A is closed in $\Omega^*(M, \mathfrak{g}_P)$. In order to have an exterior derivative on $\Omega^*(M, \mathfrak{g}_P)$ we need a connection on $\mathfrak{g}_P$, which we have as the one induced by the connection A. Again, we can be explicit by utilizing horizontal lifts:

$$d^A \colon \Omega^0(M, \mathfrak{g}_P) \to \Omega^1(M, \mathfrak{g}_P), \quad d^A(\sigma)(X) = [\widetilde{X}, \sigma],$$

for X a vector field on M. This operator d^A is then extended by requiring it to be a derivation of the module structure for the algebra of forms $\Omega^*(M)$; explicitly this requires that

$$d^A(\omega \wedge \eta) = (d\omega) \wedge s + (-1)^k \omega \wedge (d^A \eta), \quad \omega \in \Omega^k(M), \ \eta \in \Omega^1(M, \mathfrak{g}_P).$$

The *Bianchi Identity* is now the equation $d^A F_A = 0$.

As an addendum, given any vector bundle with connection (E, ∇), there is a graded vector space of E-valued forms $\Omega^*(M, E)$ and an (covariant) exterior derivative d^∇ defined analogously as in the case where $E = \mathfrak{g}_P$. In general $d^\nabla \circ d^\nabla \neq 0$. The operator d^∇ is square zero precisely when ∇ is a flat connection.

11.2 BF Theory

BF Theory is a particularly nice place to start as it is a (topological) field theory which can be defined on a manifold in any dimension. We following the description given by Alberto Cattaneo and collaborators, e.g., as in [Cat+95]; to our knowledge BF type theories were first considered in the late 1980's by Blau–Thompson [BT89] and Horowitz [Hor89], see also [Bir+91].

Let M be a closed, oriented n-manifold and G-bundle $P \to M$. There are two fields in BF theory: a connection A on P with associated curvature $F_A \in \Omega^2(M, \mathfrak{g}_P)$ and a form $B \in \Omega^{n-2}(M, \mathfrak{g}_P)$. The action is then expressed as

$$S_{BF}(A, B) = \int_M \mathrm{Tr}(B \wedge F_A),$$

where Tr denotes the point-wise pairing on the Lie algebra $\mathfrak{g}$ (we explain this a bit further for Chern–Simons Theories below). The Euler–Lagrange equations for S_{BF} are

$$F_A = 0 \quad \text{and} \quad d^A B = 0.$$

Note that depending on the dimension, there are various terms one can add to the BF action. For example, in 3-dimensions, as $B \in \Omega^1(M, \mathfrak{g}_P)$, we could consider

$$S_{BF}^\kappa(A, B) = \int_M \mathrm{Tr}(B \wedge F_A) + \kappa \, \mathrm{Tr}(B \wedge B \wedge B),$$

where $\kappa \in \mathbb{R}$ is any parameter. Similarly, in 4-dimensions, we have

$$S_{BF}^\kappa(A, B) = \int_M \mathrm{Tr}(B \wedge F_A) + \kappa \, \mathrm{Tr}(B \wedge B).$$

These additional terms are sometimes called *cosmological* terms, especially in dimension 3 where there is a relationship between BF theory and perturbative gravity.

While BF theories are topological, i.e., they have no dynamics, they still have interesting observable structure. For instance, Cattaneo and others have described invariants of links and knotted surfaces in terms of certain observables in BF theory, see [CCM95] or [CM94].

11.3 Chern–Simons Theory

The origins of Chern–Simons Theory are actually mathematical. Recall from the previous lecture the Chern–Weil homomorphism/construction which associated a closed even degree differential form $P(F)$ given an invariant polynomial P of the Lie algebra g. In 1974 Chern and Simons [CS74] described odd degree primitives for P given by traces of powers of F. More explicitly, the *Chern–Simons* p-form $CS(A) \in \Omega^p(M)$ is defined such that

$$dCS(A) = \mathrm{Tr}(F^k),$$

where $p = 2k-1$ (this is literally true on a trivial bundle and locally in general). On the trivial bundle in 3-dimensions we have

$$CS(A) = \mathrm{Tr}\left[dA \wedge A + \frac{2}{3} A \wedge A \wedge A\right].$$

This formula requires a bit of unwinding. Fortunately we assumed that we started with a trivial bundle, so the connection 1-form A is just a $\mathfrak{g}$-valued differential form, and we need to perform some algebraic operation to get a $\mathbb{C}$-valued differential form. Moreover, we assume the existence an ad-invariant pairing on $\mathfrak{g}$, so we use that pair $\mathfrak{g}$-valued elements point-wise. Introducing some notation (which we learned from José Figueroa-O'Farrill), let

$$\langle - \overset{\wedge}{,} - \rangle \colon \Omega^k(M, \mathfrak{g}) \otimes \Omega^\ell(M, \mathfrak{g}) \to \Omega^{k+\ell}(M)$$

be given by wedging form components and applying the pairing (point-wise) to the Lie algebra components. Similarly, let

$$[- \overset{\wedge}{,} -] \colon \Omega^k(M, \mathfrak{g}) \otimes \Omega^\ell(M, \mathfrak{g}) \to \Omega^{k+\ell}(M, \mathfrak{g})$$

denote the wedge of the form components and the application (again point-wise) of the Lie bracket to the Lie algebra components. Then,

$$\mathrm{Tr}[dA \wedge A] = \langle dA \overset{\wedge}{,} A \rangle \quad \text{and} \quad \mathrm{Tr}[A \wedge A \wedge A] = \frac{1}{2}\langle [A \overset{\wedge}{,} A] \overset{\wedge}{,} A \rangle.$$

In the case of a matrix Lie algebra (or in any other faithful representation), the operation Tr really is a (scalar multiple) of the standard trace.

Around 1978, Albert Schwarz realized that the work of Chern and Simons could be used to describe a (topological) quantum field theory. The *Chern–Simons action functional* on an oriented 3-manifold is then given by

$$S_{CS}(A) = \int_M CS(A) = \int_M \mathrm{Tr}\left[dA \wedge A + \frac{2}{3} A \wedge A \wedge A\right].$$

It is easy to see that the critical connections of this functional are the flat ones, i.e., connections with vanishing curvature.

In addition to trivial bundles, we can make sense of the Chern–Simons action functional when perturbing around a fixed connection on any G-bundle P. The classical (mathematical) reference for perturbative Chern–Simons is by Axelrod and Singer [AS94].

The observable theory of Chern–Simons theory and its non-perturbative aspects are very rich, but beyond our current scope. Dan Freed's overview [Fre09] is a great place to start reading. For the relationship to (topological) phases of matter, we recommend Witten's lectures [Wit16].

11.4 Chern–Simons Theories in Arbitrary Dimension

Other dimensions: dglas, this already appears in cattaneo et all 1995 and Cattaneo–Rossi 2001

11.5 Yang–Mills Theory

There are many excellent references for Yang–Mills Theory with varying degrees of detail and points of view. We will mainly draw from Donaldson and Kronheimer's text [DK90] which contains many many details and is aimed at proving Donaldson's construction of 4-manifold invariants using moduli spaces of solutions to the Yang–Mills equations. Naber's books provide some nice context and overview, especially Chapter 6 and Appendix B of [Nab11], as well as some nice examples and model computations as in Section 2.5 of [Nab00]. In addition to his original research articles, Donaldson's 1986 address at the ICM is an excellent overview [Don87]. Much of the mathematical analysis of gauge theory is predicate on Karen Uhlenbeck's beautiful work in infinite dimensional analysis; a nice summary of her influence is provided in [Don19].

To begin, let (M, g) be an oriented, closed Riemannian manifold. There is an inner product on $\Omega^*(M)$ given by

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge *\beta,$$

where $*$ is the Hodge star. Note that this pairing in non-zero on homogenous forms only if α and β have the same degree. This inner product induces a norm called the L^2-*norm*. Now, let $P \to M$ be a G-bundle with the Lie algebra $\mathfrak{g}$ being equipped with an invariant inner product. Applying the inner product on $\mathfrak{g}$ point-wise allows us to extend the L^2-norm to $\mathfrak{g}$-valued differential forms and similarly for $\mathfrak{g}_P$-valued forms.

Restrict now to the case where M is 4-dimensional. The *Yang–Mills action functional* for connections A on P is then given by the L^2-norm squared of the curvature:

$$S_{YM}(A) = \|F_A\|^2 = -\int_M \mathrm{Tr}(F_A \wedge *F_A).$$

Letting d^{A*} be the L^2-adjoint of the covariant exterior derivative, so $d^{A*} = *d^A*$, the Euler–Lagrange equation for S_{YM} is simply

$$d^{A*}F_A = 0.$$

In actuality, the variational calculus also produces the requirement that $d^A F_A = 0$, but that condition is automatically satisfied by the Bianchi Identity. The equations $d^{A*}F_A = 0$ are known as the *Yang–Mills Equations*.

Note that S_{YM} can be defined on a Lorentzian spacetime as well, though the Yang–Mills equations are now a system of hyperbolic equations rather than elliptic as in the Riemannian case. As such, much more care is needed with the analysis of them in the Lorentzian setting.

11.5.1 (Anti)Self-Dual Connections

As before, let (M, g) be an oriented Riemannian 4-manifold (not necessarily closed). The Hodge star, $*$, takes 2-forms to 2-forms and $(*)^2 = \mathrm{Id}$. Let Ω^+ and Ω^- denote the $+1$ and -1 eigenspaces of 2-forms with respect to $*$, so

$$\Omega^2_M = \Omega^+ \oplus \Omega^-.$$

(We follow convention and suppress writing '2' as well as the manifold M whenever it is clear from context.) This splitting of 2-forms extends to vector bundle forms. In particular, we can decompose the curvature 2-form of a connection on our principal bundle $P \to M$ as

$$F_A = F_A^+ \oplus F_A^- \in \Omega^+(\mathfrak{g}_P) \oplus \Omega^-(\mathfrak{g}_P).$$

A connection is *self-dual* if $F_A^- = 0$ and *anti-self-dual* (ASD) if $F_A^+ = 0$.

Using the splitting of 2-forms, we can write

$$S_{YM}(A) = \|F_A\|^2 = \int_M |F_A^+|^2 dVol + \int_M |F_A^-|^2 dVol.$$

Recall, that the Euler–Lagrange equations are

$$d^A F_A = 0 \quad \text{and} \quad d^{A*} F_A = 0,$$

where the first equation is automatically satisfied due to the Bianchi Identity. Note that if a connection is ASD then

$$d^{A*} F_A = *(d^A(*F_A^-)) = *(d^A(-F_A^-)) = - * (d^A F_A) = - * (0) = 0,$$

so ASD connections solve the Yang–Mills equations. Actually, the self-dual connections also solve the Yang–Mills equations, while a general 2-form of mixed type will not. Following [DK90] we will consider ASD connections.

It follows from Chern–Weil theory that

$$\kappa(P) = \frac{1}{8\pi^2} \int_M \operatorname{Tr}(F_A^2)$$

is a characteristic number which depends on the structure group G of P:

$$\begin{aligned} \kappa(P) &= c_2(P) && \text{for } SU(n) \text{ bundles,} \\ &= c_2(P) - \frac{1}{2} c_1(P)^2 && \text{for } U(n) \text{ bundles,} \\ &= -\frac{1}{4} p_1(P) && \text{for } SO(n) \text{ bundles.} \end{aligned}$$

Now, careful linear algebra shows that

$$\operatorname{Tr}(F_A^2) = -\left(|F_A^+|^2 - |F_A^-|^2\right) dVol.$$

Hence, on a $SU(n)$ bundle we have

$$8\pi^2 c_2(P) = \int_M |F_A^-|^2 dVol - \int_M |F_A^+|^2 dVol.$$

Consequently, if $c_2(P) > 0$, then $8\pi^2 c_2(P)$ is a lower bound for $S_{YM}(A)$ with equality obtained precisely when A is ASD. Similar analysis applies to $U(n)$ and $SO(n)$. Moreover, we have the following.

Proposition 11.5.1. *Let M be a compact, oriented Riemannian 4-manifold. If the bundle $P \to M$ admits an ASD connection, then $\kappa(P) \geq 0$. Further, if $\kappa(P) = 0$, then any ASD connection is flat.*

The characteristic number $-\kappa(P)$ is called the *instanton number*. We will see in a future lecture that the "moduli space" of ASD connections is a (disjoint) union over possible instanton numbers.

11.5.2 $SU(2)$-theory on $\mathbb{R}^4$: BPST Instantons

We consider a family of ASD connections of instanton number -1 on S^4. It will be more convenient to actually consider "finite energy" ASD connections on $\mathbb{R}^4$. Such connections will then extend to some bundle over the (conformal) compactification of $\mathbb{R}^4$ to S^4; this is non-trivial and depends on Uhlenbeck's Removable Singularities Theorem. The finite energy condition is simply

$$\int_{\mathbb{R}^4} |F_A|^2 dVol < \infty.$$

These instantons were first considered in [Bel+75], with further mathematical description in [AHS78]. We follow the presentation of Naber [Nab00] Section 2.5, see also [Nab11] Sections 6.3–6.5.

The ADHM (after Atiyah, Drinfeld, Hitchin, and Manin) is a beautiful construction which works for all instanton numbers, see [Don22] or the original [Ati+78].

It will be convenient to identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$. Recall that $S^3 \subset \mathbb{H}$ as the unit quaternions. Moreover, we have an identification $S^4 \cong \mathbb{H}P^1$. Next, via the clutching construction and the Hopf Degree Theorem

$$\frac{\{SU(2) - \text{bundles on } S^4\}}{\text{isomorphism}} \cong \pi_3(SU(2)) \cong \pi_3(S^3) \cong \mathbb{Z},$$

so there is actually an integers worth of isomorphism classes of $SU(2)$ bundles on S^4. The preceding isomorphism is given by the second Chern class/number. We are only dealing with the isomorphism class consisting of those whose second Chern number is 1. Recall that $SU(2) \cong S^3 \subset \mathbb{H}$; a representative for the equivalence class of such bundles is given by the quaternionic Hopf fibration $S^3 \hookrightarrow S^7 \to S^4$.

Recall that $q \in \mathbb{H}$ can be expanded as $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, where $q_i \in \mathbb{R}$. The quaternions admit a conjugation operation

$$\overline{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}.$$

The *real quaternions* are fixed by conjugation, so only have component q_0. The complement of real quaternions is called the set of *imaginary quaternions*. It is useful to identify the Lie algebra $\mathfrak{su}(2)$ with the imaginary quaternions. The imaginary quaternions are real 3-vectors and the Lie bracket is given by the cross product in $\mathbb{R}^3$. This Lie algebra can also be recovered as the Lie algebra of the (non-Abelian) Lie group $S^3 \subset \mathbb{H}$.

Definition 11.5.2. Let $\lambda \in \mathbb{R}_{>0}$ and $n \in \mathbb{H}$. The *BPST instanton with scale λ and center n* is the connection

$$A_{\lambda,n}(q) = \mathrm{Im}\left(\frac{\overline{q} - \overline{n}}{\lambda^2 + |q-n|^2} dq\right).$$

Given an instanton $A_{\lambda,n}$, its curvature/field strength is the 2-form

$$F_{\lambda,n}(q) = \frac{\lambda^2}{\left(\lambda^2 + |q-n|^2\right)^2} d\overline{q} \wedge dq.$$

These instantons are ASD and hence we have $S_{YM}(A_{\lambda,n}) = 8\pi^2$. Note that the Yang–Mills action is independent of λ and n as we would expect since for ASD (respectively self-dual) connections, the action

S_{YM} is a characteristic number of the $SU(2)$ bundle. One computes that

$$|F_{\lambda,n}(q)|^2 = \frac{48\lambda^2}{(\lambda^2 + |q-n|^2)^4}.$$

Note that $|F_{\lambda,n}(q)|$ is maximized at $q = n$ and as $\lambda \to 0$ the field strength becomes localized at the point n, this explains the terminology of "scale" and "center."

The BPST potentials $A_{\lambda,n}$ all arise as pullbacks, via stereographic projection, of natural connections on the Hopf bundle over $\mathbb{H}P^1 \cong S^4$. Let $[q^0, q^1]$ be a homogeneous coordinates for $\mathbb{H}P^1$, then

$$\omega = \mathrm{Im}\left(\frac{\overline{q}^0}{|q^0|^2} dq^0 + \frac{\overline{q}^1}{|q^1|^2} dq^1\right)$$

is a connection on the quaternionic Hopf bundle. The $A_{\lambda,n}$ are then just scaled and shifted versions of the pullback of ω.

11.5.3 Short Physical Story of Yang–Mills

11.6 Exercises

1. Let $\omega_{MC} \in \Omega^1(G, \mathfrak{g})$ be the Maurer–Cartan form on a Lie group G.

   (a) Suppose X is a left-invariant vector field on G, show that $\omega_{MC}(X)$ is constant as a $\mathfrak{g}$-valued function on G.

   Solution. Let G be a Lie group. Let $\omega_{MC} \in \Omega^1(G, \mathfrak{g})$ denote the Maurer–Cartan form on G. Let $X \in \mathfrak{X}(G)$ be a left-invariant vector field. This means that for all $g, h \in G$, we have:

   $$(L_g)_* X_h = X_{gh},$$

   where $L_g : G \to G$ is left multiplication by g, and $(L_g)_*$ is its pushforward on tangent vectors.

   [**We want to show that** $\omega_{MC}(X) : G \to \mathfrak{g}$ is constant as a $\mathfrak{g}$-valued function on G. That is, we want to prove tha $g_1, g_2 \in G$, we have $\omega_{MC}(X_{g_1}) = \omega_{MC}(X_{g_2})$.]

   Proof. Recall that the Maurer–Cartan form is defined by

   $$\omega_{MC}(v_g) = (L_{g^{-1}})_* v_g \in \mathfrak{g},$$

   for any $v_g \in T_g G$.

   Let X be a left-invariant vector field. Then for each $g \in G$, we have:

   $$X_g = (L_g)_* X_e.$$

   Now apply the Maurer–Cartan form to X_g:

   $$\omega_{MC}(X_g) = \omega_{MC}((L_g)_* X_e) = (L_{g^{-1}})_*(L_g)_* X_e = (L_{g^{-1}} \circ L_g)_* X_e = (L_e)_* X_e = \mathrm{id}_* X_e = X_e.$$

So we find:

$$\omega_{MC}(X_g) = X_e \quad \text{for all } g \in G.$$

Hence,

$$\omega_{MC}(X_{g_1}) = \omega_{MC}(X_{g_2}) = X_e \quad \text{for all } g_1, g_2 \in G.$$

Conclusion. The map $\omega_{MC}(X) : G \to \mathfrak{g}$ is constant, with value equal to $X_e \in \mathfrak{g}$. $\square$

(b) Next, show that if Y is also a left-invariant vector field, then

$$\omega_{MC}\left([X, Y]\right) = [\omega_{MC}(X), \omega_{MC}(Y)].$$

Solution. Let G be a Lie group, and let $\omega_{MC} \in \Omega^1(G, \mathfrak{g})$ denote the Maurer–Cartan form on G. Let $X, Y \in \mathfrak{X}(G)$ be two left-invariant vector fields. This means that for all $g \in G$, we have:

$$X_g = (L_g)_* X_e, \quad Y_g = (L_g)_* Y_e,$$

where $X_e, Y_e \in \mathfrak{g} \cong T_eG$, and $L_g : G \to G$ denotes left multiplication. Recall also that the Maurer–Cartan form is defined by:

$$\omega_{MC}(v_g) = (L_{g^{-1}})_* v_g \in \mathfrak{g}, \quad \text{for } v_g \in T_gG.$$

[**We want to show that** $\omega_{MC}([X, Y]) = [\omega_{MC}(X), \omega_{MC}(Y)]$ as an identity of $\mathfrak{g}$-valued functions on G.]

Proof. First, note that the Lie bracket $[X, Y]$ of two left-invariant vector fields is again left-invariant. In particular,

$$[X, Y]_g = (L_g)_*[X_e, Y_e],$$

and so,

$$\omega_{MC}([X, Y]_g) = (L_{g^{-1}})_*(L_g)_*[X_e, Y_e] = [X_e, Y_e].$$

On the other hand, since $\omega_{MC}(X) = X_e$ and $\omega_{MC}(Y) = Y_e$ as shown previously, we compute:

$$[\omega_{MC}(X), \omega_{MC}(Y)] = [X_e, Y_e].$$

Thus,

$$\omega_{MC}([X, Y]_g) = [\omega_{MC}(X), \omega_{MC}(Y)] \quad \text{for all } g \in G,$$

which proves the identity.

Conclusion. The Maurer–Cartan form preserves the Lie bracket of left-invariant vector fields:

$$\omega_{MC}([X, Y]) = [\omega_{MC}(X), \omega_{MC}(Y)]. \qquad \square$$

(c) Now verify that

$$d\omega_{MC}(X,Y) = X\omega_{MC}(Y) - Y\omega_{MC}(X) - \omega_{MC}\left([X,Y]\right).$$

(This actually holds for arbitrary vector fields X and Y, they need not be left-invariant.)

Solution. Let $\omega \in \Omega^1(M)$ be a smooth 1-form on a smooth manifold M, and let $X, Y \in \mathfrak{X}(M)$ be smooth vector fields.

[We want to show that the exterior derivative of a 1-form satisfies the formula

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).]$$

Proof. Any smooth 1-form ω can be written locally as a linear combination of forms of the type $f\,dg$, where f and g are smooth functions. Since both sides of the identity are linear in ω, it suffices to prove the result for the case $\omega = f\,dg$.

Compute $d\omega(X,Y)$. We begin by computing the exterior derivative:

$$d\omega = d(f\,dg) = df \wedge dg.$$

Then:

$$\begin{aligned} d\omega(X,Y) &= (df \wedge dg)(X,Y) \\ &= df(X)\,dg(Y) - df(Y)\,dg(X) \\ &= (Xf)(Yg) - (Yf)(Xg). \end{aligned}$$

Compute the right-hand side.

First term:

$$X(\omega(Y)) = X(f \cdot dg(Y)) = X(f \cdot Yg) = (Xf)(Yg) + f \cdot X(Yg).$$

Second term:

$$Y(\omega(X)) = Y(f \cdot dg(X)) = Y(f \cdot Xg) = (Yf)(Xg) + f \cdot Y(Xg).$$

Third term:

$$\omega([X,Y]) = f \cdot dg([X,Y]) = f \cdot [X,Y]g = f \cdot (X(Yg) - Y(Xg)).$$

Now put all three terms together:

$$\begin{aligned} &X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \\ &= [(Xf)(Yg) + f \cdot X(Yg)] - [(Yf)(Xg) + f \cdot Y(Xg)] - f \cdot (X(Yg) - Y(Xg)) \\ &= (Xf)(Yg) - (Yf)(Xg) + f \cdot X(Yg) - f \cdot Y(Xg) - f \cdot X(Yg) + f \cdot Y(Xg) \\ &= (Xf)(Yg) - (Yf)(Xg). \end{aligned}$$

Compare both sides. We find:

$$d\omega(X,Y) = (Xf)(Yg) - (Yf)(Xg) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

Conclusion. Since the identity holds for any 1-form of the form $f\,dg$, and all 1-forms are locally linear combinations of such terms, it follows by linearity that:

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

for all smooth 1-forms ω and smooth vector fields X, Y. □

(d) Finally, under the assumption that X and Y are left-invariant, prove the Maurer–Cartan equation

$$d\omega_{MC}(X,Y) + [\omega_{MC}(X), \omega_{MC}(Y)] = 0.$$

This equality can also be extended to arbitrary vector fields since the invariant ones span the tangent space at each point. The Maurer–Cartan equation is significant in many settings and is generally written as

$$d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

Solution. Let G be a Lie group, and let $\omega_{MC} \in \Omega^1(G, \mathfrak{g})$ denote the Maurer–Cartan form on G. Let $X, Y \in \mathfrak{X}(G)$ be left-invariant vector fields.

[We want to show that the Maurer–Cartan form satisfies the identity

$$d\omega_{MC}(X,Y) + [\omega_{MC}(X), \omega_{MC}(Y)] = 0, \text{ known as the Maurer–Cartan equation. }]$$

Proof. We begin by recalling the general formula for the exterior derivative of a 1-form:

$$d\omega_{MC}(X,Y) = X(\omega_{MC}(Y)) - Y(\omega_{MC}(X)) - \omega_{MC}([X,Y]).$$

Since X and Y are left-invariant, we have previously shown that $\omega_{MC}(X)$ and $\omega_{MC}(Y)$ are constant as $\mathfrak{g}$-valued functions. Therefore, the directional derivatives $X(\omega_{MC}(Y))$ and $Y(\omega_{MC}(X))$ vanish. It follows that

$$d\omega_{MC}(X,Y) = -\omega_{MC}([X,Y]).$$

But the Lie bracket $[X, Y]$ is also a left-invariant vector field, and we have previously established that

$$\omega_{MC}([X,Y]) = [\omega_{MC}(X), \omega_{MC}(Y)].$$

Substituting this into the expression above gives

$$d\omega_{MC}(X,Y) = -[\omega_{MC}(X), \omega_{MC}(Y)],$$

and thus,

$$d\omega_{MC}(X,Y) + [\omega_{MC}(X), \omega_{MC}(Y)] = 0.$$

Since both sides of the equation are tensorial in X and Y, and left-invariant vector fields span the tangent space at each point of G, the identity extends to all smooth vector fields. □

- Show that $*$ and the splitting of 2-forms into self-dual and anti-self-dual forms only depends on the conformal class of the metric.

  Solution. We want to show that the Hodge star operator on 2-forms is invariant under conformal scaling of the metric. Specifically, let $\mathcal{V}$ be an oriented 4-dimensional real inner product space. Then the decomposition

$$\Lambda^2(\mathcal{V}) = \Lambda^2_+(\mathcal{V}) \oplus \Lambda^2_-(\mathcal{V}),$$

  into self-dual and anti-self-dual 2-forms, is conformally invariant. That is, it depends only on the conformal class of the inner product on $\mathcal{V}$, not on the particular representative.

  Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathcal{V}$, and let $* : \Lambda^2(\mathcal{V}) \to \Lambda^2(\mathcal{V})$ be the Hodge star operator defined with respect to this inner product and orientation. Then for all $\Omega, \Psi \in \Lambda^2(\mathcal{V})$,

$$\langle *\Omega, \Psi \rangle = \langle \Omega, *\Psi \rangle.$$

  Hence $*$ is a symmetric operator with respect to the inner product on 2-forms. Since $*^2 = \text{id}$, the eigenvalues of $*$ on $\Lambda^2(\mathcal{V})$ are ± 1, and the space of 2-forms decomposes orthogonally into the ± 1 eigenspaces:

$$\Lambda^2_+(\mathcal{V}) = \{\Omega \in \Lambda^2(\mathcal{V}) \mid *\Omega = \Omega\}, \quad \Lambda^2_-(\mathcal{V}) = \{\Omega \in \Lambda^2(\mathcal{V}) \mid *\Omega = -\Omega\}.$$

  Let $\{e^1, e^2, e^3, e^4\}$ be an oriented orthonormal basis for $\mathcal{V}^*$. Then explicit bases for the self-dual and anti-self-dual subspaces are given by:

$$\Lambda^2_+(\mathcal{V}) = \text{Span}\left\{e^1 \wedge e^2 + e^3 \wedge e^4,\ e^1 \wedge e^3 + e^4 \wedge e^2,\ e^1 \wedge e^4 + e^2 \wedge e^3\right\},$$
$$\Lambda^2_-(\mathcal{V}) = \text{Span}\left\{e^1 \wedge e^2 - e^3 \wedge e^4,\ e^1 \wedge e^3 - e^4 \wedge e^2,\ e^1 \wedge e^4 - e^2 \wedge e^3\right\}.$$

  Every $\Omega \in \Lambda^2(\mathcal{V})$ has a unique decomposition:

$$\Omega = \Omega_+ + \Omega_-, \quad \text{where } \Omega_\pm \in \Lambda^2_\pm(\mathcal{V}).$$

  Moreover, we have the projection formulas:

$$\Omega_+ = \frac{1}{2}(\Omega + *\Omega), \qquad \Omega_- = \frac{1}{2}(\Omega - *\Omega).$$

  This implies $*\Omega = \Omega$ if and only if $\Omega_- = 0$, and $*\Omega = -\Omega$ if and only if $\Omega_+ = 0$. That is, Ω is self-dual if and only if it lies in $\Lambda^2_+(\mathcal{V})$, and anti-self-dual if and only if it lies in $\Lambda^2_-(\mathcal{V})$.

  Now suppose the inner product on $\mathcal{V}$ is rescaled conformally:

$$\langle \cdot, \cdot \rangle' := \lambda \langle \cdot, \cdot \rangle,$$

  for some fixed $\lambda > 0$. Let $*'$ denote the Hodge star defined using $\langle \cdot, \cdot \rangle'$. Then the same oriented basis $\{e_i\}$ for $\mathcal{V}$ satisfies:

$$\tilde{e}_i := \frac{1}{\sqrt{\lambda}} e_i \quad \text{is orthonormal with respect to } \langle \cdot, \cdot \rangle'.$$

The dual basis transforms as $\tilde{e}^i = \sqrt{\lambda} e^i$, and hence the wedge products $e^i \wedge e^j$ scale by λ. The volume form scales by λ^2. As a result, the defining equation for the Hodge star:

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, \mathrm{vol}$$

remains unchanged under this rescaling.

Therefore, for all $\Omega \in \Lambda^2(\mathcal{V})$, we have:

$$*'\Omega = *\Omega.$$

This proves that the Hodge star operator on 2-forms is conformally invariant under rescaling of the inner product. As a consequence, the decomposition

$$\Lambda^2(\mathcal{V}) = \Lambda^2_+(\mathcal{V}) \oplus \Lambda^2_-(\mathcal{V})$$

depends only on the conformal class of the inner product. In particular, the notions of self-duality and anti-self-duality for 2-forms are conformally invariant.

- Here we will see how Chern–Simons can arise as a boundary (theta) term in Yang–Mills. For simplicity, let us restrict to the Abelian case.

  (a) Let M be a 4-manifold with boundary X. Use Stokes' Theorem to show that the Yang–Mills action on M is the same as the Chern–Simons action on X. Let us denote this quantity as S^M_{CS}.

  Solution. To show how the Chern–Simons action arises as a boundary term in Yang–Mills theory, we use Stokes' theorem to relate the 4-dimensional Yang–Mills action on a manifold M to a 3-dimensional Chern–Simons action on its boundary $X = \partial M$, focusing on the Abelian case.

  The Yang–Mills action in 4 dimensions is given by

$$S_{\mathrm{YM}} = -\frac{1}{4} \int_M F \wedge *F,$$

  where $F = dA$ is the field strength 2-form of the Abelian connection $A \in \Omega^1(M)$.

  Substituting $F = dA$, we rewrite the action as

$$S_{\mathrm{YM}} = -\frac{1}{4} \int_M dA \wedge *dA.$$

  We now add a topological term to the action:

$$S'_{\mathrm{YM}} = S_{\mathrm{YM}} + \frac{\theta}{8\pi^2} \int_M F \wedge F.$$

  This term does not affect the classical equations of motion because it is a total derivative.

  Expanding the topological term in the Abelian case:

$$\frac{\theta}{8\pi^2} \int_M F \wedge F = \frac{\theta}{8\pi^2} \int_M dA \wedge dA = \frac{\theta}{8\pi^2} \int_M d(A \wedge dA).$$

Applying Stokes' theorem to the compact manifold M with boundary X, we get:

$$\frac{\theta}{8\pi^2}\int_M d(A \wedge dA) = \frac{\theta}{8\pi^2}\int_X A \wedge dA.$$

The boundary term

$$S_{\mathrm{CS}}^M := \frac{\theta}{8\pi^2}\int_X A \wedge dA$$

is precisely the Abelian Chern–Simons action on X.

Thus, we can write:

$$S'_{\mathrm{YM}} = S_{\mathrm{YM}} + S_{\mathrm{CS}}^M,$$

demonstrating that the 3D Chern–Simons action naturally arises as a boundary contribution in 4D Yang–Mills theory with the topological term included.

(b) Let X be an oriented 3-manifold. Suppose M and M' are oriented compact 4-manifolds with common boundary X. Consider the closed 4-manifold N obtained by gluing M and M' along X. Show that

$$S_{CS}^M - S_{CS}^{M'} = \frac{k}{4\pi}\int_N F \wedge F.$$

(c) In the previous step we computed the dependence (as a difference element) of the Chern–Simons action of a 3-manifold as a function of choice of a *bulk* manifold. In order to have a well-defined *partition function* the difference in (b) need not vanish, but rather be an integer multiple of 2π, prove that this is so if N is a spin-manifold.

- Let $A_{\lambda,n}$ be a BPST instanton/gauge potential/connection.

(a) Verify that $A_{\lambda,n}$ is ASD.

Goal. Show that $F_{\lambda,n}$ is anti-self-dual; that is,

$$*F_{\lambda,n} = -F_{\lambda,n}.$$

Solution. We observe that the curvature 2-form is expressed as a scalar function times the 2-form $d\bar{q} \wedge dq$, which is quaternion-valued and depends on the coordinate $q \in \mathbb{H}$. The function

$$\frac{\lambda^2}{\left(\lambda^2 + |q-n|^2\right)^2}$$

is a smooth, positive real-valued scalar function, so the Hodge star operator acts only on the differential form part.

Now we recall a standard identity from quaternionic geometry: the 2-form $d\bar{q} \wedge dq$ on $\mathbb{R}^4$ is anti-self-dual. That is,

$$*(d\bar{q} \wedge dq) = -d\bar{q} \wedge dq.$$

Therefore,

$$*F_{\lambda,n} = *\left(\frac{\lambda^2}{\left(\lambda^2 + |q-n|^2\right)^2}\, d\bar{q} \wedge dq\right) = \frac{\lambda^2}{\left(\lambda^2 + |q-n|^2\right)^2} \cdot *(d\bar{q} \wedge dq) = -F_{\lambda,n}.$$

Conclusion. The BPST instanton satisfies

$$*F_{\lambda,n} = -F_{\lambda,n},$$

so it is anti-self-dual, as claimed. □

(b) By explicitly computing the integral

$$\int_{\mathbb{R}^4} \frac{48\lambda^2}{(\lambda^2 + |q-n|^2)^4} dq$$

verify that the total field strength is indeed $8\pi^2$.

Solution. The integrand depends only on $r = |q-n|$, so we may assume $n = 0$ by translation invariance. The volume form in spherical coordinates on $\mathbb{R}^4$ is $dq = \omega_3 r^3 dr$, where $\omega_3 = \mathrm{Vol}(S^3) = 2\pi^2$. Therefore,

$$\int_{\mathbb{R}^4} \frac{48\lambda^2}{(\lambda^2 + |q|^2)^4}\, dq = 48\lambda^2 \cdot 2\pi^2 \int_0^\infty \frac{r^3}{(\lambda^2 + r^2)^4}\, dr.$$

Let $u = r^2$, so $r^3\, dr = \frac{1}{2} u\, du$. Then,

$$\int_0^\infty \frac{r^3}{(\lambda^2 + r^2)^4}\, dr = \frac{1}{2} \int_0^\infty \frac{u}{(\lambda^2 + u)^4}\, du.$$

Let $v = u + \lambda^2$, then

$$\int_0^\infty \frac{u}{(\lambda^2 + u)^4}\, du = \int_{\lambda^2}^\infty \frac{v - \lambda^2}{v^4}\, dv = \int_{\lambda^2}^\infty \frac{1}{v^3}\, dv - \lambda^2 \int_{\lambda^2}^\infty \frac{1}{v^4}\, dv.$$

Computing these:

$$\int_{\lambda^2}^\infty \frac{1}{v^3}\, dv = \frac{1}{2\lambda^4}, \quad \int_{\lambda^2}^\infty \frac{1}{v^4}\, dv = \frac{1}{3\lambda^6}.$$

So the integral becomes:

$$\frac{1}{2}\left(\frac{1}{2\lambda^4} - \lambda^2 \cdot \frac{1}{3\lambda^6}\right) = \frac{1}{2}\left(\frac{1}{2\lambda^4} - \frac{1}{3\lambda^4}\right) = \frac{1}{12\lambda^4}.$$

Therefore,

$$\int_{\mathbb{R}^4} \frac{48\lambda^2}{(\lambda^2 + |q|^2)^4}\, dq = 48\lambda^2 \cdot 2\pi^2 \cdot \frac{1}{12\lambda^4} = 8\pi^2.$$

Conclusion. The total field strength of the BPST instanton integrates to $8\pi^2$, as expected. □

- Show that isomorphism classes of $SU(2)$-bundles on S^4 are indeed classified by their second Chern number.

12 Simply Connected 4-Manifolds

These notes mostly follow along section 1.2 of the[GS99] while also pulling from [DK01],[Sco05],[GP74].

12.1 Classification of Integral Forms

Let A be a finitely generated free $\mathbb{Z}$-module and Q a bilinear form

$$Q : A \times A \to \mathbb{Z}$$

Definition 12.1.1. The bilinear form Q is *symmetric* if

$$Q(x, y) = Q(y, x)$$

for all $x, y \in A$. The form is *skew-symmetric* if

$$Q(x, y) = -Q(y, x)$$

for all $x, y \in A$.

For the rest of this section let Q be a symmetric bilinear form. By choosing a basis of A the form Q can be represented by a square matrix and $Q(x, y) = x^T Qy$.

Definition 12.1.2. The *rank* $\mathrm{rk}(Q)$ is the rank of A as a free $\mathbb{Z}$-module.
Diagonalizing Q over $A \otimes_{\mathbb{Z}} \mathbb{R}$ we can find the number of $+1$s and -1s on the diagonal, denoted b_2^+ and b_2^-, respectively. The *signature* $\sigma(Q) = b_2^+ - b_2^-$.
The type of parity of Q is *even* iff $Q(x, x) \equiv 0 \pmod 2$ for all $x \in A$; otherwise Q is *odd*.
Additionally, if $\mathrm{rk}(Q) = \sigma(Q)$ or $\mathrm{rk}(Q) = -\sigma(Q)$ then Q is called *positive definite* or *negative definite*, respectively. If Q is neither then it is called *indefinite*.

We can build a form on the direct sum of $\mathbb{Z}$-modules as follows:
Let Q_1, Q_2 be bilinear forms defined on free $\mathbb{Z}$-modules A_1, A_2, respectively. Elements $x, y \in A = A_1 \oplus A_2$ can be written as $x = x_1 + x_2$ and $y = y_1 + y_2$. Then $Q = Q_1 \oplus Q_2$ is defined as

$$Q(x, y) = Q_1(x_1, y_y) + Q_2(x_2, y_2)$$

Definition 12.1.3. We call a form Q *unimodular* if $\det Q = \pm 1$.

We can characterize unimodular form by the following: Let $A^\vee$ denote the dual space of A. Define the homomorphism $\varphi : A \to A^\vee$ as $x \mapsto Q(x, -)$.

Lemma 12.1.4. *The form Q is unimodular iff φ is an isomorphism.*

Proof. Pick a basis $\{x_1, \dots, x_n\}$ of A. The dual basis $\{\xi_1, \dots, \xi_n\}$ of $A^\vee$ is characterized by $\xi_j(x_k) = \delta_{jk}$. Hence, $\varphi(x_j) = \sum_k Q(x_j, x_k)\xi_k$, i.e. φ is represented by the matrix with coefficients $Q(x_j, x_k)$. A matrix P over $\mathbb{Z}$ is invertible iff $\det P = \pm 1$. □

The following theorem shows that indefinite forms are classified by their signature, parity, and rank.

Theorem 12.1.5. *If Q_1, Q_2 are two indefinite forms defined on A_1, A_2, respectively, with the same signature, parity, and rank then Q_1 is equivalent to Q_2.*

Let H be the unimodualr form defined by the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Lemma 12.1.6. *If $\sigma(Q) = 0$, then if Q is even*

$$Q \cong \bigoplus_k H$$

(the k-fold direct sum of H) and if Q is odd

$$Q \cong \bigoplus_m (1) \bigoplus_m (-1)$$

(the direct sum of m copies of 1 and m copies of -1) for $k, m \in \mathbb{N}$.

Definition 12.1.7. An element $x \in A$ is *characteristic* if $Q(x, a) \equiv Q(a, a) \pmod 2$ for all $a \in A$.

Lemma 12.1.8. *If $x \in A$ is characteristic then $Q(x, x) \equiv \sigma(Q)$ (mod 8); in particular if Q is even then 8 divides $\sigma(Q)$.*

Let

$$E_8 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

be the matrix defining a bilinear form on $\mathbb{Z}^8$. This form is even, positive definite, unimodular and $\sigma(E_8) = 8$.

Theorem 12.1.9. *Suppose Q is an indefinite, unimodular form. If Q is odd then*

$$Q \cong \bigoplus_{b_2^+} (1) \bigoplus_{b_2^-} (-1)$$

If Q is even then

$$Q \cong \bigoplus_s E_8 \bigoplus_r H$$

where

$$s = \frac{\sigma(Q)}{8} \quad r = \frac{rk(Q) - |\sigma(Q)|}{2}$$

12.2 Intersection Forms

12.2.1 Oriented Intersection

Let M be a compact, closed, oriented n-dimensional manifold. Let A, B be oriented submanifolds of dimensions k and m, respectively. Assume A intersects B transversely, that is, for every $p \in A \cap B$ we have $T_pA + T_pB = T_pM$. In this case $A \cap B$ is a submanifold of dimension $n - k - m$. The orientation of $A \cap B$ comes from the short exact sequence

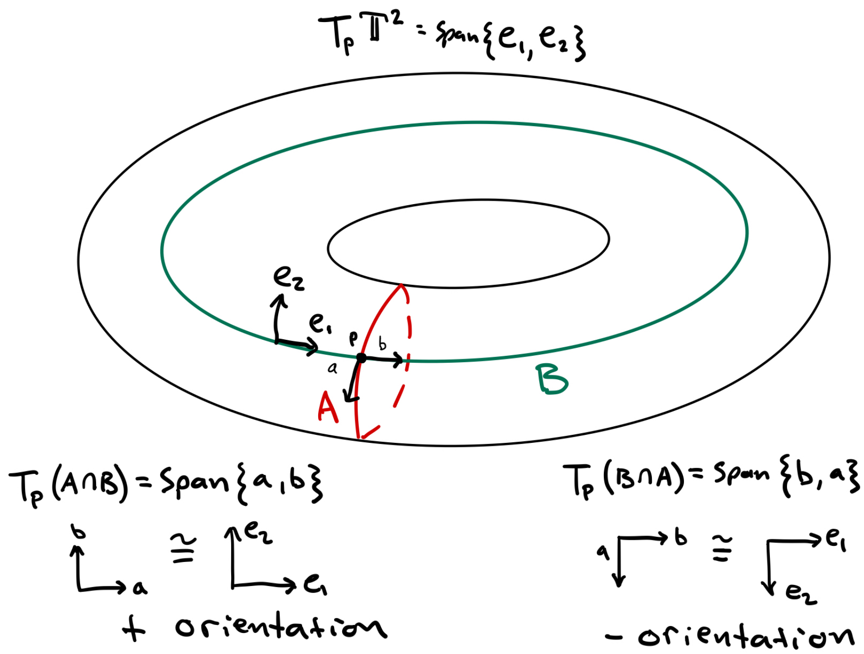
$$0 \to T_p(A \cap B) \to T_pA \oplus T_pB \to T_pM \to 0$$

Choose orientations of $T_p(A \cap B), T_pA \oplus T_pB, T_pM$. Then the orientation of $T_p(A \cap B)$ is positive if the isomorphism

$$T_pA \oplus T_pB \cong T_pM$$

is orientation preserving and negative if it is orientation reversing.

Example 12.2.1. Consider the torus $\mathbb{T}^2$. Let A, B be the meridional and latitudinal embedded circles and $p = A \cap B$. Let $\{e_1, e_2\}$ be an ordered basis of $T_p\mathbb{T}^2$ oriented positively. Choose an orientation of $T_pA = \mathsf{span}\{a\}, T_pB = \mathsf{span}\{b\}$ as below. Then the isomorphism $T_pA \oplus T_pB \to T_pM$ is given by concatenating basis vectors.



Concretely for $T_p(A \cap B)$ we have $a \mapsto e_1, b \mapsto e_2$ which agrees with the chosen orientation of $T_p\mathbb{T}^2$, thus $A \cap B$ is positively oriented. Conversely, for $T_p(B \cap A)$ the isomorphism is $b \mapsto e_1, a \mapsto e_2$ which is opposite the chosen orientation, so $B \cap A$ is negatively oriented.

Of most interest to us is the case dimension of $A \cap B$ is 0. In which case $A \cap B$ is just a oriented collection of points, i.e. each point is assigned ± 1.

Definition 12.2.2. The oriented intersection number $A \cdot B = I(A, B)$ is the signed count of points in $A \cap B$. Note: the order matters

$$I(A, B) = (-1)^{(\dim A)(\dim B)} I(B, A)$$

This confirms what we saw in the torus example. If one or both of the submanifolds is even then this is symmetric, as we will see for 4-manifolds.

12.2.2 Poincaré Duality

Let's recall some definitions from algerbaic topology.

Definition 12.2.3. The *cap product* is a bilinear map

$$\frown : H_j(M, \mathbb{Z}) \times H^k(M, \mathbb{Z}) \to H_{j-k}(M, \mathbb{Z})$$

that takes a j-chain $\sigma : \Delta^j \to M$ and a k-cochain α and sends it to the $k - j$-chain given by

$$\sigma \frown \alpha = \alpha(\sigma|_{[0,1,\ldots,j]})\sigma|_{[j,\ldots,k]}$$

where $\sigma|_{[0,1,\ldots,j]}$ is the restriction of the simplex to the first $j + 1$ vertices.
The *cup product* is the bilinear map on cohomology

$$\smile : H^l(M, \mathbb{Z}) \times H^m(M, \mathbb{Z}) \to H^{l+m}(M, \mathbb{Z})$$

where $(\alpha \smile \beta)(\sigma) = \alpha(\sigma|_{[0,1,\ldots,l]})\beta(\sigma|_{[l,\ldots,l+m]})$ for $l + m$-chain σ.

Every oriented n-manifold M admits a *fundamental class* $[M] \in H_n(M, \mathbb{Z})$. Using this we have the Poincaré duality isomorphism

$$H^k(M, \mathbb{Z}) \xrightarrow{\cong} H_{n-k}(M, \mathbb{Z})$$

$$\alpha \mapsto [M] \frown \alpha$$

12.2.3 Intersection form on Manifolds

Let M be a closed, compact, oriented n-dimensional manifold. Define a bilinear form

$$Q : H^p(M, \mathbb{Z}) \times H^{n-p}(M, \mathbb{Z}) \to \mathbb{Z}$$

by

$$Q(\alpha, \beta) = \langle \alpha \smile \beta, [M] \rangle = (\alpha \smile \beta)([M])$$

For any $\alpha \in T$, the torsion submodule, $Q(\alpha, \beta) = 0$ for all $\beta \in H^{n-p}$, similarly for any torsion element β. Hence, Q descends to a well defined form on $H^p(M, \mathbb{Z})/T \times H^{n-p}(M, \mathbb{Z})/T$.
This is called the *intersection pairing* on M and we will see that this can be described by the intersection of submanifolds.
Let's focus on the case where M is an even dimensional manifold, i.e. $\dim(M) = 2k$. In this case

$$Q : H^k(M, \mathbb{Z})/T \times H^k(M, \mathbb{Z})/T \to \mathbb{Z}$$

is called the *intersection form* of M. This form is unimodular, and can be represented by a matrix once a basis of $H^k(M, \mathbb{Z})/T$ is chosen.
If k is odd then Q is skew-symmetric. In which case there exists a symplectic basis of $H^k(M, \mathbb{Z})/T$ such that Q is a direct sum of the matrix

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Thus, unimodular skew-symmetric forms over $\mathbb{Z}$ are classified only by their rank which is precisely the rank of $H^k(M, \mathbb{Z})/T$.
If k is even then Q is symmetric. As we saw above symmetric unimodular forms are classified by parity, rank, and signature.

Let A, B be dimension k, oriented submanifolds of M, a compact, closed, oriented manifold of dimension $2k$. The inclusion maps of A, B into M induce maps on homology, which map the fundamental classes of A and B to homology classes $[A], [B]$ in $H_k(M, \mathbb{Z})$. Similarly, the fundamental class of $A \cap B$ is mapped to $[A \cap B] \in H_0(M, \mathbb{Z})$.
Form Poincaré duality we have

Theorem 12.2.4. *Let $\alpha, \beta \in H^k(M, \mathbb{Z})$ be the Poincaré duals of $[A]$ and $[B]$, respectively. Then*

$$A \cdot B = \langle \alpha \smile \beta, [M] \rangle$$

This justifies the terminology of intersection form and makes the form easier to compute explicitly.

Example 12.2.5. Continuing our example of the torus. Let $\{[A], [B]\}$ be a basis for $H_1(\mathbb{T}^2, \mathbb{Z})$. Then for any $x, y_1(\mathbb{T}^2, \mathbb{Z})$

$$Q(x, y) = x^T Q y$$

We already know from above that Q is skew-symmetric, thus a direct sum of $\frac{\mathrm{rk}(Q)}{2} = 1$ copies of X, but let's confirm this by hand.
Each entry of the matrix Q is given by the intersection number of the corresponding basis elements. That is,

$$Q_{11} = A \cdot A = 0 \quad Q_{12} = A \cdot B = 1$$

$$Q_{21} = B \cdot A = -1 \quad Q_{22} = B \cdot B = 0$$

There are no self intersections of the loops so the only intersections we have are between A and B, which is skew-symmetric.

Example 12.2.6. Consider the genus two surface $M = \mathbb{T}^2 \# \mathbb{T}^2$.
By a result that is proved in the exercises

$$Q_M \cong Q_{\mathbb{T}^2} \oplus Q_{\mathbb{T}^2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

12.3 Simply connected 4-manifolds

Now let's pursue this notion of intersection form in the context of 4-manifolds.

Proposition 12.3.1. *If M is closed, oriented, smooth 4-manifold then every element $\alpha \in H_2(M, \mathbb{Z})$ can be represented by an embedded surface Σ_α.*

In the simply connected case Hurewicz isomorphism theorem states $\pi_2(M) \cong H_2(M)$ so every class is represented by an immersed sphere. These immersions only fail to be embeddings at transverse double points. Therefore our earlier definition of the intersection form holds

$$Q_M : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \to \mathbb{Z}$$

$$Q(\alpha, \beta) = \Sigma_\alpha \cdot \Sigma_\beta$$

Proposition 12.3.2. *If M is a closed, 4-manifold ($\partial M = \emptyset$) then Q_M is unimodular.*

The proof of this statement is left as an exercise. This proposition can actually be extended to the case where ∂M is a homology sphere.

Now let's consider a simply connected, closed 4-manifold M. By Poincaré duality

$$\pi_1(M)_{\mathsf{ab}} \cong H_1(M) \cong H^4(M) = 0$$

$$H_3(M) \cong H^1(M) \cong \mathsf{Hom}(H_1(M), \mathbb{Z}) = \mathsf{Hom}(0, \mathbb{Z}) = 0$$

so all (co)homological data is contained in $H^2(M) \cong H_2(M)$. This allows for a homotpical classification of such manifolds.

Theorem 12.3.3. *(Whitehead) Two closed, simply connected, topological 4-manifolds X, Y are homotopy equivalent $X \simeq Y$, iff $Q_X \cong Q_Y$.*

The topological strengthening of this is a theorem by Freedman

Theorem 12.3.4. *(Freedman) For every unimodular symmetric bilinear form Q there exists a closed, simply connected, topological 4-manifold M such that $Q \cong Q_M$. If Q is even, this manifold is unique up to homeomorphism. If Q is odd, then there exist exactly two distinct homeomorphism classes of manifolds, with at most one carrying a smooth structure.*

That is closed, simply connected, topological 4-manifolds are determined up to homeomorphism by their intersection forms.

12.4 Exercises

(a) Prove that if M is a closed 4-manifold then Q_M is unimodular. (Hint: Poincaré duality).

(b) Let M, N be 4-manifolds with intersection forms Q_M, Q_N. Prove that $Q_{M\#N} \cong Q_M \oplus Q_N$. (Hint: Mayer-Vietoris).

(c) Determine the intersection for for $\mathbb{CP}^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, where the overline denotes the oppositely oriented manifold.

(d) Determine the intersection form of $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ which denotes the unique nontrivial sphere bundle over $\mathbb{S}^2$. This construction is given by gluing the two hemispherical trivial bundles together along the equator of the base sphere with a π twist to the fiber spheres. What is the relation between

$$Q_{\mathbb{S}^2 \tilde{\times} \mathbb{S}^2} \text{ and } Q_{\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2}$$

13 Donaldson's Theorem

Theorem 13.0.1 (Donaldson 1983)**.** *Let M be a closed, simply connected 4-manifold with intersection form Q_M. If M is smooth, then Q_M is diagonalizable (over $\mathbb{Z}$).*

Theorem 13.0.2 (Milnor (1958) following Whitehead (1949))**.** *Let M and N be closed, simply connected, oriented 4-manifolds. There is an (oriented) homotopy equivalence between M and N if and only if $Q_M \cong Q_N$.*

Theorem 13.0.3 (Wall 1964)**.** *If M and N are simply connected 4-manifolds with $Q_M \cong Q_N$, then M and N are h-cobordant.*

If the h-Cobordism Theorem held in dimension 4, as Smale proved in dimensions five and higher, then we could deduce that under the assumption that $Q_M \cong Q_N$ then M and N are diffeomorphic. However, it's a corollary of Donaldson's Theorem that there are h-cobordisms between simply connected 4-manifolds which are not diffeomorphic to products.

Nonetheless, by the following result of Freedman, the h-Cobordism Theorem does hold in the topological category in dimension 4. (Smoothness can be removed in the following theorem if one also considers the *Kirby–Siebenmann invariant*.)

Theorem 13.0.4 (Freedman 1982)**.** *Let M and N be closed, smooth, simply connected 4-manifolds. The manifolds M and N are homeomorphic if and only if $Q_M \cong Q_N$.*

Corollary 13.0.5 (Topological Poincaré Conjecture in Dimension 4)**.** *Any homotopy 4-sphere is homeomorphic to S^4.*

While it was already known at the time that there were simply-connected topological 4-manifolds which aren't smoothable, e.g., Freedman's E_8 manifold, Donaldson's Theorem is still remarkable in its scope and in the wave of applications of mathematical gauge theory to topology and geometry that it help to start, e.g., fake $\mathbb{R}^4$s.

This lecture borrows heavily from Donaldson and Kronheimer's tome [DK90]. As the author's note there, a lot of credit is also due to Freed and Uhlenbeck [FU91], especially regarding key analytic details in building moduli spaces of ASD connections.

13.1 Sketch of Proof

Let us summarize Section 8.3.1 of [DK90] (combined with the original argument in [Don83] or [FU91]).

Let M be a simply connected, closed (oriented) 4-manifold with negative definite intersection form. Let P be a $SU(2)$-bundle with second Chern class $c_2 = 1$ and $\mathcal{M}_1$ the associated moduli space of ASD connections on P.

It is quite technical, but one can show that there is a compactification of $\mathcal{M}_1$ to a manifold with boundary, $\widetilde{\mathcal{M}}_1$. Moreover, the boundary of $\widetilde{\mathcal{M}}_1$ is given by

$$\partial\widetilde{\mathcal{M}}_1 \cong M \coprod \underbrace{\mathbb{C}P^2 \amalg \cdots \amalg \mathbb{C}P^2}_{s \text{ summands}} \coprod \underbrace{\overline{\mathbb{C}P}^2 \amalg \cdots \amalg \overline{\mathbb{C}P}^2}_{r \text{ summands}},$$

where $r + s = n$ is an integer that will be defined below. So, $\widetilde{M_1}$ defines a cobordism between M and n copies of $\mathbb{C}P^2$ (of various orientations). In 1954, Thom proved that the signature is a cobordism invariant. Moreover, he proved that the signature is actually a ring homomorphism $\sigma\colon \Omega_O \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$, so in particular, $\sigma(X \amalg Y) = \sigma(X) + \sigma(Y)$. (This last assertion also follows from the discussion of intersection forms as in the previous lecture.) To summarize,

$$\sigma(M) = s\sigma(\mathbb{C}P^2) + r\sigma(\overline{\mathbb{C}P}^2) = s - r \leq n.$$

Next, one considers classes $\Sigma \in H^2(M, \mathbb{Z})$ such that

$$\Sigma \smile \Sigma = -1 \in \mathbb{Z} \cong H^4(M, \mathbb{Z}),$$

where the last isomorphism is determined by the orientation of M. One determines that the number of such Σ is precisely the non-negative integer n. Of course, as Q_M is (negative) definite, $n \leq \mathrm{rk}(Q_M)$, with equality precisely when Q_M is diagonalizable over $\mathbb{Z}$. (This last assertion is a simple exercise in linear algebra.)

At this point we have that $\sigma(M) \leq n$ and $n \leq \mathrm{rk}(Q_M)$. Again by (negative) definiteness, $\sigma(M) = \mathrm{rk}(Q_M)$, so $n = \mathrm{rk}(Q_M)$ and by the previous observation Q_M is diagonalizable over $\mathbb{Z}$.

Alternatively, as in [DK90], one could bypass the cobordism argument and show directly, through some intricate Poincaré duality arguments that the classes Σ with $\Sigma^2 = -1$ actually generate $H^2(M, \mathbb{Z})$ and that n of them are needed to form a $\mathbb{Z}$-basis, hence $\mathrm{rk}(Q_M) = n$.

13.2 Topology of Moduli Space

We would like to now outline the construction of instanton moduli spaces. We begin with recalling a model for infinite dimensional manifolds.

13.2.1 Banach Manifolds

Just as manifolds are defined in terms of Euclidean spaces and smooth maps between them, Banach manifolds are locally modeled on Banach spaces and smooth maps between them. Recall that a Banach space is a normed linear space which is complete, this is sufficient structure on which to model "infinite dimensional manifolds", there are other models, e.g., Fréchet manifolds, Hilbert manifolds, convenient manifolds, etc.

In order to glue local models, we recall the notion of Fréchet derivative. A nice reference for this material is Chapter 1 of [DGV16].

Definition 13.2.1. Let V and W be normed linear spaces, $U \subseteq V$ an open subeset and $x \in U$. A continuous linear map $f\colon U \to W$ is *Fréchet* differentiable at x if there exists a continuous (bounded) linear map $Df(x)\colon V \to W$ such that

$$\lim_{\|h\|_V \to 0} \frac{\|(f(x+h) - f(x) - [Df(x)](h)\|_W}{\|h\|} = 0.$$

A function f is *differentiable* if it is *Fréchet* differentiable at each point.

One can check that if a Fréchet derivative exists, then it is unique. The (total) derivative is a map $Df\colon U \to \mathcal{B}(V, W)$ to the space of bounded linear operators. If Df is continuous, then f is of class C^1. Higher regularity is defined recursively, so f is smooth if $D^k f$ exists and is continuous for all k.

It is a standard exercise to show that differentiation is a linear operation and satisfies the chain rule. Moreover, if V and W are finite dimensional, then Df is the standard derivative given (in coordinates) by the Jacobian matrix.

We can now mimic the definition of smooth manifolds to define Banach manifolds.

Definition 13.2.2. Given a Banach space V, a set X is a *Banach manifold modeled on* V if X is equipped with an equivalence class of smooth atlas where each chart (U_i, ϕ_i) is a bijection onto an open subset $\phi_i(U_i) \subseteq V$.

It is immediate that any Banach space or any open subset thereof has a natural Banach manifold structure.

The setting of Banach manifolds allows for many constructions/notions from finite dimensional geometry, e.g., vector fields, flows, bundles, and infinite dimensional Lie groups. All of these objects are defined in [DGV16].

13.2.2 Outline of the Moduli Space Construction

Let us restrict to the case of $G = SU(n)$ for some n and P a given $SU(n)$ bundle on our Riemannian 4-manifold (M, g). As before, $\mathfrak{g}$ will denote the Lie algebra of G.

The idea, essentially following Section 4.2 of [DK90], is to define an infinite dimensional manifold of connections $\mathcal{A}$, then quotient out by infinite dimensional Lie group of gauge transformations $\mathcal{G}$ to obtain the moduli space of connections modulo gauge. We will then use the ASD equations to find a finite dimensional submanifold of $\mathcal{A}/\mathcal{G}$ which will be the instanton moduli space.

Recall that $\mathcal{A}$ is an infinite dimensional affine space modeled on $\Omega^1(M, \mathfrak{g}_P)$. If we fix a reference connection A on P, i.e., choose a base point, then the have an isomorphism

$$\mathcal{A} \cong \Omega^1(M, \mathfrak{g}_P), \quad B \mapsto B - A.$$

Hence, we can equip $\mathcal{A}$ with the structure of a vector space; this structure depends on A, but only up to (non-canonical) isomorphism. For this presentation, let us conflate $\mathcal{A}$ and $\Omega^1(M, \mathfrak{g}_P)$.

Recall that $\Omega^1(M, \mathfrak{g}_P)$ has a natural L^2 coming from the metric g and the invariant pairing on our Lie algebra $\mathfrak{g}$. Therefore, for $\ell > 2$, we can consider the Sobolev space of $L^2_{\ell-1}$ sections of $T^*M \otimes \mathfrak{g}_P$. This Sobolev space can be defined in local charts (with care as the atlas must be *adapted* or define a *fine cover*). Alternatively, we can give a chart independent definition by using jet expansion and the jet bundle. Either way, we construct a Banach space of connections $\mathcal{A}(\ell)$.

We can similarly model the gauge group as a Banach Lie group, $\mathcal{G}(\ell)$, by considering the Sobolev space of L^2_ℓ sections of the adjoint bundle $\mathfrak{g}_P$.

One then needs to check a few technical details to show that for each $\ell > 2$, we have a Banach manifold of connections modulo gauge: $\mathcal{B}(\ell) := \mathcal{A}(\ell)/\mathcal{G}(\ell)$. Proposition 4.2.16 of [DK90] then shows that the space of solutions to the ASD equations inside of $\mathcal{B}(\ell)$, up to homeomorphism, does not depend on ℓ. Hence, from this point on we will suppress the Sobolev index ℓ from notation.

Now for $A \in \mathcal{A}$ and $\epsilon > 0$, define a formal tangent space

$$T_{A,\epsilon} := \{a \in \Omega^1(\mathfrak{g}_P) : d^{A*}a = 0, \ \|a\|_{L^2_{\ell-1}} < \epsilon\}.$$

One shows that a neighborhood of $[A] \in \mathcal{B}$ is obtained by taking the quotient of $T_{A,\epsilon}$ by the gauge group for sufficiently small ϵ.

Now define $\mathcal{M} \subseteq \mathcal{B}$ to be the subspace of connections modulo gauge that satisfy the ASD equations. Let $A \in \mathcal{A}$ be ASD, define

$$\Psi \colon T_{A,\epsilon} \to \Omega^+(\mathfrak{g}_P), \quad \Psi(a) = F^+(A+a).$$

Then the zero set, $Z(\Psi)$, is a neighborhood of $[A] \in \mathcal{M}$. (If the connection A is *reducible* then one further has to quotient by a certain *isotropy subgroup* Γ_A.)

Finally, one shows that Ψ is a smooth Fredholm map, i.e., it is a smooth map of Banach spaces and its derivative at 0

$$d_A^+ \colon \ker d^{A*} \to \Omega^+(\mathfrak{g}_P)$$

is a (linear) Fredholm operator. The consequence of this is huge: the zero set $Z(\Psi)$ is actually a finite-dimensional vector space. That is, most points (the open subset of irreducible connections) of $\mathcal{M}$ we have a local chart which is a finite dimensional vector space, i.e., they are manifold points. Index theory allows one to compute the dimension at such a manifold point. In the case of $SU(2)$, this dimension is $8c_2(P) - 3(1 - b_1(M) + b_+(M))$, e.g., if $c_2(P) = 1$ over S^4, then the moduli space, $\mathcal{M} = \mathcal{M}_1$, has dimension $8 - 3 = 5$.

13.3 Examples of Moduli Space

We summarize the following examples from Chapter 4 of [DK90] of instanton moduli spaces and their compactifications. We pay special attention to instanton number -1, which Donaldson and Kronheimer call "one-instantons."

13.3.1 $SU(2)$ of One-Instantons over S^4

From our previous discussion of Yang–Mills, we have the family of BPST instantons $A_{\lambda,n}$ with $\lambda \in \mathbb{R}_{>0}$ and $n \in \mathbb{H} \cong \mathbb{R}^4$. It is a theorem of Atiyah, Hitchin, and Singer [AHS78] that these give a complete description of the moduli space $\mathcal{M}_1$. That is, $\mathcal{M}_1 \cong \mathbb{R}_{>0} \times \mathbb{R}^4$, so it is homeomorphic to an open five-ball D^5. This five-ball has a natural compactification by adjoining a copy of S^4 to the boundary. Recall that

$$|F_{\lambda,n}(q)| = \frac{48\lambda^2}{(\lambda^2 + |q-n|)^4}.$$

For fixed n, the λ-family of connections converges to the point $n \in S^4$ as $\lambda \to 0$, where this boundary "connection" is singular with field strength the delta function centered at n.

13.3.2 $SU(2)$ of One-Instantons over $\mathbb{C}P^2$ and $\overline{\mathbb{C}P}^2$

On $\mathbb{C}P^2$, there are actually no ASD connections with instanton number -1, the moduli space is empty. The moduli spaces are non-empty on $SU(2)$ bundles P with $c_2(P) > 1$.

The one-instanton moduli space over $\overline{\mathbb{C}P}^2$ is an open (real) cone on $\overline{\mathbb{C}P}^2$. Indeed, choose a base point $x \in \overline{\mathbb{C}P}^2$, then for $t \in [0, 1)$ consider the $\mathfrak{su}(2) \cong \operatorname{Im}\mathbb{H}$ connection matrix

$$J_t = \frac{1}{1 + |x|^2 - t^2}(\theta_1 \mathbf{i} + t\theta_2 \mathbf{j} + t\theta_3 \mathbf{k}),$$

where, in local (real) coordinates,

$$\begin{aligned}
\theta_1 &= x_1 dx_2 - x_2 dx_1 - x_3 dx_4 + x_4 dx_3 \\
\theta_2 &= x_1 dx_3 - x_3 dx_1 - x_4 dx_2 + x_2 dx_4 \\
\theta_3 &= x_1 dx_4 - x_4 dx_1 - x_2 dx_3 + x_3 dx_2.
\end{aligned}$$

The natural compactification of $\mathcal{M}_1$ in this case is again given by gluing a copy of $\overline{\mathbb{C}P}^2$ at the boundary $t = 1$.

13.3.3 $SO(3)$ ASD Instantons over $\mathbb{C}P^2$

Let P be a $SO(3)$ bundle over $\mathbb{C}P^2$ with Stiefel–Whitney number $w_2(P) \neq 0$. We will consider bundles P with $p_1(P) = -(3 + 4j)$. (The condition that $p_1(P) \equiv 1 \pmod 4$ is explained, among other things, by Buchdahl in [Buc86].) For $j = 0$, there is a unique ASD connection (up to gauge), so the moduli space is a point. If $j = 1$, then the moduli space can be identified with the unordered configuration of two points in $\mathbb{C}P^2$: $\mathrm{Conf}_2(\mathbb{C}P^2) := (\mathbb{C}P^2 \times \mathbb{C}P^2 \setminus \Delta\mathbb{C}P^2)/S_2$. This example is best understand from a complex geometric/algebro-geometric viewpoint of stable bundles over Kähler manifolds as described in Chapter 6 of [DK90].

13.4 A Bit About Compactification

Let X be a locally compact, Hausdorff space which itself is non-compact. A *compactification* of X is a compact Hausdorff space Y together with a continuous map $c\colon X \to Y$ which is a homeomorphism onto its image and such that the image $c(X)$ is dense in Y. Under our hypotheses, every such X has a compactification. In fact, there is a whole poset of compactifications with minimal element *the one-point compactification* and maximal element *the Stone–Čech compactification*.

In actuality, one should take more care. That is, we consider the partially ordered set of equivalence classes of compactifications. If (Y_1, c_1) and (Y_2, c_2) are compactifications, then they are *equivalent* if there is a homeomorphism $\varphi\colon Y_1 \to Y_2$ such that $f \circ c_1 = c_2$. The partial order is then defined as follows: $(Y_1, c_1) \geq (Y_2, c_2)$ if there exists a continuous map $f\colon Y_1 \to Y_2$ such that

$f \circ c_1 = c_2$. One can readily check that $(Y_1, c_1) \geq (Y_2, c_2)$ and $(Y_2, c_2) \geq (Y_1, c_1)$ if and only if the compactifications are equivalent.

The above notions are standard and contained in many first year topology texts. If our space X is a manifold, we are most interested in compactifications Y which are also manifolds. There are elementary point-set requirements for the one-point compactification X^+ to be a manifold. As an example $\mathbb{C}^+ \cong S^2$, while $(\mathbb{C} \setminus \mathbb{Z})^+$ is not a manifold.

More generally, given a non-compact manifold X we would like a compact manifold Y such that there is a smooth map (or of whatever regularity) $c \colon X \to Y$ which is a homeomorphism onto an open subset of Y. The inverse to stereographic projection from the north pole is such a map for $\mathbb{C} \hookrightarrow S^2$. Not every manifold admits a manifold compactification, e.g., a surface of infinite genus does not. The question of compactification/tameness of ends for manifolds is a classic question in geometric topology; a good reference with which to start is Larry Siebenmann's thesis.

13.5 Exercises

(a) Consider the *first-order formulation* of Yang–Mills, i.e., let $E \in \Omega^2(M, \mathfrak{g}_P)$ be an auxiliary field and consider the functional

$$S_{YM}^{FO}(A, E) = i \int_M \mathrm{Tr}(E \wedge F_A) + \mathrm{Tr}(E \wedge *E).$$

  (a) Show that the Euler–Lagrange equations for S_{YM}^{FO} are given by

$$\begin{aligned} i * F_A + 2E &= 0 \\ i * d_A E &= 0. \end{aligned}$$

  (b) Show that there is a bijection between solutions to these Euler–Lagrange equations and those for the standard Yang–Mills action. (Hint: apply $*d_A$ to the first equation.)

(b) Describe an explicit example to show that the h-Cobordism Theorem indeed fails in dimension 4.

(c) Let X and Y be locally compact, Hausdorff spaces.

  (a) Show that if X and Y are homeomorphic, then X^+ and Y^+ are homeomorphic.

  (b) Compute $\pi_1((S^1 \times (0,1))^+)$ and $\pi_1(\mathbb{M}^+)$, where $\mathbb{M}$ is the open Möbius band.

  (c) Congratulate yourself for completing a funky proof that $\mathbb{M}$ is not homeomorphic to an open cylinder.

(d) Let $\|-\| \colon \mathcal{H} \to \mathbb{R}$ be the norm on a Hilbert space $\mathcal{H}$. Prove that $\|-\|$ is Fréchet differentiable at $x \neq 0$. Prove that the norm is not differentiable at 0. What is the regularity of the norm on $\mathcal{H} \setminus \{0\}$?

14 Gravity as a Gauge Theory: 3D

14.1 Introduction to General Relativity

Newtonian Gravity Let's consider the force on the mass m at $\mathbf{r}$ from the mass m_0 at $\mathbf{r}_0$:

$$\mathbf{F}(\mathbf{r}) = -\frac{Gmm_0}{|\mathbf{r}-\mathbf{r}_0|^3}(\mathbf{r}-\mathbf{r}_0),$$

where G is Newton's gravitational constant. We can write the above as $\mathbf{F}(\mathbf{r}) = -m\nabla\Phi(\mathbf{r})$ with the gravitational potential:

$$\Phi(\mathbf{r}) = -\frac{Gm_0}{|\mathbf{r}-\mathbf{r}_0|} = -G\int \frac{\rho(\mathbf{r}')\, d^3x'}{|\mathbf{r}-\mathbf{r}'|}$$

when the general mass distribution is $\rho(\mathbf{r}') = m_0\delta(\mathbf{r}'-\mathbf{r}_0)$. From the above integral we use $\nabla^2(1/|\mathbf{r}-\mathbf{r}'|) = -4\pi\delta(\mathbf{r}-\mathbf{r}')$ to get

$$\nabla^2\Phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r}),$$

which is a Poisson equation governing Newtonian gravity.

This formulation of gravity, even with a great amount of applications, is not very satisfactory from the point of Relativity. First, it implies an instantaneous action at a distance, which violates the statement of Relativity that no information or physical object can travel faster than the finite speed of light. Second, the above lacks in a covariant framework, that is, it is not a tensor equation. In Relativity every Physics law should be written in a tensor equation so that it becomes covariant under transformations. The above Poisson equation is invariant under Galilean transformations, but it is not covariant under Lorentz transformations, let alone diffeomorphisms.

The ideas of General Relativity We recall that in Special Relativity energy and momentum are not separately covariant under Lorentz transformations, but together they form a covariant four momentum. Thus we ask the question: is it possible that we can find a true tensor equation which has the above Poisson equation as a component? Also recall that in Special Relativity the space and time are in the same footing: we use coordinate $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$. Then Δx^μ or dx^μ is a four-vector, a type of tensor. Let's consider a particle whose world line in the spacetime is parametrized by $x^\mu(\lambda)$ in an inertial frame. Then it can be reparametrized by using its proper time τ, which is given by

$$c^2d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

The four-velocity vector of the particle is given by

$$u^\mu = \frac{dx^\mu}{d\tau} = (\gamma c, \gamma\mathbf{v}), \quad \gamma = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2/c^2}}.$$

Then the four-momentum is given by

$$p^\mu = mu^\mu = (\gamma mc, \gamma m\mathbf{v}) = (E/c, \mathbf{p}),$$

where m is the rest mass and E is the energy of the particle. Observe that γm can be regarded as the mass of the moving particle.

Now we look at the right hand side of the Poisson equation. There we find the mass distribution ρ, which we regard as the proper mass density, that is, the mass density viewed in the frame of the moving particle. Observe that this frame varies as τ varies. In our inertial frame, the fluid element or the particle is moving at speed v at an instance. Then, the mass density in our frame is $\gamma^2\rho$, where γ is the Lorentz factor given above. Here, one factor of γ is from the length contraction, hence a smaller volume, and the other is from the mass of the moving particle. We naturally consider the rank two tensor, the energy momentum tensor, given by,

$$T^{\mu\nu} = \rho u^\mu u^\nu.$$

Here $T^{00} = \rho u^0 u^0 = \gamma^2 \rho c^2$ is the energy density,

$$c\,T^{0i} = \gamma^2 \rho c^2 u^i = \frac{(\gamma^2 \rho A\, dx^i)c^2}{Ad\tau},$$

is the energy flux in the i-th direction, $\frac{1}{c}T^{i0} = \gamma^2 \rho u^i$ is the density of the i-th component of the momentum and $T^{ij} = \gamma^2 \rho u^i u^j$ is the flux of the i-th component of the momentum in the j-th direction. Notice that $T^{\mu\nu}$ is a symmetric tensor. Then, our Poisson equation $\nabla^2\Phi = 4\pi G\rho$ becomes the 00-th component of the "tensor-like" equation

$$u^\mu u^\nu \nabla^2 \Phi = 4\pi G\, T^{\mu\nu},$$

where both sides have the unit of c^4/m^2 in SI.

Another idea of General Relativity is that the gravity can be explained geometrically. Imagine a mass in an otherwise empty space. In view of Newton's first law of motion we, in an inertial frame, expect that the mass will move at a constant velocity in a straight line. As soon as we introduce a massive object not far from our mass, our mass will feel a Newton's gravitational attraction, thereby it will move in a curved path with acceleration. Einstein thought that the presence of a massive object distorted the flat Minkowski spacetime $(\mathbb{R}^4, \eta_{\mu\nu})$ into a Lorentzian manifold $(\mathbb{R}^4, g_{\mu\nu})$ and that our mass would still move as straight as possible in this "curved" spacetime, that is, it would move along a geodesic. The geodesics, in this Lorentzian manifold, are not necessarily straight lines in our frame. That's why we observe gravitational attractions. Once we have a metric $g_{\mu\nu}$, we can think of its Levi-Civita connection

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}),$$

thus, the covariant derivative of a tensor $h^\mu{}_\nu$

$$\nabla_\alpha h^\mu{}_\nu = \partial_\alpha h^\mu{}_\nu + \Gamma^\mu{}_{\alpha\beta} h^\beta{}_\nu - \Gamma^\beta{}_{\nu\alpha} h^\mu{}_\beta$$

and the derivative of a vector field $w^\mu(\lambda)$ along a curve $x^\mu(\lambda)$

$$\frac{Dw^\mu}{D\lambda} = \frac{dw^\mu}{d\lambda} + w^\alpha \Gamma^\mu{}_{\alpha\beta} \frac{dx^\beta}{d\lambda}.$$

Here, $h^\mu{}_\nu$ is a tensor in the sense that if $h^\mu{}_\nu$ is the component of the tensor in the coordinate x^μ, then

$$h^{\mu'}{}_{\nu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}} h^\mu{}_\nu$$

is the component of the same tensor in the coordinate $x^{\mu'}$. Notice the minus sign in front of the connection coefficient corresponding to the covariant index ν. One can check that $\nabla_\alpha h^\mu{}_\nu$ is a tensor if $h^\mu{}_\nu$ is. Also, we have $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$ and $\nabla_\alpha g_{\mu\nu} = 0$, that is, our connection is torsion-free and metric compatible.

We define that the curve $x^\mu(\tau)$ of a particle is a geodesic if its four-velocity $u^\mu = dx^\mu/d\tau$ along the path doesn't change, that is,

$$\frac{Du^\mu}{D\tau} = \frac{d^2x^\mu}{d\tau^2} + u^\alpha \Gamma^\mu{}_{\alpha\beta} u^\beta = 0.$$

One can define a geodesic in a more general way so that it can include light-like geodesics. In that case one uses affine parameters instead of proper time parametrization.

Weak gravitational fields and the Newtonian limit We ask the question: how is the gravitational potential Φ related to the metric $g_{\mu\nu}$? Let's assume the Newtonian approximation, that is, our Lorentzian manifold $(\mathbb{R}^4, g_{\mu\nu})$ is not much different from the flat spacetime, the metric $g_{\mu\nu}$ is static and our particle moves

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \partial_0 g_{\mu\nu} = 0 \quad \text{and} \quad \left|\frac{dx^i}{dt}\right| \ll c \text{ for } i = 1, 2, 3.$$

Here we assume that $h_{\mu\nu}$ and $\partial_\alpha h_{\mu\nu}$ are small as well. Of course we postulate that our particle follows a geodesic. Since $dx^i/d\tau \ll c\,dt/d\tau$, our geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{00} c^2 \left(\frac{dt}{d\tau}\right)^2 = 0$$

in first order. Observe that $\partial_0 g_{\mu\nu} = 0$ implies

$$\Gamma^\mu{}_{00} = \frac{1}{2} g^{\mu\nu} (\partial_0 g_{0\nu} + \partial_0 g_{0\nu} - \partial_\nu g_{00}) = -\frac{1}{2} g^{\mu\nu} \partial_\nu g_{00} = -\frac{1}{2} \eta^{\mu\nu} \partial_\nu h_{00},$$

hence, $\Gamma^0{}_{00} = 0$ and $\Gamma^i{}_{00} = -\frac{1}{2}\delta^{ij}\partial_j h_{00}$. Notice that we are using the convention $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Thus the geodesic equations become

$$\frac{d^2t}{d\tau^2} = 0 \quad \text{and} \quad \frac{d^2x^i}{d\tau^2} = \frac{1}{2} c^2 \left(\frac{dt}{d\tau}\right)^2 \partial^i h_{00},$$

from which we see that $\frac{dt}{d\tau}$ is a constant, hence that the gravitational acceleration is

$$-\nabla\Phi = \frac{d^2\mathbf{r}}{dt^2} = \frac{1}{2}c^2\nabla h_{00}.$$

Since Φ tends to zero at infinity, we have $h_{00} = -2\Phi/c^2$, or

$$g_{00} = -1 - \frac{2\Phi}{c^2}.$$

Thus our Poisson equation $\nabla^2\Phi = 4\pi G\rho$ becomes

$$-\nabla^2 g_{00} = \frac{2}{c^2}\nabla^2\Phi = \frac{2}{c^2}4\pi G\rho = \frac{8\pi G}{c^4}\rho\gamma^2c^2 = \frac{8\pi G}{c^4}\rho u_0 u_0 = \frac{8\pi G}{c^4}T_{00},$$

where we used the low velocity limit $\gamma = 1$. We want to find a rank two tensor $G_{\mu\nu}$ such that $G_{00} = -\nabla^2 g_{00}$ in the weak gravitational field approximation. Then the sought after equation would be $G_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$.

Einstein's field equations The curvature is how much the spacetime deviates from being flat and one way of measuring it is to compute the difference between two mixed second order derivatives, that is, the Riemann curvature tensor $R^{\alpha}{}_{\beta\mu\nu}$ is defined by

$$[\nabla_\nu, \nabla_\mu]w_\beta = w_\alpha R^{\alpha}{}_{\beta\mu\nu}.$$

In Exercise 1 we check that $R^{\alpha}{}_{\beta\mu\nu}$ is indeed a tensor and

$$R^{\alpha}{}_{\beta\mu\nu} = (\partial_\mu\Gamma^{\alpha}{}_{\nu\beta} + \Gamma^{\alpha}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\beta}) - (\mu \leftrightarrow \nu).$$

From the Riemann tensor we define the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R by

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \quad \text{and} \quad R = R^{\mu}{}_{\mu}.$$

In Exercise 2 we derive the Bianchi identity

$$\nabla_{[\gamma}R_{\alpha\beta]\mu\nu} = 0, \quad \text{where} \quad R_{\alpha\beta\mu\nu} = g_{\alpha\eta}R^{\eta}{}_{\beta\mu\nu}.$$

In the same exercise we see that the Bianchi identity implies $\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla^\nu R$.

Suppose that the gravity is expressed by the equation

$$R_{\mu\nu} + aRg_{\mu\nu} = \kappa T_{\mu\nu}.$$

Conservation of energy and momentum implies $\nabla^\mu T_{\mu\nu} = 0$, which implies that $a = \frac{1}{2}$. Since

$$\kappa T = \kappa g^{\mu\nu}T_{\mu\nu} = g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = R - \frac{1}{2}R\delta^{\mu}_{\mu} = -R$$

we can write the above equation as

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right).$$

In order to fix the constant κ we again use the Newtonian approximation, that is,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \partial_0 g_{\mu\nu} = 0 \quad \text{and} \quad \left|\frac{dx^i}{dt}\right| \ll c \text{ for } i = 1, 2, 3.$$

As before we assume that $h_{\mu\nu}$ and $\partial_\alpha h_{\mu\nu}$ are small. Since we consider weak gravity we are assuming that $T_{\mu\nu}$ is small as well. We also consider a perfect fluid with negligible pressure, hence $T^{\mu\nu} = \rho u^\mu u^\nu$. Now we compute both sides of

$$R_{00} = \kappa \left(T_{00} - \frac{1}{2} T g_{00}\right).$$

First, in first order, we have

$$\begin{aligned} R_{00} = R^\alpha_{\ 0\alpha 0} &= \partial_\alpha \Gamma^\alpha_{\ 00} - \partial_0 \Gamma^\alpha_{\ 0\alpha} - \Gamma^\alpha_{\ 0\beta}\Gamma^\beta_{\ 0\alpha} + \Gamma^\alpha_{\ \alpha\beta}\Gamma^\beta_{\ 00} \\ &= \partial_i \Gamma^i_{\ 00} = \frac{1}{2}\partial_i g^{i\beta}(\partial_0 g_{\beta 0} + \partial_0 g_{0\beta} - \partial_\beta g_{00}) \\ &= -\frac{1}{2}\eta^{i\beta}\partial_i\partial_\beta h_{00} = -\frac{1}{2}\delta^{ij}\partial_j h_{00} = -\frac{1}{2}\nabla^2 h_{00}. \end{aligned}$$

Second, $T_{00} = \gamma^2 \rho c^2 = \rho c^2$, $T = \rho g_{\mu\nu} u^\mu u^\nu = \gamma^2 \rho(-c^2 - v^2) = -\rho c^2$ and $T g_{00} = (-\rho c^2)(-1 + h_{00}) = \rho c^2$. Hence, using $h_{00} = -2\Phi/c^2$, we get

$$\begin{aligned} 4\pi G \rho = \nabla^2 \Phi = -\frac{c^2}{2}\nabla^2 h_{00} = c^2 R_{00} = c^2 \kappa \left(T_{00} - \frac{1}{2} T g_{00}\right) \\ = c^2 \kappa (\rho c^2 - \frac{1}{2}\rho c^2) = \frac{1}{2}\kappa \rho c^4, \end{aligned}$$

from which we get

$$\kappa = \frac{8\pi G}{c^4}$$

and the Einstein equation of gravity becomes

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

In Exercise 3 we consider Einstein's equation with the cosmological constant and the relationship with the vacuum energy.

Schwarzschild Solution Let's consider a solution of the Einstein equation outside a static spherical body. Thus, the equation becomes

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) = 0.$$

Since the source is static the metric is invariant under the time inversion $t \to -t$, hence there is no mixed terms like $c\,dt\,dx^i$ in the metric. Using the spherical symmetry we can use the coordinate $x^\mu = (ct, \bar{r}, \theta, \phi)$ and write the metric as

$$ds^2 = -e^{2A} c^2 dt^2 + e^{2\bar{B}} d\bar{r}^2 + e^{2C} \bar{r}^2 d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2,$$

where $A, \bar{B}, C$ are functions of $\bar{r}$. Now, we use a new coordinate $r = e^C \bar{r}$ which satisfies

$$dr^2 = \left(1 + \bar{r}\frac{dC}{d\bar{r}}\right)^2 d\bar{r}^2.$$

Then, with

$$e^{2B} = \left(1 + \bar{r}\frac{dC}{dr}\right)^{-2} e^{2\bar{B}-2C}$$

we get

$$ds^2 = -e^{2A}c^2dt^2 + e^{2B}dr^2 + r^2 d\Omega^2,$$

where A, B are now functions of r. Here we identify that $g_{00} = -e^{2A}$, $g_{11} = e^{2B}$, $g_{22} = r^2$, $g_{33} = r^2 \sin^2\theta$ and $g^{00} = -e^{-2A}$, $g^{11} = e^{-2B}$, $g^{22} = r^{-2}$, $g^{33} = r^{-2}\sin^{-2}\theta$. Thus, we have

$$\Gamma^0{}_{01} = \frac{1}{2}g^{0\alpha}(\partial_0 g_{\alpha 1} + \partial_1 g_{0\alpha} - \partial_\alpha g_{01}) = \frac{dA}{dr} = A'$$

and similarly we have

$$\Gamma^1{}_{00} = A'e^{2(A-B)}, \quad \Gamma^1{}_{11} = B', \quad \Gamma^1{}_{22} = -re^{-2B}, \quad \Gamma^1{}_{33} = -re^{-2B}\sin^2\theta$$
$$\Gamma^2{}_{12} = \frac{1}{r}, \quad \Gamma^2{}_{33} = -\sin\theta\cos\theta, \quad \Gamma^3{}_{13} = \frac{1}{r}, \quad \Gamma^3{}_{23} = \cot\theta,$$

where the symmetry of $\Gamma^\alpha{}_{\mu\nu}$ is assumed and all the other terms are zero. Now we have

$$\begin{aligned} R_{00} &= \partial_\alpha \Gamma^\alpha{}_{00} - \partial_0 \Gamma^\alpha{}_{\alpha 0} + \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{00} - \Gamma^\alpha{}_{0\beta}\Gamma^\beta{}_{\alpha 0} \\ &= \left(A'' + (A')^2 - A'B' + \frac{2}{r}A'\right)e^{2(A-B)}. \end{aligned}$$

Similarly we get

$$R_{11} = -A'' - (A')^2 + A'B' + \frac{2}{r}B', \; R_{22} = 1 + [r(B' - A') - 1]e^{-2B}, \; R_{33} = R_{22}\sin^2\theta.$$

From $R_{00} = R_{11} = 0$ we get $A' + B' = 0$, hence $A + B$ is a constant. Since we expect the space to be flat at infinite we have $A = -B$. Then from $R_{22} = 0$ we get $e^{2B} + 2rB' = 1$, or $-g_{00} = e^{2B} = 1 + C/r$ for a constant C. In the weak-field limit we have $h_{00} = -2\Phi/c^2$, hence

$$-\frac{C}{r} = h_{00} = -\frac{2}{c^2}\Phi = \left(-\frac{2}{c^2}\right)\left(-\frac{GM}{r}\right) = \frac{2GM}{c^2}\frac{1}{r}.$$

Thus we have $C = -2GM/c^2$ and the Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}dr^2 + r^2 d\Omega^2.$$

If we consider a charged rotating body with mass M, electric charge Q and angular momentum J, then it is known that we get the Kerr-Newman solution with the metric in Boyer-Lindquist coordinates (ct, r, θ, ϕ) given by:

$$ds^2 = -\frac{\Delta}{\rho^2}\left(c\,dt - a\sin^2\theta\,d\theta\right)^2 + \rho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right)$$
$$+\frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\phi - ac\,dt\right]^2,$$

where $a = \frac{J}{Mc}$ is the rotation parameter and

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 - \frac{2GMr}{c^2} + \frac{Q^2G}{4\pi\epsilon_0 c^4}.$$

Observe that we recover the Schwarzschild solution when both Q and J are zero. The case $J = 0$ is called the Reissner-Nordström solution and the case $Q = 0$ is called the Kerr solution.

14.2 Field Theory Approach

We consider an oriented manifold M of dimension n on which we have matter fields ϕ. Let $g_{\mu\nu}$ be a metric with Lorentzian signature. Observe that covariant derivative, Riemannian tensor and all other tensor calculus can be generalized to this Lorentzian spacetime $(M, g_{\mu\nu})$. We consider two Lagrangian densities

$$\mathcal{L}_{\mathrm{EH}}(g_{\mu\nu}) = R - 2\Lambda \quad \text{and} \quad \mathcal{L}_{\mathrm{matter}}(g_{\mu\nu}, \phi)$$

and the action

$$S = S(g_{\mu\nu}) = S_{\mathrm{EH}} + S_{\mathrm{matter}},$$

where the Einstein-Hilbert action and the matter action are given by

$$S_{EH} = \frac{1}{2\kappa}\int d^n x\,\sqrt{-g}\,\mathcal{L}_{\mathrm{EH}} \quad \text{and} \quad S_{\mathrm{matter}} = \int d^n x\,\sqrt{-g}\,\mathcal{L}_{\mathrm{matter}}.$$

Here $\kappa = \frac{8\pi G}{c^4}$, g is the determinant of the metric tensor $g_{\mu\nu}$ and Λ is the cosmological constant. We also assume that $\mathcal{L}_{\mathrm{matter}}$ is a tensor. Since the volume form $dx^n\sqrt{-g}$ is diffeomorphism invariant, so is our action.

Our goal in this section is to show that the above action yields, as the Euler-Lagrange equation, the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu},$$

if the energy-momentum tensor $T_{\mu\nu}$ is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{matter}})}{\delta g^{\mu\nu}}.$$

Since

$$\delta S = \delta S_{\mathrm{EH}} + \int d^n x \frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{matter}})}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \delta S_{\mathrm{EH}} - \int d^n x\sqrt{-g}\,\frac{1}{2}\,T_{\mu\nu}\delta\, g^{\mu\nu},$$

it suffices to check

$$\delta S_{\text{EH}} = \frac{1}{2\kappa}\int d^n x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} + \Lambda g_{\mu\nu}\right).$$

In fact, we have

$$\begin{aligned} 2\kappa\,\delta S_{\text{EH}} &= \int d^n x\delta(\sqrt{-g}(R-2\Lambda)) \\ &= \int d^n x\delta(\sqrt{-g})\,(R-2\Lambda)) + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}(\delta R_{\mu\nu})g^{\mu\nu}. \end{aligned}$$

In Exercise 4 we check that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.$$

Hence, the integral of the first two terms becomes

$$\int d^n x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}.$$

Also, in the same exercise, we check that there is a tensor $V^\alpha = V^\alpha(g_{\mu\nu})$ such that $(\delta R_{\mu\nu})g^{\mu\nu} = \nabla_\alpha V^\alpha$, thus

$$\int d^n x\sqrt{-g}(\delta R_{\mu\nu})g^{\mu\nu} = \int d^n x\sqrt{-g}\nabla_\alpha V^\alpha = 0$$

if M is closed. Even if M has boundary, the above integral is zero by the Stokes theorem if we are varying Lorentzian metrics $g_{\mu\nu}$ so that $\delta g_{\mu\nu}$ and $\nabla_\alpha \delta g_{\mu\nu}$ vanish on the boundary of M, hence that V^α also vanishes on the boundary. Physically it may not be natural to assume vanishing $\nabla_\alpha \delta g_{\mu\nu}$. In that case we can still get the same Einstein equation by adding the Gibbons-Hawking-York action term S_{GHY} to our action. The action is given by

$$S_{\text{GHY}} = \frac{1}{\kappa}\int_{\partial M} d^{n-1}y\,\sqrt{|h|}\,K,$$

where $K = h^{ab}K_{ab}$ is the trace of the extrinsic curvature of the boundary, defined as $K_{ab} = h_a{}^\mu h_b{}^\nu \nabla_\mu n_\nu$, with h_{ab} the induced metric and n_ν the normal vector to the boundary.

14.3 Exercises

1. We define $R^\alpha{}_{\beta\mu\nu}$ by the equation

$$[\nabla_\nu, \nabla_\mu]w_\beta = w_\alpha R^\alpha{}_{\beta\mu\nu},$$

which holds for every tensor w_α.

(a) Show that $R^\alpha{}_{\beta\mu\nu}$ is a tensor.

(b) Using the definition of covariant derivative derive that

$$R^\alpha{}_{\beta\mu\nu} = (\partial_\mu \Gamma^\alpha{}_{\nu\beta} + \Gamma^\alpha{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\beta}) - (\mu \leftrightarrow \nu).$$

2. (a) Suppose that we are falling in a gravitational field. If we use the frame of our path, then we will find that all objects near us are falling at the same rate. This follows from

$$\mathbf{F} = m_I \mathbf{a} = -m_G \nabla\Phi \quad \text{or} \quad \frac{d^2\mathbf{r}}{dt^2} = -\frac{m_G}{m_I}\nabla\Phi$$

and the experimental fact that m_G/m_I is the same for every object. Here m_I is the inertia mass and m_G is the gravitational mass. So, there is no acceleration in the relative motion between objects around us, that is, we are in a flat spacetime if we use the freely falling coordinate. Mathematically, in a Lorentzian manifold $(\mathbb{R}^4, g_{\mu\nu})$ show the following: for each event P there is a coordinate $x^{\mu'}$ near P such that

$$g_{\mu'\nu'} = \eta_{\mu'\nu'} \quad \text{and} \quad \Gamma^{\alpha'}_{\ \mu'\nu'} = 0$$

at P. This coordinate is called a normal coordinate at P. [Hint. Start with any coordinate x^μ near P and define a new coordinate $x^{\alpha'}$ by

$$x^\mu(x^{\alpha'}) = x^\mu_P + \left(\frac{\partial x^\mu}{\partial x^{\alpha'}}\right)_P (x^{\alpha'} - x^{\alpha'}_P) + \frac{1}{2}\left(\frac{\partial^2 x^\mu}{\partial x^{\alpha'}\partial x^{\beta'}}\right)_P (x^{\alpha'} - x^{\alpha'}_P)(x^{\beta'} - x^{\beta'}_P)$$

and count the number of coefficients in the above so that we can impose $g_{\mu'\nu'} = \eta_{\mu'\nu'}$ and $\partial_{\alpha'} g_{\mu'\nu'} = 0$ at P.]

(b) Prove that

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}, \quad R_{\alpha[\beta\mu\nu]} = 0.$$

(c) Prove the Bianchi identity

$$\nabla_{[\gamma} R_{\alpha\beta]\mu\nu} = 0.$$

(d) Show that $\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$.

3. Consider the Einstein equation with cosmological constant

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}.$$

(a) Show that the above equation is consistent with the energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$. Show that $-R + 4\Lambda = \kappa T$.

(b) In the weak-field Newtonian approximation show that the above equation becomes

$$\nabla^2\Phi = 4\pi G\rho - \Lambda c^2.$$

Also, with $\rho(\mathbf{r}) = M\delta(\mathbf{r})$ show that

$$-\nabla\Phi = -\frac{GM}{r^3}\mathbf{r} + \frac{c^2\Lambda}{3}\mathbf{r}.$$

(c) Write the general metric for a spacetime with homogeneous and isotropic spatial structure with a constant curvature K. Using a perfect fluid distribution, impose the Einstein equation with cosmological constant to get Friedmann equations. When is this universe expanding?

4. Let $g_{\mu\nu}$ be a Lorentzian metric on a manifold M.

(a) Show that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.$$

(b) Show that there is a tensor $V^\alpha = V^\alpha(g_{\mu\nu})$ such that $(\delta R_{\mu\nu})g^{\mu\nu} = \nabla_\alpha V^\alpha$.

15 Gravity (cont): Palatini–Cartan

15.1 Palatini Action

In the previous section we fixed an oriented manifold M of dimension n, a matter field ϕ on M and a function $\mathcal{L}_{\text{matter}} = \mathcal{L}_{\text{matter}}(\cdot, \phi)$. And for each Lorentzian metric $g_{\mu\nu}$ on M we considered the action

$$S[g_{\mu\nu}] = \frac{1}{2\kappa}\int_M d^n x\sqrt{-g}(R - 2\Lambda) + \int_M d^n x\sqrt{-g}\mathcal{L}_{\text{matter}}(g_{\mu\nu}, \phi).$$

Here $g = \det(g_{\mu\nu})$, $\kappa = 8\pi G/c^4$, Λ is the cosmological constant, $\mathcal{L}_{\text{matter}}(g_{\mu\nu}, \phi)$ is a Lagrangian density and $R = R(g_{\mu\nu})$ is the scalar curvature determined by the metric $g_{\mu\nu}$. Imposing that $g_{\mu\nu}$ is a critical metric with respect to the action S we showed that $g_{\mu\nu}$ satisfies the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu},$$

where the energy momentum tensor $T_{\mu\nu}$ is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\text{matter}})}{\delta g^{\mu\nu}}.$$

Now we introduce the Palatini formalism: a good reference is Lecture Notes on General Relativity by Matthias Blau. In this formalism we view the metric $g_{\mu\nu}$ and the connection $\Gamma^\alpha_{\ \mu\nu}$ as independent variables, where the Ricci tensor is defined from the connection as

$$R_{\mu\nu} = R_{\mu\nu}(\Gamma^\alpha_{\ \beta\gamma}) = \partial_\alpha\Gamma^\alpha_{\ \nu\mu} + \Gamma^\alpha_{\ \alpha\beta}\Gamma^\beta_{\ \nu\mu} - (\text{lower } \alpha \leftrightarrow \nu).$$

Thus, the Palatini action is defined as

$$\begin{aligned} S = S[g_{\mu\nu}, \Gamma^\alpha_{\ \beta\gamma}] = {} & \frac{1}{2\kappa}\int_M d^n x\sqrt{-g}\left(g^{\mu\nu}R_{\mu\nu}(\Gamma^\alpha_{\ \beta\gamma}) - 2\Lambda\right) \\ & + \int_M d^n x\sqrt{-g}\,\mathcal{L}_{\text{matter}}(g_{\mu\nu}, \phi). \end{aligned}$$

Then, as in the previous section we get the variation of the action

$$\begin{aligned}
\delta S = &\frac{1}{2\kappa}\int_M d^n x(\delta\sqrt{-g})\,(R-2\Lambda) \\
&+\frac{1}{2\kappa}\int_M d^n x\sqrt{-g}\,[(\delta g^{\mu\nu})R_{\mu\nu}+g^{\mu\nu}\delta R_{\mu\nu}] \\
&+\int_M d^n x\delta\left(\sqrt{-g}\,\mathcal{L}_{\text{matter}}(g_{\mu\nu},\phi)\right) \\
=&\frac{1}{2\kappa}\int_M d^n x\left(-\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}\right)(R-2\Lambda) \\
&+\frac{1}{2\kappa}\int_M dx^n\sqrt{-g}\,R_{\mu\nu}\delta g^{\mu\nu}+\frac{1}{2\kappa}\int_M d^n x\sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu} \\
&+\int_M d^n x\left(-\frac{1}{2}\sqrt{-g}\,T_{\mu\nu}\right)\delta g^{\mu\nu} \\
=&\frac{1}{2\kappa}\int_M d^n x\sqrt{-g}\,\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}-\kappa T_{\mu\nu}\right)\delta g^{\mu\nu} \\
&+\frac{1}{2\kappa}\int_M d^n x\sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}.
\end{aligned}$$

We will show that

$$\frac{1}{2\kappa}\int_M d^n x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}=\frac{1}{2\kappa}\int_M d^n x\sqrt{-g}A_\alpha^{\beta\gamma}\delta\Gamma^\alpha_{\ \beta\gamma}$$

for some expression $A_\alpha^{\beta\gamma}=A_\alpha^{\beta\gamma}(g_{\mu\nu},C^\mu_{\ \nu\lambda})$, where

$$C^\mu_{\ \nu\lambda}=\Gamma^\mu_{\ \nu\lambda}-\tilde\Gamma^\mu_{\ \nu\lambda}\quad\text{and}\quad\tilde\Gamma^\mu_{\ \nu\lambda}=\frac{1}{2}g^{\mu\sigma}\left(\partial_\nu g_{\sigma\lambda}+\partial_\lambda g_{\nu\sigma}-\partial_\sigma g_{\nu\lambda}\right).$$

Thus, critical metric $g_{\mu\nu}$ and connection $\Gamma^\alpha_{\ \mu\nu}$ satisfies the equations

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}-\kappa T_{\mu\nu}\quad\text{and}\quad A_\alpha^{\beta\gamma}(g_{\mu\nu},C^\mu_{\ \nu\lambda})=0.$$

Clearly the first is the Einstein equation. Assuming that $\Gamma^\alpha_{\ \beta\gamma}=\Gamma^\alpha_{\ \gamma\beta}$, we will show that the second equation implies that $\Gamma^\alpha_{\ \beta\gamma}$ is the Levi-Civita connection of the metric $g_{\mu\nu}$. Indeed, as in Exercise 4 of Section 14, we compute

$$\begin{aligned}
\delta R_{\mu\nu}&=\delta(\partial_\lambda\Gamma^\lambda_{\ \nu\mu}+\Gamma^\lambda_{\ \lambda\eta}\Gamma^\eta_{\ \nu\mu})-(\lambda\leftrightarrow\nu) \\
&=\partial_\lambda\delta\Gamma^\lambda_{\ \nu\mu}+(\delta\Gamma^\lambda_{\ \lambda\eta})\Gamma^\eta_{\ \nu\mu}+\Gamma^\lambda_{\ \lambda\eta}\delta\Gamma^\eta_{\ \nu\mu}-\partial_\nu\delta\Gamma^\lambda_{\ \lambda\mu}-(\delta\Gamma^\lambda_{\ \nu\eta})\Gamma^\eta_{\ \lambda\mu}-\Gamma^\lambda_{\ \nu\eta}\delta\Gamma^\eta_{\ \lambda\mu}.
\end{aligned}$$

Let $\tilde\nabla_\lambda$ be the covariant derivative with respect the Levi-Civita connection $\tilde\Gamma^\alpha_{\ \beta\gamma}$. Then we have $\partial_\lambda\delta\Gamma^\lambda_{\ \nu\mu}=\tilde\nabla_\lambda\delta\Gamma^\lambda_{\ \nu\mu}-\tilde\Gamma^\lambda_{\ \lambda\eta}\delta\Gamma^\eta_{\ \nu\mu}+\tilde\Gamma^\eta_{\ \lambda\nu}\delta\Gamma^\lambda_{\ \eta\mu}+\tilde\Gamma^\eta_{\ \lambda\mu}\delta\Gamma^\lambda_{\ \nu\eta}$, hence

$$\begin{aligned}
\delta R_{\mu\nu}=&\,\tilde\nabla_\lambda\delta\Gamma^\lambda_{\ \nu\mu}-\tilde\Gamma^\lambda_{\ \lambda\eta}\delta\Gamma^\eta_{\ \nu\mu}+\tilde\Gamma^\eta_{\ \lambda\nu}\delta\Gamma^\lambda_{\ \eta\mu}+\tilde\Gamma^\eta_{\ \lambda\mu}\delta\Gamma^\lambda_{\ \nu\eta}+(\delta\Gamma^\lambda_{\ \lambda\eta})\Gamma^\eta_{\ \nu\mu}+\Gamma^\lambda_{\ \lambda\eta}\delta\Gamma^\eta_{\ \nu\mu} \\
&-\tilde\nabla_\nu\delta\Gamma^\lambda_{\ \lambda\mu}+\tilde\Gamma^\lambda_{\ \nu\eta}\delta\Gamma^\eta_{\ \lambda\mu}-\tilde\Gamma^\eta_{\ \nu\lambda}\delta\Gamma^\lambda_{\ \eta\mu}-\tilde\Gamma^\eta_{\ \nu\mu}\delta\Gamma^\lambda_{\ \lambda\eta}-(\delta\Gamma^\lambda_{\ \nu\eta})\Gamma^\eta_{\ \lambda\mu}-\Gamma^\lambda_{\ \nu\eta}\delta\Gamma^\eta_{\ \lambda\mu} \\
=&\,\tilde\nabla_\lambda\delta\Gamma^\lambda_{\ \nu\mu}-\tilde\nabla_\nu\delta\Gamma^\lambda_{\ \lambda\mu}+(\delta\Gamma^\lambda_{\ \lambda\eta})C^\eta_{\ \nu\mu}+C^\lambda_{\ \lambda\eta}\delta\Gamma^\eta_{\ \nu\mu}-(\delta\Gamma^\lambda_{\ \nu\eta})C^\eta_{\ \lambda\mu}-C^\lambda_{\ \nu\eta}\delta\Gamma^\eta_{\ \lambda\mu}.
\end{aligned}$$

Thus,

$$g^{\mu\nu}\delta R_{\mu\nu} = \tilde{\nabla}_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\ \nu\mu}) - \tilde{\nabla}_\nu(g^{\mu\nu}\delta\Gamma^\lambda_{\ \lambda\mu}) + A_\alpha^{\ \beta\gamma}\delta\Gamma^\alpha_{\ \beta\gamma},$$

where

$$\begin{aligned} A_\alpha^{\ \beta\gamma}\delta\Gamma^\alpha_{\ \beta\gamma} &= g^{\mu\nu}\left((\delta\Gamma^\lambda_{\ \lambda\eta})C^\eta_{\ \nu\mu} + C^\lambda_{\ \lambda\eta}\delta\Gamma^\eta_{\ \nu\mu} - (\delta\Gamma^\lambda_{\ \nu\eta})C^\eta_{\ \lambda\mu} - C^\lambda_{\ \nu\eta}\delta\Gamma^\eta_{\ \lambda\mu}\right) \\ &= \left(g^{\mu\nu}\delta^\beta_\alpha C^\gamma_{\ \nu\mu} + g^{\beta\gamma}C^\lambda_{\ \lambda\alpha} - g^{\beta\mu}C^\gamma_{\ \alpha\mu} - g^{\nu\gamma}C^\beta_{\ \nu\alpha}\right)\delta\Gamma^\alpha_{\ \beta\gamma}. \end{aligned}$$

In the above we made index changes like

$$\delta\Gamma^\lambda_{\ \lambda\eta}C^\eta_{\ \nu\mu} = \delta^\lambda_\sigma\delta\Gamma^\sigma_{\ \lambda\eta}C^\eta_{\ \nu\mu} = \delta^\beta_\alpha C^\gamma_{\ \nu\mu}\delta\Gamma^\alpha_{\ \beta\gamma}.$$

In the above we have divergence terms and $\delta\Gamma^\alpha_{\ \beta\gamma} = 0$ on ∂M, hence we get

$$\frac{1}{2\kappa}\int_M d^nx\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \frac{1}{2\kappa}\int_M d^nx\sqrt{-g}A_\alpha^{\ \beta\gamma}\delta\Gamma^\alpha_{\ \beta\gamma}$$

and the Euler-Lagrange equations $A_\alpha^{\ \beta\gamma} = 0$. Contracting this equation with $g_{\beta\gamma}$ and using the symmetry $C^\alpha_{\ \beta\gamma} = C^\alpha_{\ \gamma\beta}$, we get

$$\begin{aligned} 0 &= g_{\beta\gamma}A_\alpha^{\ \beta\gamma} \\ &= g_{\beta\gamma}\left(g^{\mu\nu}\delta^\beta_\alpha C^\gamma_{\ \nu\mu} + g^{\beta\gamma}C^\lambda_{\ \lambda\alpha} - g^{\beta\mu}C^\gamma_{\ \alpha\mu} - g^{\nu\gamma}C^\beta_{\ \nu\alpha}\right) \\ &= C_{\alpha\ \mu}^{\ \mu} + 4C^\lambda_{\ \lambda\alpha} - C^\gamma_{\ \alpha\gamma} - C^\nu_{\ \nu\alpha} \\ &= C_{\alpha\ \lambda}^{\ \lambda} + 2C^\lambda_{\ \lambda\alpha}. \end{aligned}$$

Similarly, multiplying δ^α_β to the same equation, we get

$$2C_{\alpha\ \lambda}^{\ \lambda} + C^\lambda_{\ \lambda\alpha} = 0.$$

Hence $C_{\alpha\ \lambda}^{\ \lambda} = C^\lambda_{\ \lambda\alpha} = 0$, which implies

$$\begin{aligned} 0 &= A_\alpha^{\ \beta\gamma} \\ &= g^{\mu\nu}\delta^\beta_\alpha C^\gamma_{\ \nu\mu} + g^{\beta\gamma}C^\lambda_{\ \lambda\alpha} - g^{\beta\mu}C^\gamma_{\ \alpha\mu} - g^{\nu\gamma}C^\beta_{\ \nu\alpha} \\ &= \delta^\beta_\alpha g^{\lambda\gamma}C_{\lambda\ \mu}^{\ \mu} + g^{\beta\gamma}C^\lambda_{\ \lambda\alpha} - g^{\beta\mu}C^\gamma_{\ \alpha\mu} - g^{\nu\gamma}C^\beta_{\ \nu\alpha} \\ &= -g^{\beta\mu}C^\gamma_{\ \alpha\mu} - g^{\nu\gamma}C^\beta_{\ \nu\alpha} \\ &= -(C^\beta_{\ \alpha\lambda}g^{\lambda\gamma} + C^\gamma_{\ \alpha\lambda}g^{\beta\lambda}). \end{aligned}$$

Now, since $\tilde{\nabla}_\alpha g^{\beta\gamma} = 0$, the compatibility of the connection $\Gamma^\alpha_{\ \beta\gamma}$ with respect to the metric $g_{\mu\nu}$ follows from

$$\nabla_\alpha g^{\beta\gamma} = \tilde{\nabla}_\alpha g^{\beta\gamma} + C^\beta_{\ \alpha\lambda}g^{\lambda\gamma} + C^\gamma_{\ \alpha\lambda}g^{\beta\lambda} = 0.$$

15.2 Tetrads and Spin Connections

Let's consider a four dimensional manifold M. Given a Lorentzian metric $g_{\mu\nu}$ on M we can find a local orthonormal basis e_a for the tangent space at each point in the manifold: this was shown in Exercise 2.(a) in Section 14. Thus writing $e_a = e^\mu{}_a \partial_\mu$ we have

$$e^\mu{}_a e^\nu{}_b g_{\mu\nu} = \eta_{ab},$$

the metric of flat Minkowski spacetime. Since $e^\mu{}_a$ is invertible it has the inverse $e_\mu{}^a$, which is called a tetrad. For tetrads and Cartan formalism see Theory of Gravitational Interactions by Maurizio Gasperini or Spacetime and Geometry by Sean Carroll. Observe that the original metric is recovered from the tetrad as

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}.$$

In Exercise 1 we show that $\det(e_\mu{}^a) = \sqrt{-g}$; our convention is that $\det(e_\mu{}^a) > 0$. In the tetrad $e_\mu{}^a$ the index μ is called a curved index, while a is a flat index. We can raise and lower flat indices by multiplying η^{ab} or η_{ab}. If $V = V^\mu \partial_\mu = V^a e_a$ is a vector field, then we have $V^a = V^\mu e_\mu{}^a$. Here V^a can be viewed as the e_a component of the projection of the vector V on the tangent Minkowski space. A local Lorentz transformation is a transformation $\Lambda^{a'}{}_a(x)$ such that

$$\Lambda^{a'}{}_a \Lambda^{b'}{}_b \eta_{a'b'} = \eta_{ab}.$$

Observe that the tetrad related to $g_{\mu\nu}$ is not unique; if $e_\mu{}^a$ is related to $g_{\mu\nu}$, then so is $e_\mu{}^{a'} = \Lambda^{a'}{}_a e_\mu{}^a$: see Exercise 1. For a tensor with mixed indices $T^\mu{}_a$ we have

$$T^{\mu'}{}_{a'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \Lambda_{a'}{}^a T^\mu{}_a,$$

where $\Lambda_{a'}{}^a$ is the inverse transformation of $\Lambda^{a'}{}_a$. The requirement of general covariance under coordinate transformations(diffeomorphisms) in a curved space–time manifold, thus translates—within the tetrads formalism—into the requirement of local Lorentz invariance.

Besides the tetrads $e_\mu{}^a$ we will consider spin connections $\omega_\mu{}^{ab}$ so that we can defines a new covariant derivative D_μ as

$$D_\mu A_\nu{}^a := \partial_\mu A_\nu{}^a - \Gamma^\lambda{}_{\mu\nu} A_\lambda{}^a + \omega_\mu{}^a{}_b A_\nu{}^b = \nabla_\mu A_\nu{}^a + \omega_\mu{}^a{}_b A_\nu{}^b.$$

Recall that, related to the metric $g_{\mu\nu}$ we have a special connection, that is, the Levi-Civita connection $\Gamma^\alpha{}_{\mu\nu}$, which is metric compatible and torsion-free. Similarly on our spin connection we impose "tetrad postualte" and "local Lorentz invariance", that is,

$$D_\mu e_\nu{}^a = 0 \quad \text{and} \quad D_\mu \eta_{ab} = 0.$$

The first requirement fixes the spin connection in terms of the tetrad and the Levi-Civita connection:

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma^\lambda{}_{\mu\nu} e_\lambda{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0,$$

which implies

$$\omega_\mu{}^{ab} = \left(\Gamma^\lambda{}_{\mu\nu} e_\lambda{}^a - \partial_\mu e_\nu{}^a\right) e^{\nu b}.$$

The second requirement yields

$$0 = D_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega_\mu{}^c{}_a \eta_{cb} - \omega_\mu{}^c{}_b \eta_{ca} = -(\omega_{\mu ba} + \omega_{\mu ab}),$$

showing that ω_μ^{ab} is anti-symmetric in a, b.

Similarly as above we get in Exercise 2 that

$$R^\alpha_{\ \beta\mu\nu} = e^\alpha{}_a e_\beta{}^b \left[\partial_\mu \omega_\nu{}^a{}_b - \omega_\mu{}^c{}_b \omega_\nu{}^a{}_c\right] - (\mu \leftrightarrow \nu).$$

We introduce Cartan formalism to the above tetrad and spin connection. For this we first make 1-forms:

$$e^a = e_\mu^{\ a} dx^\mu \quad \text{and} \quad \omega^{ab} = \omega_\mu{}^{ab} dx^\mu.$$

Cartan structure equations are

$$T^a = De^a := de^a + \omega^a{}_b \wedge e^b \quad \text{and} \quad R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^c{}_b.$$

So, in Cartan formalism we use the coframe 1-form, the spin-connection 1-form and the curvature 2-from, e^a, ω^{ab} and R^{ab}, instead of the metric, Levi-Civita connection and the Riemann tensor, $g_{\mu\nu}, \Gamma^\rho{}_{\mu\nu}$ and $R^\rho{}_{\sigma\mu\nu}$. We can check that the above usage of the curvature 2-orm R^{ab} is consistent with the Riemann curvature tensor $R^\alpha_{\ \beta\mu\nu}$ introduced earlier: see Exercise 3. Here T^a is called the torsion because

$$T_{\mu\nu}{}^\lambda = e^\lambda{}_a T_{\mu\nu}{}^a = e^\lambda{}_a \left(\partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b\right) - (\mu \leftrightarrow \nu) = \Gamma^\lambda_{\ \mu\nu} - \Gamma^\lambda_{\ \nu\mu},$$

where we used the formula for the Levi-Civita connection which is obtained from the tetrad postulate.

In Exercise 3 we check two Bianchi identities

$$DT^a = R^a{}_b \wedge e^b \quad \text{and} \quad DR^{ab} = 0.$$

One can also show that the above two identities are equivalent to our old identities

$$R_{\alpha[\beta\mu\nu]} = 0 \quad \text{and} \quad \nabla_{[\gamma} R_{\alpha\beta]\mu\nu} = 0.$$

Observe that the torsion-free condition implies

$$-de^a = \omega^a{}_b \wedge e^b,$$

which, along with the anti-symmetry of ω_{ab}, enables us to compute the spin connections ω_{ab}. As an example let's consider the Schwarzschild metric

$$ds^2 = -f^2 c^2 dt^2 + f^{-2} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2),$$

where

$$f = f(r) = \sqrt{1 - \frac{2GM}{c^2 r}}.$$

From $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ we can propose the tetrads

$$e^0 = f\, cdt,\ e^1 = f^{-1}\, dr,\ e^2 = r\ d\theta,\ e^3 = r \sin\theta\, d\phi.$$

Here

$$\begin{aligned}
de^0 &= f'\, dr \wedge c\, dt,\\
de^1 &= 0,\\
de^2 &= dr \wedge d\theta,\\
de^3 &= (dr \sin\theta + r\cos\theta\, d\theta) \wedge d\phi.
\end{aligned}$$

From

$$\begin{aligned}
de^3 &= \frac{f}{r} e^1 \wedge e^3 + \frac{\cot\theta}{r} e^2 \wedge e^3\\
&= -\omega^3{}_0 \wedge e^0 - \omega^3{}_1 \wedge e^1 - \omega^3{}_2 \wedge e^2
\end{aligned}$$

we propose that $\omega^{03} = \omega^3{}_0$ is proportional to e^0 and

$$\omega^{13} = \omega^1{}_3 = -\frac{f}{r} e^3,\ \ \omega^{23} = \omega^2{}_3 = -\frac{\cot\theta}{r} e^3.$$

Similar computation shows

$$\omega^{01} = \frac{GM}{c^2 r^2} \frac{1}{f}\, e^0,\ \omega^{02} = \omega^{03} = 0,\ \omega^{12} = -\frac{f}{r}\, e^2.$$

This describes the spin connection completely since it is anti-symmetric.

15.3 Palatini-Cartan Action

Recall that for a sourceless case the Einstein-Hilbert action takes the form

$$S = \frac{1}{2\kappa} \int_M dx^n \sqrt{-g}(R - 2\Lambda).$$

Now let's take the general tetrads and spin connection as our independent variables: we are assuming neither the torsion-free condition nor the tetrad postulate, but we assume that the spin connection is anti-symmetric. We will need some knowledge about tensor densities and Levi-Civita symbols.

First the Levi-Civita symbol of rank 4 is $\epsilon^{\mu\nu\rho\sigma}$ satisfying

$$\epsilon^{0123} = 1 \quad \text{and} \quad \epsilon^{\mu\nu\alpha\beta} = \epsilon^{[\mu\nu\alpha\beta]}.$$

Even though we used curved indices in the above definition we can equally work with flat indices. Observe that for a coordinate transformation matrix $J^\alpha{}_\mu$ with determinant J we have

$$J = J^0{}_\mu J^1_\nu J^2{}_\rho J^3{}_\sigma \epsilon^{\mu\nu\rho\sigma}.$$

Since we impose that $\epsilon^{\mu\nu\rho\sigma}$ take the same form in each coordinate, we have

$$\epsilon^{\alpha'\beta'\gamma'\delta'} = \epsilon^{\alpha\beta\gamma\delta} = J^{-1} J^\alpha{}_\mu J^\beta{}_\nu J^\gamma{}_\rho J^\delta{}_\sigma \epsilon^{\mu\nu\rho\sigma}.$$

Thus $\epsilon^{\mu\nu\rho\sigma}$ is a tensor density with weight -1. We know that $\sqrt{-g}$ is also a scalar density with weight -1. Therefore we have a tensor(tensor density with weight 0)

$$\eta^{\alpha\beta\gamma\delta} = \frac{\epsilon^{\alpha\beta\gamma\delta}}{\sqrt{-g}}.$$

In Exercise 4 we show that lower index tensor has the form $\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}$, where $\epsilon_{0123} = -1$ and $\epsilon_{\alpha\beta\gamma\delta} = \epsilon_{[\alpha\beta\gamma\delta]}$. In the same exercise we also check that

$$\eta^{\alpha\beta\gamma\delta}\eta_{\mu\nu\gamma\delta} = -2(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu), \;\; \eta^{\mu\nu\gamma\delta}\eta_{\mu\nu\gamma\delta} = -24.$$

Now we will show that

$$\begin{aligned}
S = S[e_\mu{}^a, \omega_\mu{}^{ab}] &= \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}(R - 2\Lambda) \\
&= -\frac{1}{4\kappa}\int_M d^4x\, \epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\left(\frac{1}{2}R_{\mu\nu}{}^{ab} - \frac{\Lambda}{6}e_\mu{}^a e_\nu{}^b\right) e_\alpha{}^c e_\beta{}^d \\
&= -\frac{1}{4\kappa}\int_M \epsilon_{abcd}\left(R^{ab} - \frac{\Lambda}{6}e^a\wedge e^b\right)\wedge e^c\wedge e^d.
\end{aligned}$$

Indeed we compute

$$\begin{aligned}
\eta^{\mu\nu\alpha\beta}e_\rho{}^a e_\sigma{}^b e_\alpha{}^c e_\beta{}^d \epsilon_{abcd} &= \eta^{\mu\nu\alpha\beta}\epsilon_{\rho\sigma\alpha\beta}(-e_0{}^a e_1{}^b e_2{}^c e_3{}^d)\epsilon_{abcd} \\
&= \eta^{\mu\nu\alpha\beta}\epsilon_{\rho\sigma\alpha\beta}\sqrt{-g} = \eta^{\mu\nu\alpha\beta}\eta_{\rho\sigma\alpha\beta} = -2(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho),
\end{aligned}$$

thus

$$\begin{aligned}
\epsilon^{\mu\nu\alpha\beta}e_\alpha{}^c e_\beta{}^d\epsilon_{abcd} &= \sqrt{-g}\eta^{\mu\nu\alpha\beta}e^\rho{}_a e^\sigma{}_b\epsilon_{abcd} = -2\sqrt{-g}e^\rho{}_a e^\sigma{}_b(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho) \\
&= -4\sqrt{-g}e^{[\mu}{}_a e^{\nu]}{}_b.
\end{aligned}$$

Then

$$\begin{aligned}
\sqrt{-g}R = \sqrt{-g}R_{\mu\nu}{}^{\mu\nu} &= \sqrt{-g}e^\mu{}_a e^\nu{}_b R_{\mu\nu}{}^{ab} = \sqrt{-g}e^{[\mu}{}_a e^{\nu]}{}_b R_{\mu\nu}{}^{ab} \\
&= -\frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}R_{\mu\nu}{}^{ab}e_\alpha{}^c e_\beta{}^d
\end{aligned}$$

and

$$\begin{aligned}
d^4x\sqrt{-g}R &= -\frac{1}{4}dx^0\wedge dx^1\wedge dx^2\wedge dx^3\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}R_{\mu\nu}{}^{ab}e_\alpha{}^c e_\beta{}^d \\
&= -\frac{1}{4}dx^\mu\wedge dx^\nu\wedge dx^\alpha\wedge dx^\beta\epsilon_{abcd}R_{\mu\nu}{}^{ab}e_\alpha{}^c e_\beta{}^d \\
&= -\frac{1}{2}\epsilon_{abcd}R^{ab}\wedge e^c\wedge e^d.
\end{aligned}$$

On the other hand, we have

$$\epsilon^{\mu\nu\alpha\beta}e_\mu{}^a e_\nu{}^b e_\alpha{}^c e_\beta{}^d \epsilon_{abcd} = \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\alpha\beta}(-e_0{}^a e_1{}^b e_2{}^c e_3{}^d)\epsilon_{abcd} = -24\sqrt{-g}$$

and

$$\begin{aligned} d^4x\sqrt{-g} &= -\frac{1}{24}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \epsilon^{\mu\nu\alpha\beta}e_\mu{}^a e_\nu{}^b e_\alpha{}^c e_\beta{}^d \epsilon_{abcd} \\ &= -\frac{1}{24}dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta e_\mu{}^a e_\nu{}^b e_\alpha{}^c e_\beta{}^d \epsilon_{abcd} \\ &= -\frac{1}{24}e^a \wedge e^b \wedge e^c \wedge e^d \epsilon_{abcd}. \end{aligned}$$

From these the action formulas for the Palatini-Cartan follow.

In order to get the Euler-Lagrange equations we take the variation of the action that

$$S = -\frac{1}{4\kappa}\int_M \epsilon_{abcd}\left(R^{ab} - \frac{\Lambda}{6}e^a \wedge e^b\right) \wedge e^c \wedge e^d,$$

that is ,

$$-4\kappa\,\delta S = \int_M \epsilon_{abcd}\left[\delta R^{ab} \wedge e^c \wedge e^d\right] + 2\int_M \epsilon_{abcd}\left[R^{ab} - \frac{\Lambda}{3}e^a \wedge e^b\right] \wedge e^c \wedge \delta e^d.$$

For the first integral we observe that

$$\begin{aligned} \delta R^{ab} &= \delta(d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}) \\ &= d(\delta\omega^{ab}) + (\delta\omega^a{}_c) \wedge \omega^{cb} + \omega^a{}_c \wedge (\delta\omega^{cb}) \\ &= d(\delta\omega^{ab}) + \omega^b{}_c \wedge \delta\omega^{ac} + \omega^a{}_c \wedge (\delta\omega^{cb}) \\ &= D(\delta\omega^{ab}), \end{aligned}$$

where we used the antisymmetry of ω^{ab}. Using the same antisymmetry we get $D\epsilon_{abcd} = 0$, hence

$$\begin{aligned} &\epsilon_{abcd}\left[\delta R^{ab} \wedge e^c \wedge e^d\right] = \epsilon_{abcd}D(\delta\omega^{ab}) \wedge e^c \wedge e^d \\ &= D\left[\epsilon_{abcd}(\delta\omega^{ab}) \wedge e^c \wedge e^d) + \epsilon_{abcd}(\delta\omega^{ab}) \wedge D(e^c \wedge e^d\right] \\ &= d\left[\epsilon_{abcd}(\delta\omega^{ab}) \wedge e^c \wedge e^d\right] + 2\epsilon_{abcd}(\delta\omega^{ab}) \wedge e^c \wedge T^d, \end{aligned}$$

where we used the definition of torsion T^a and the fact that $\epsilon_{abcd}(\delta\omega^{ab}) \wedge e^c \wedge e^d$ is a scalar. Hence, when the Stokes theorem is applied, the first integral becomes

$$2\int_M (\delta\omega^{ab}) \wedge \epsilon_{abcd}\, e^c \wedge T^d$$

and our Euler-Lagrange equations are

$$\epsilon_{abcd}\, e^c \wedge T^d = 0 \quad \text{and} \quad \epsilon_{abcd}\left[R^{ab} - \frac{\Lambda}{3}e^a \wedge e^b\right] \wedge e^c = 0.$$

We want to check that the first is just $T^a = 0$, the torsion-free condition and the second is the Einstein equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$.

Indeed, wedging dx^β with the expression

$$0 = \epsilon_{abcd}\, e^c \wedge T^d = \epsilon_{abcd}\, e_\alpha{}^c \frac{1}{2} T_{\mu\nu}{}^d dx^\alpha \wedge dx^\mu \wedge dx^\nu$$

we get

$$0 = \epsilon_{abcd}\, T_{\mu\nu}{}^d e_\alpha{}^c dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta = \epsilon_{abcd}\, T_{\mu\nu}{}^d e_\alpha{}^c dx^\mu \epsilon^{\mu\nu\alpha\beta} d^4x.$$

Since

$$\begin{aligned}\epsilon_{abcd}\, \epsilon^{\mu\nu c\beta} &= -e^{\mu\nu\beta}_{abd} \\ &:= e^\mu{}_a e^\nu{}_b e^\beta{}_d + e^\nu{}_a e^\beta{}_b e^\mu{}_d + e^\beta{}_a e^\mu{}_b e^\nu{}_d \\ &\quad - e^\nu{}_a e^\mu{}_b e^\beta{}_d - e^{\nu\beta}{}_a e^\nu{}_b e^\mu{}_d - e^\mu{}_a e^\beta{}_b e^\nu{}_d,\end{aligned}$$

we get

$$\begin{aligned}0 &= T_{ab}{}^d e^\beta{}_d + T_{bd}{}^d e^\beta{}_a + T_{da}{}^d e^\beta{}_b - T_{ba}{}^d e^\beta{}_d - T_{db}{}^d e^\beta{}_a - T_{ad}{}^d e^\beta{}_b \\ &= 2(T_{ab}{}^d e^\beta{}_d + S_b e^\beta{}_a - S_a e^\beta{}_b),\end{aligned}$$

where $S_a = T_{ad}{}^d$ and we used the antisymmetry of $T_{ab}{}^d$ in a, b. Multiplying the above by $e_\beta{}^a$ we get $S_B = 0$, hence $T_{ab}{}^d = 0$ as wanted to show. We leave the equivalence of the second Euler-Lagrange equation to the Einstein equation to Exercise 5.

15.4 Lie Algebra $\mathfrak{so}(2,2)$ and an Invariant Pairing

For the Chern-Simons formulation of 2+1-dimensional gravity we will need the Lie algebra $\mathfrak{so}(2,2)$, which is the Lie algebra of the Lie group $SO(2,2)$. In order to define this Lie algebra let η be the diagonal matrix $\eta = \text{diag}(-1, 1, 1, -1)$. Now let

$$SO(2,2) = \left\{ g \in \text{Mat}_4(\mathbb{R}) \mid \det g = 1,\ g^T \eta g = \eta \right\}.$$

By definition a matrix A is in $\mathfrak{so}(2,2)$ iff $e^{tA} \in SO(2,2)$ for all $t \in \mathbb{R}$. Observe that

$$\det e^{tA} = e^{t\text{Tr}A} = 1 \text{ iff } \text{Tr}A = 0.$$

Also,

$$\left(e^{tA}\right)^T \eta e^{tA} = e^{tA^T} \eta e^{tA} = \eta \ \text{ for all } \ t \in \mathbb{R}$$

iff

$$0 = \frac{d}{dt}\left(e^{tA^T} \eta e^{tA}\right) = e^{tA^T}(A^T\eta + \eta A)e^{tA} \ \text{ for; all } \ t \in \mathbb{R}$$

iff $A^T\eta + \eta A = 0$.

Writing

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{bmatrix},$$

we have

$$A^T\eta + \eta A = \begin{bmatrix} -2A_{00} & A_{10}-A_{01} & A_{20}-A_{02} & -A_{30}-A_{03} \\ A_{10}-A_{01} & 2A_{11} & A_{21}+A_{12} & -A_{31}+A_{13} \\ -A_{02}+A_{20} & A_{12}+A_{21}+ & 2A_{22} & -A_{32}+A_{23} \\ -A_{03}-A_{30} & A_{13}-A_{31} & A_{23}-A_{32} & -2A_{33} \end{bmatrix}.$$

Thus, from $A^T\eta + \eta A = 0$ we get

$$\mathfrak{so}(2,2) = \left\{ \begin{bmatrix} 0 & a_1 & a_2 & -b_3 \\ a_1 & 0 & -a_3 & b_2 \\ a_2 & a_3 & 0 & b_1 \\ b_3 & b_2 & b_1 & 0 \end{bmatrix} \;\middle|\; a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R} \right\}.$$

Let's write the generators of $\mathfrak{so}(2,2)$ as

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ B_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then we have

$[X,Y]$	A_1	A_2	A_3	B_1	B_2	B_3
A_1	0	$-A_3$	$-A_2$	0	$-B_3$	$-B_2$
A_2	A_3	0	$-A_1$	$-B_3$	0	$-B_1$
A_3	A_2	A_1	0	$-B_2$	$-B_1$	0
B_1	0	B_3	B_2	0	A_3	A_2
B_2	B_3	0	$-B_1$	$-A_3$	0	A_1
B_3	B_2	B_1	0	$-A_2$	$-A_1$	$0.$

Recall that the adjoint operators $\mathrm{ad}_X : \mathfrak{so}(2,2) \to \mathfrak{so}(2,2)$ is given by $\mathrm{ad}_X Y = [X,Y]$. Thus in the basis $A_1, A_2, A_3, B_1, B_2, B_3$ we have matrix representations

$$\mathrm{ad}_{A_1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \ \mathrm{ad}_{A_2} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\mathrm{ad}_{A_3} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathrm{ad}_{B_1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{ad}_{B_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathrm{ad}_{B_3} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, with the pairing from the Killing form

$$\langle X, Y \rangle = \frac{1}{4} \operatorname{Tr} [\mathrm{ad}_X \circ \mathrm{ad}_Y] .$$

we have

$\langle X, Y \rangle$	A_1	A_2	A_3	B_1	B_2	B_3
A_1	1	0	0	0	0	0
A_2	0	1	0	0	0	0
A_3	0	0	−1	0	0	0
B_1	0	0	0	1	0	0
B_2	0	0	0	0	1	0
B_3	0	0	0	0	0	−1

Since the table is diagonal with non-zero diagonal entries, we see that the pairing is non-degenerate. Also, being the Killing form, the pairing is invariant, that is,

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle.$$

From the above computation we easily get Lie algebra $\mathfrak{so}(2,1)$ of the Lie group $SO(2,1)$. With $\eta = \text{diag}(-1,1,1)$ we let

$$SO(2,1) = \left\{ g \in \text{Mat}_3(\mathbb{R}) \mid \det g = 1,\ g^T \eta g = \eta \right\}$$

to get

$$\mathfrak{so}(2,1) = \left\{ \begin{bmatrix} 0 & a_1 & a_2 \\ a_1 & 0 & -a_3 \\ a_2 & a_3 & 0 \end{bmatrix} \middle| \, a_1, a_2, a_3 \in \mathbb{R} \right\}$$

and the generators

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

As above we have

$[X,Y]$	A_1	A_2	A_3
A_1	0	$-A_3$	$-A_2$
A_2	A_3	0	$-A_1$
A_3	A_2	A_1	0

Also, with the pairing from the Killing form

$$\langle X, Y \rangle = \frac{1}{2} \text{Tr} \left[\text{ad}_X \circ \text{ad}_Y \right].$$

we have

$\langle X,Y \rangle$	A_1	A_2	A_3
A_1	1	0	0
A_2	0	1	0
A_3	0	0	-1

15.5 BTZ Black Hole

We first review AdS_3 spacetime. For this we consider the hyperboloid

$$-u^2 + x^2 + y^2 - v^2 = -l^2$$

in the flat spacetime $\mathbb{R}^4$ with the metric $\eta = \text{diag}(-1,1,1-1)$, that is,

$$ds^2 = -du^2 + dx^2 + dy^2 - dv^2.$$

Here l is a parameter representing the radius of curvature of our AdS spacetime. Using the coordinates $t \in \mathbb{R},\ r \geq 0,\ 0 \leq \theta < 2\pi$ and putting

$$u = \sqrt{l^2 + r^2}\cos\frac{ct}{l},\ x = r\cos\theta,\ y = r\sin\theta,\ v = \sqrt{l^2+r^2}\sin\frac{ct}{l},$$

we get the induced metric on the hyperboloid

$$\begin{aligned} ds^2 &= \left(\frac{rdr}{\sqrt{l^2+r^2}}\cos\frac{ct}{l} - \sqrt{1+\frac{r^2}{l^2}}\sin\frac{ct}{l}cdt\right)^2 + (dr\ \cos\theta - r\sin\theta)^2 \\ &\quad + (dr\ \sin\theta + r\cos\theta)^2 - \left(\frac{rdr}{\sqrt{l^2+r^2}}\sin\frac{ct}{l} + \sqrt{1+\frac{r^2}{l^2}}\cos\frac{ct}{l}cdt\right)^2 \\ &= -\left(1+\frac{r^2}{l^2}\right)c^2dt^2 + \frac{dr^2}{1+\frac{r^2}{l^2}} + r^2d\theta^2. \end{aligned}$$

The hyperboloid with this induced metric is called the anti-de Sitter space of dimension 3. Just like the flat spacetime this space has the maximal symmetry $SO(2,2)$. One can check that AdS_3 solves the Einstein field equation with the cosmological constant $\Lambda = -1/l^2$ and

$$R_{\mu\nu\alpha\beta} = \Lambda(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}),\ R_{\mu\nu} = 2\Lambda g_{\mu\nu},\ R = 6\Lambda.$$

In 1992 Bañados, Teitelboim and Zanelli introduced (2+1)-dimensional black hole with the metric

$$ds^2 = f^2c^2dt^2 + \frac{1}{f^2}dr^2 + r^2(d\theta - \Omega c\,dt)^2,$$

where

$$f^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2},\ \Omega = \frac{J}{2r^2}.$$

here the coordinates t, r, θ are as in AdS_3, M is the mass and J is the angular momentum of the black hole. Notice that we are using the units so that M is dimensionless and J has unit of length. If $lM > J$, then the black hole has horizons at $r = r_\pm$, where $f(r_\pm) = 0$. One can also write

$$f^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2r^2} \quad \text{and} \quad J = \frac{2r_+r_-}{l}.$$

Notice that BTZ black hole is reduced to AdS_3 if $J = 0$ and we adjust the reference energy level so that $M = -1$. BTZ black hole is locally AdS_3, hence it satisfies the same Einstein equations as AdS_3. Good references for BTZ black hole and $3D$ gravity as a gauge theory are the papers "The (2+1)-Dimensional Black Hole" in 1995 by S. Carlip and "2+1 Dimensional Gravity as an Exactly Soluble System" in 1988 by Edward Witten.

From the metric one can easily find a triad

$$e^0 = fc\,dt,\ e^1 = f^{-1}\,dr,\ e^2 = r(d\theta - \Omega c\,dt).$$

As before we can use $T^a = de^a + \omega^a{}_b \wedge e^b = 0$ and $\omega^{ab} = -\omega^{ba}$ to get the corresponding spin connection

$$\omega^{01} = f'e^0 - \Omega e^2,\ \omega^{02} = -\Omega e^1,\ \omega^{12} = -\Omega e^0 - \frac{f}{r}e^2.$$

Notice that since ω^{ab} is antisymmetric, hence it is a 1-form with values in $\mathfrak{so}(2,1)$.

15.6 Chern-Simons Functional

For the BTZ black hole we computed a triads and its spin connection. In general whenever we have a triads e^a and a spin connection ω^{ab} on a closed oriented $3D$ manifold with values in $\mathfrak{so}(2,1)$ we can combine them to form the cooresponding Chern-Simons connection A with values in $\mathfrak{so}(2,2)$:

$$A^{AB} = \begin{bmatrix} \omega^{00} & \omega^{01} & \omega^{02} & e^0/l \\ \omega^{10} & \omega^{11} & \omega^{12} & e^1/l \\ \omega^{20} & \omega^{21} & \omega^{22} & e^2/l \\ -e^0/l & -e^1/l & -e^2/l & 0 \end{bmatrix}$$

The argument below works in general with any Lie algebra $\mathfrak{g}$ with a non-degenerate invariant pairing $\langle X, Y\rangle$ in place of $\mathfrak{so}(2,2)$. Let's first define a few operations on Lie algebra valued forms. Let $\alpha \in \Omega^p(M), \beta \in \Omega^q(M)$ and $X, Y \in \mathfrak{g}$. Then we define

$$\begin{aligned} \langle\langle \alpha \otimes X, \beta \otimes Y \rangle\rangle &= \langle X, Y\rangle (\alpha \wedge \beta) \\ (\alpha \otimes X) \wedge (\beta \otimes Y) &= (\alpha \wedge \beta) \otimes [X, Y] \\ [\alpha \otimes X, \beta \otimes Y] &= 2(\alpha \wedge \beta) \otimes [X, Y]. \end{aligned}$$

Of course these operations can be extended bilinearly. We observe following properties for $A \in \Omega^p(M, \mathfrak{g})$ and $B \in \Omega^q(M, \mathfrak{g})$:

$$\begin{aligned} &\text{(i)} \;\; \langle\langle A, B\rangle\rangle = (-1)^{pq}\langle\langle B, A\rangle\rangle \\ &\text{(ii)} \;\; d\langle\langle A, B\rangle\rangle = \langle\langle dA, B\rangle\rangle + (-1)^p \langle\langle A, dB\rangle\rangle \\ &\text{(iii)} \;\; \langle\langle A, B \wedge C\rangle\rangle = \langle\langle A \wedge B, C\rangle\rangle \\ &\text{(iv)} \;\; A \wedge B = -(-1)^{pq} B \wedge A. \end{aligned}$$

In particular, the last one follows from the invariance of the pairing.

Now we can consider the Chern-Simons form $\mathrm{CS}(A)$ and the Chern-Simons action $S_{\mathrm{CS}}[A]$:

$$\mathrm{CS}(A) = \langle\langle dA, A\rangle\rangle + \frac{2}{3}\langle\langle A \wedge A, A\rangle\rangle, \quad S_{\mathrm{CS}}[A] = \int_M \mathrm{CS}(A).$$

We then have the following

Theorem 1. (a) $d\,\mathrm{CS}(A) = F$, where $F = dA + A \wedge A$ is the curvature of the connection A.

(b) $\delta \mathrm{CS}(A) = d\,\langle\langle \delta A, A\rangle\rangle + 2\langle\langle \delta A, F\rangle\rangle$ and the Euler-Lagrange equation of the Chern-Simons action is $F = 0$.

(c) $S_{\mathrm{CS}(A)} = S_{\mathrm{CS}(A+\delta A)}$ under the gauge transformation

$$\delta A = DX = dX + [A, X],$$

where $X \in \Omega^0(M, \mathfrak{g})$.

We leave the proof of (a) and (b) as Exercise 6. In order to prove (c) we use the result in (b)

$$\delta \mathrm{CS}(A) = d\,\langle\langle \delta A, A\rangle\rangle + 2\langle\langle \delta A, dA + A\wedge A\rangle\rangle$$

and the assumption that M is closed oriented to have

$$\begin{aligned}\delta S_{\mathrm{CS}} = S_{\mathrm{CS}}(A+\delta A) - S_{\mathrm{CS}}(A) = 2\int_M \langle\langle \delta A, dA + A\wedge A\rangle\rangle \\ = 2\int_M \langle\langle [A,X], dA\rangle\rangle + \langle\langle [A,X], A\wedge A\rangle\rangle + \langle\langle dX, dA\rangle\rangle + \langle\langle \delta dX, A\wedge A\rangle\rangle.\end{aligned}$$

Using (ii) we get

$$\int_M \langle\langle dX, dA\rangle\rangle = \int_M d\langle\langle X, dA\rangle\rangle - \langle\langle X, d^2A\rangle\rangle = 0.$$

Also,

$$\langle\langle [A,X], A\wedge A\rangle\rangle = \langle\langle A\wedge X, A\wedge A\rangle\rangle - \langle\langle X\wedge A, A\wedge A\rangle\rangle = 0$$

since

$$\begin{aligned}\langle\langle A\wedge X, A\wedge A\rangle\rangle = \langle\langle A\wedge A, A\wedge X\rangle\rangle = \langle\langle A\wedge A\wedge A, X\rangle\rangle = \langle\langle X, A\wedge A\wedge A\rangle\rangle \\ = \langle\langle X\wedge A, A\wedge A\rangle\rangle.\end{aligned}$$

Thus

$$\begin{aligned}\delta S_{\mathrm{CS}} &= 2\int_M \langle\langle [A,X], dA\rangle\rangle + \langle\langle dA, A\wedge A\rangle\rangle \\ &= 2\int_M \langle\langle A\wedge X - X\wedge A, dA\rangle\rangle + \langle\langle dX, A\wedge A\rangle\rangle \\ &= 2\int_M \langle\langle A, X\wedge dA\rangle\rangle - \langle\langle X\wedge A, dA\rangle\rangle + \langle\langle (dA)\wedge A, A\rangle\rangle \\ &= \int_M \langle\langle X\wedge dA + (dX)\wedge A, A\rangle\rangle\langle\langle X\wedge A, dA\rangle\rangle \\ &= 2\int_M d\,\langle\langle X\wedge A, A\rangle\rangle \\ &= 0.\end{aligned}$$

In case A is made of a triad e^a and its spin connection ω^{ab} we have

Theorem 2. (a) The Euler-Lagrange equations are $R^{ab} - \Lambda e^a\wedge e^b = 0$ and $T^a = 0$, which, in turn, are equivalent to the Einstein equation and torsion-free condition.

(b) The action is written as

$$S_{\mathrm{CS}}[e,\omega] = \langle\langle d\omega, \omega\rangle\rangle + \frac{2}{3}\langle\langle \omega\wedge\omega, \omega\rangle\rangle - \Lambda\langle\langle de + \omega\wedge e, e\rangle\rangle.$$

Proof of (a) We use indices a, b, c for $0, 1, 2$. Then, we have

$$A^{ab} = \omega^{ab} \quad \text{and} \quad A^{a3} = \frac{1}{l}e^a.$$

Thus,

$$F^{ab} = dA^{ab} + A^a{}_c \wedge A^{cb} + A^a{}_3 \wedge A^{3b} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} + \frac{1}{l^2} e^a \wedge e^b = R^{ab} - \Lambda e^a \wedge e^b$$

and

$$F^{a3} = dA^{a3} + A^a{}_c \wedge A^{c3} + A^a{}_3 \wedge A^{3b} = \frac{1}{l} de^a + \omega^a{}_c \wedge \left(\frac{1}{l}\right) e^c = \frac{1}{l} T^a,$$

from which we get the desired Euler-Lagrange equations. We have checked that $T^a = 0$ is equivalent to torsion-free condition in 15.2 right after we introduced Cartan structure equations. To show that the first is equivalent to Einstein equations we follow the argument in Exercise 5. Thus, using similar properties of Levi-Civita symbol in dimension 3, we have

$$\begin{aligned}
0 &= \epsilon_{abc} \left(R^{ab} - \Lambda\, e^a \wedge e^b\right) \wedge dx^\alpha \\
&= \epsilon_{abc} \left(\frac{1}{2} R_{\mu\nu}{}^{ab} - \Lambda\, e_\mu{}^a e_\nu{}^b\right) dx^\mu \wedge dx^\nu \wedge dx^\alpha \\
&= \epsilon_{abc} \left(\frac{1}{2} R_{\mu\nu}{}^{ab} - \Lambda\, e_\mu{}^a e_\nu{}^b\right) \epsilon^{\mu\nu\alpha}\, d^3x \\
&= -e^{\mu\nu\alpha}_{abc} \left(\frac{1}{2} R_{\mu\nu}{}^{ab} - \Lambda\, e_\mu{}^a e_\nu{}^b\right) d^3x,
\end{aligned}$$

or

$$\begin{aligned}
0 &= -\frac{1}{2} \left(R_{\mu\nu}{}^{\mu\nu} e^\alpha{}_c + R_{\mu\nu}{}^{\nu\alpha} e^\mu{}_c + R_{\mu\nu}{}^{\alpha\mu} e^\nu{}_c - R_{\mu\nu}{}^{\nu\mu} e^\alpha{}_c - R_{\mu\nu}{}^{\alpha\nu} e^\mu{}_c - R_{\mu\nu}{}^{\mu\alpha} e^\nu{}_c\right) \\
&\quad + \Lambda \left(e^\mu{}_a e^\nu{}_b e^\alpha{}_c + e^\nu{}_a e^\alpha{}_b e^\mu{}_c + e^\alpha{}_a e^\mu{}_b e^\nu{}_c - e^\nu{}_a e^\mu{}_b e^\alpha{}_c - e^\alpha{}_a e^\nu{}_b e^\mu{}_c - e^\mu{}_a e^\alpha{}_b e^\nu{}_c\right) e_\mu{}^a e_\nu{}^b \\
&= -\frac{1}{2} \left(R e^\alpha{}_c - R_c{}^\alpha - R_c{}^\alpha + R e^\alpha{}_c - R_c{}^\alpha - R_c{}^\alpha\right) \\
&\quad + \Lambda \left(9 e^\alpha{}_c + e^\alpha{}_c + e^\alpha{}_c - 3 e^\alpha{}_c - 3 e^\alpha{}_c - 3 e^\alpha{}_c\right) \\
&= 2 \left(R_c{}^\alpha - \frac{1}{2} R e^\alpha{}_c + \Lambda e^\alpha{}_c\right) \\
&= 2 \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}\right) e^\mu{}_c g^{\alpha\nu}.
\end{aligned}$$

completing the proof.

Proof of (b) We can write

$$A = \omega^{01} A_1 + \omega^{02} A_2 - \omega^{12} A_3 - \frac{1}{l} e^0 B_3 + \frac{1}{l} e^1 B_2 + \frac{1}{l} e^2 B_1,$$

hence

$$\begin{aligned}
\langle\langle dA, A\rangle\rangle &= d\omega^{01} \wedge \omega^{01} + d\omega^{02} \wedge \omega^{02} - d\omega^{12} \wedge \omega^{12} - \frac{1}{l^2} e^0 \wedge e^0 + \frac{1}{l^2} e^1 \wedge e^0 + \frac{1}{l^2} e^2 \wedge e^2 \\
&= \langle\langle d\omega, \omega\rangle\rangle - \Lambda \langle\langle de, e\rangle\rangle.
\end{aligned}$$

Also, from

$$\begin{aligned}
\frac{1}{2} A \wedge A &= -\omega^{01} \wedge \omega^{02} A_3 + \omega^{01} \wedge \omega^{02} A_2 - \omega^{02} \wedge \omega^{12} A_1 \\
&\quad + \frac{1}{l} \left[-\omega^{01} \wedge e^1 B_3 + \omega^{01} \wedge e^0 B_2 - \omega^{02} \wedge e^2 B_3 + \omega^{02} \wedge e^0 B_1 + \omega^{12} \wedge e^2 B_2 - \omega^{12} \wedge e^1 B_1\right]
\end{aligned}$$

we get

$$\frac{1}{2}\langle\langle A\wedge A, A\rangle\rangle = -3\omega^{01}\wedge\omega^{02}\wedge\omega^{12} + \frac{1}{l^2}\left[\omega^{01}\wedge e^0\wedge e^1 + \omega^{02}\wedge e^0\wedge e^2 - \omega^{12}\wedge e^1\wedge e^2\right]$$
$$= \frac{1}{2}\langle\langle\omega\wedge\omega,\omega\rangle\rangle - \frac{1}{2}\Lambda\langle\langle\omega\wedge e, e\rangle\rangle.$$

15.7 Exercise

1. (a) Show that if ${e_\mu}^a$ is related to $g_{\mu\nu}$, then so is ${e_\mu}^{a'} = {\Lambda^{a'}}_a {e_\mu}^a$.

(b) Assuming that ${e_\mu}^a$ is related to $g_{\mu\nu}$, show that $\det({e^\mu}_a) = \sqrt{-g}$.

(c) If V^μ is a vector in the curved spacetime, show that V^a is a tensor in flat Lorentzian spacetime, but a scalar in the curved spacetime.

2. Show that

$$R^\alpha_{\beta\mu\nu} = {e^\alpha}_a {e_\beta}^b\left[\partial_\mu {\omega_\nu}^a{}_b - {\omega_\mu}^c{}_b {\omega_\nu}^a{}_c\right] - (\mu\leftrightarrow\nu).$$

3. (a) Show

$$R_{\mu\nu}{}^{ab} dx^\mu dx^\nu = d\omega^{ab} + {\omega^a}_c\wedge{\omega^c}_b.$$

(b) Prove two Bianchi identities $DT^a = {R^a}_b\wedge e^b$ and $DR^{ab} = 0$.

4. (a) Prove that

$$\eta^{\alpha\beta\gamma\delta}\eta_{\mu\nu\gamma\delta} = -2(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu),\ \ \eta^{\mu\nu\gamma\delta}\eta_{\mu\nu\gamma\delta} = -24 \quad \eta^{\alpha\beta\gamma\delta}\eta_{abc\delta} = -e^{\alpha\beta\gamma}_{abc}.$$

(b) Show that $\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}$.

5. Show that the Euler-Lagrange equation $\epsilon_{abcd}\left(R^{ab} - \frac{\Lambda}{3}e^a\wedge e^b\right)\wedge e^c = 0$ is equivalent to the Einstein equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$.

6. (a) Show that $d\,\mathrm{CS}(A) = \langle\langle F, F\rangle\rangle$, where $F = dA + A\wedge A$ is the curvature of the connection A.

(b) Prove that $\delta\mathrm{CS}(A) = d\,\langle\langle\delta A, A\rangle\rangle + 2\langle\langle\delta A, F\rangle\rangle$ and that the Euler-Lagrange equation of the Chern-Simons action is $F = 0$.

16 The BV Formalism

Our presentation follows Costello [Cos11] Section 5.2, for an in-depth modern presentation from another perspective, see Mnev's text [Mne19].

16.1 Two key maneuvers

One mathematical route to Batalin–Vilkovisky (BV) Theory is through two homological algebra maneuvers: resolving quotients and approaching integration on manifolds (co)homologically. This approach essentially goes back to Albert Schwarz and his work with culminated in [Sch93]. The physical motivations for Batalin and Vilkovisky was the quantization of gauge theories and the development of techniques beyond the existent BRST and Fadeev–Popov methods, the original article on BV is [BV81].

16.1.1 Resolution of Quotients

Perturbation techniques, e.g., stationary phase, work best for functionals with isolated critical points. Let $S\colon \mathcal{F} \to \mathbb{C}$ be a functional with gauge symmetry group G. If $\varphi \in \mathcal{F}$ is a critical point of S, then the entire gauge orbit $G \cdot \varphi \subseteq \mathcal{F}$ is also critical. Under such conditions, and provided G is neither discrete nor trivial, critical points are not isolated. It was realized early on that one solution is *gauge fixing* which in simplest terms is the choice of a "slice" in $\mathcal{F}$, containing ϕ, which is transverse to gauge orbits. BRST, Fadeev–Popov, and symplectic reduction are all approaches to gauge fixing.

A conceptual way of thinking about gauge fixing is the selection of a representative for each equivalence class in the quotient $\mathcal{F}/G$. By the universal property of quotients, if our functional S is G-invariant, then it descends to a functional on $\mathcal{F}/G$. If $\mathcal{F}/G$ were a nice mathematical object/space and supported a sufficiently rich class of functionals, we could just attempt to work directly with the quotient. However, the quotient space $\mathcal{F}/G$ is typically a wild object in that its topology may not be Hausdorff or have any nice linear properties. Such "bad" quotients are already familiar to us from topology, e.g., the quotient of S^1 or T^2 by an irrational rotation or slope.

One (of many) standard approach to working with quotients like $\mathcal{F}/G$ is to work with the algebra of functions and find a resolution thereof by "tame" objects. While this is a common theme throughout algebra, geoemtry, and algebraic geometry, it is perhaps best illustrated through a particular construction. For simplicity, we will work in the "linearized" setting, so our fields will be simply a vector space V and gauge symmetries will be implemented by an action of the gauge (Lie) algebra $\mathfrak{g}$ acting on V. Functions on fields will be modeled by the (completed) symmetric algebra, $\mathcal{O}(V) := \widehat{\mathrm{Sym}}(V^\vee)$, on the linear dual of V; thus functions inherit an action of the gauge algebra $\mathfrak{g}$.

Now, we would like to understand functions on V/G as $\mathfrak{g}$-invariant functions of V, i.e.,

$$\mathcal{O}(V)^{\mathfrak{g}} := \{\psi \in \widehat{\mathrm{Sym}}(V^\vee) : X \cdot \psi = 0, \text{ for all } X \in \mathfrak{g}\}.$$

It will be more convenient (and in line with the yoga of homological algebra) to find a resolution, i.e., a cochain complex, such that the zeroth cohomology recovers $\mathcal{O}(V)^{\mathfrak{g}}$ and satisfies other desirable properties. Such a desire is satisfied by *Lie algebra cohomology*. Following [Wei94], for $\mathfrak{g}$ a Lie algebra and M a $\mathfrak{g}$-module, we will define a cochain complex, $CE^*(\mathfrak{g}, M)$, such that

$H^0(CE^*(\mathfrak{g}, M)) \cong M^{\mathfrak{g}}$. More generally, we define $H^k_{\mathrm{Lie}}(\mathfrak{g}, M) := H^k(CE^*(\mathfrak{g}, M))$. (Using the language of derived functors, there is a more universal approach to Lie algebra cohomology. It standard that our definition agrees with the one obtained via derived functors.)

Definition 16.1.1. Let $\mathfrak{g}$ be a Lie algebra (over a field $\mathbb{K}$) and M a $\mathfrak{g}$-module. The *Chevalley–Eilenberg complex*, $CE^*(\mathfrak{g}, M)$, has underlying graded vector space

$$CE^\sharp(\mathfrak{g}, M) = \prod_n \mathrm{Hom}_{\mathbb{K}}(\Lambda^n \mathfrak{g}, M)[-n],$$

with differential d_{CE} given by

$$\begin{aligned}(d_{CE} f)(X_1, \ldots, X_{n+1}) = &\sum (-1)^{i+1} X_i \cdot f(X_1, \ldots, \widehat{X_i}, \ldots) \\ &+ \sum (-1)^{i+j} f([X_i, X_j]_{\mathfrak{g}}, X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots).\end{aligned}$$

While the Chevalley–Eilenberg differential looks complicated, it really is just a sum of all reasonable ways of taking an alternating multilinear functional of n entries and producing an alternating linear functional of $n+1$ entries. Let us unpack the differential d_{CE} in low degrees. Starting in degree zero,

$$d^0_{CE} \colon M \to \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, M), \quad \left(d^0_{CE} m\right)(X) = X \cdot m.$$

Hence, we see that $H^0_{\mathrm{Lie}}(\mathfrak{g}, M) = M^{\mathfrak{g}}$ as desired. In degree one we have

$$\begin{aligned}&d^1_{CE} \colon \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, M) \to \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g} \wedge \mathfrak{g}, M), \\ &\left(d^1_{CE} f\right)(X_1, X_2) = X_1 \cdot f(X_2) - X_2 \cdot f(X_1) - f([X_1, X_2]).\end{aligned}$$

Returning to our motivation, we have our gauge algebra $\mathfrak{g}$ and our $\mathfrak{g}$-module is functions on fields, so $\mathcal{O}(V)$. Hence, we are interested in the Chevalley–Eilenberg complex $CE^*(\mathfrak{g}, \mathcal{O}(V))$. Following the conventions of graded multilinear algebra, note that

$$\widehat{\mathrm{Sym}}(\mathfrak{g}[1] \oplus V) \cong \prod_n \Lambda^n \mathfrak{g}^\vee \otimes \widehat{\mathrm{Sym}}(V^\vee)[-n] \cong CE^\sharp(\mathfrak{g}, \mathcal{O}(V)).$$

The upshot of these isomorphisms is that $CE^\sharp(\mathfrak{g}, \mathcal{O}(V))$ models functions on the graded space $\mathfrak{g}[1] \oplus V$. The differential d_{CE} can then be thought of as a vector field (of degree $+1$) on $\mathfrak{g}[1] \oplus V$.

16.1.2 Integration via (co)Homology

We now discuss a reinterpretation of integration on a smooth manifold in such a way that it will apply to infinite dimensional spaces. Historically, much of this approach goes back to Koszul [Kos85].

For simplicity, let X be a connected, orientable, smooth manifold without boundary of dimension n.[3] Every top form $\mu \in \Omega^n(X)$ then defines a linear functional

$$\begin{array}{rccc}\int_\mu : & {}_c(X) & \to & \mathbb{R} \\ & f & \mapsto & \int_X f\mu\end{array},$$

[3] Using densities, the following arguments can be adopted to unoriented manifolds.

which is a natural object from several perspectives. First, from this linear functional — the distribution associated to μ — we can completely reconstruct the top form μ. Second, if μ is a probability measure, then $\int_\mu$ is precisely the expected value map. Our goal is now to rephrase $\int_\mu$ in a way that does not explicitly depend on ordinary integration and thus to obtain a version of volume form that can be extended to Ł8 spaces.

We can understand $\int_\mu$ in a purely homological way, as follows. We know that integration over X vanishes on total derivatives $d\omega \in \Omega^n_c(X)$, by Stokes' Theorem, so we have a commutative diagram

$$\begin{array}{ccc} \Omega^n_c(X) & \xrightarrow{\int_X} & \mathbb{R} \\ & {\scriptstyle [-]}\searrow \quad \nearrow{\scriptstyle \cong} & \\ & H^n_c(X) & \end{array}$$

where $[\omega]$ denotes the cohomology class of the top form ω. (The cohomology group $H^n_c(X)$ is 1-dimensional by Poincaré duality.) In consequence, we can identify $\int_\mu$ with the composition

$$\begin{array}{ccc} \Omega^n_c(X) & \xrightarrow{[-]} & H^n_c(X) \\ {\scriptstyle \iota_\mu}\uparrow & \nearrow{\scriptstyle \int_\mu} & \\ {}_c(X) & & \end{array}$$

where ι_μ denotes "multiplication by μ" (or "contraction with μ"). We thus have a purely homological version of integration against μ.

It is natural to extend the map "contract with μ" to the whole de Rham complex, and not just the top forms:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \Omega^{n-2}_c(X) & \xrightarrow{d} & \Omega^{n-1}_c(X) & \xrightarrow{d} & \Omega^n_c(X) & \xrightarrow{\int_X} & \mathbb{R}\ , \\ & & \uparrow{\scriptstyle \iota_\mu} & & \uparrow{\scriptstyle \iota_\mu} & & \uparrow{\scriptstyle \iota_\mu} & \nearrow{\scriptstyle \int_\mu} & \\ \cdots & \longrightarrow & PV^2_c T(X) & \xrightarrow{div_\mu} & PV^1_c(X) & \xrightarrow{div_\mu} & C^\infty_c(X) & & \end{array}$$

where $PV^k_c(X) := \Gamma_c(X, \Lambda^k T_X)$ denotes the compactly-supported *polyvector fields* and div_μ denotes "divergence with respect to μ." We require now that μ is nowhere-vanishing, so that the divergence is well-defined. This map of cochain complexes ι_μ is then an isomorphism.

The significance of the bottom row is that it fully encodes integration against μ but the relevant data of μ is contained in the differential div_μ. This kind of integration notion works even for infinite-dimensional spaces, for which there are no top forms but there is a ring of functions and polyvector fields.

The complex $(PV(X), div_\mu)$ is a fundamental example of a BV algebra, a notion we recall in the next section, the associated bracket $\{-,-\}$ is the Schouten bracket. Recall that the Schouten bracket is the extension of the Lie bracket of vector fields to polyvector fields by requiring it to be a derivation of the wedge product. Further, note that we have an equivalence

$$\mathcal{O}(T^*[-1]X) \cong PV(X),$$

where $T^*[-1]X$ is the shifted cotangent bundle of the space X.

16.1.3 Summary

Given a classical field theory with field content $\mathcal{F}$, action $S\colon \mathcal{F} \to \mathbb{C}$, and gauge symmetries G, the (perturbative) BV formalism can be understood as a two step procedure:

(a) Given $\varphi \in \mathcal{F}$, the tangent space to the gauge orbit $[\varphi] \in \mathcal{F}/G$ can be modeled on $\mathcal{F}/\mathfrak{g}$. More precisely, we resolve the quotient $\mathcal{F}/\mathfrak{g}$ using the Chevalley–Eilenberg complex, $CE^*(\mathfrak{g}, \mathcal{O}(\mathcal{F}))$, which can be understood as functions on the graded space $\mathcal{F} \oplus \mathfrak{g}[1]$ equipped with a vector field corresponding to the differential d_{CE}.

(b) Even in the linear case, the spaces $\mathcal{F}$ and $\mathfrak{g}$ are often infinite dimensional, so it is convenient to interpret integration (co)homologically. Hence, we consider the shifted cotangent bundle

$$T^*[-1](\mathcal{F} \oplus \mathfrak{g}[1]) \cong \mathcal{F} \oplus \mathfrak{g}[1] \oplus \mathcal{F}^\vee[-1] \oplus \mathfrak{g}^\vee[-2].$$

Functions on $T^*[-1](\mathcal{F} \oplus \mathfrak{g}[1])$ model polyvector fields, so if we can find a divergence operator, we have a well-behaved theory of integration. The main goal of our approach to BV quantization will then be to find such a divergence operator; this operator will be called the *BV Laplacian* and is denoted by Δ.

16.2 BV Theory a la Costello

Having discussed the BV formalism in some generality, following Costello [Cos11], we now specialize to the case where our field theory is determined by a vector bundle on spacetime equipped with a local functional. To begin, it will be useful to codify the algebraic structure we observed in the case of polyvector fields. Our presentation follows that of [GLL17] and [Li23].

16.2.1 BV Algebras and Observables

Definition 16.2.1. A BV algebra is a pair $(\mathcal{A}, \Delta)$ where

– $\mathcal{A}$ is a $\mathbb{Z}$-graded commutative associative unital algebra.

– $\Delta : \mathcal{A} \to \mathcal{A}$ is a second-order operator of degree 1 such that $\Delta^2 = 0$.

Here Δ is called the BV operator. Δ being "second-order" means the following: define the *BV bracket* $\{-,-\}_\Delta$ as the measuring of the failure of Δ being a derivation

$$\{a,b\}_\Delta := \Delta(ab) - (\Delta a)b - (-1)^{|a|} a\Delta b.$$

In this section we will suppress Δ from the notation, simply writing $\{-,-\}$. Then $\{-,-\}$: $\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ defines a Poisson bracket of degree 1 satisfying

- $\{a,b\} = (-1)^{|a||b|}\{b,a\}$.
- $\{a,bc\} = \{a,b\}c + (-1)^{(|a|+1)|b|} b\{a,c\}$.
- $\Delta\{a,b\} = -\{\Delta a, b\} - (-1)^{|a|}\{a,\Delta b\}$.

Definition 16.2.2. A differential BV algebra is a triple $(\mathcal{A}, Q, \Delta)$ where

- $(\mathcal{A},\Delta)$ is a BV algebra (see Definition 16.2.1).
- $Q:\mathcal{A}\to\mathcal{A}$ is a derivation of degree 1 such that $Q^2=0$ and $[Q,\Delta]=0$.

Definition 16.2.3. Let $(\mathcal{A}, Q, \Delta)$ be a differential BV algebra. A degree 0 element $I_0\in\mathcal{A}$ is said to satisfy the *classical master equation* (CME) if

$$QI_0 + \frac{1}{2}\{I_0, I_0\} = 0.$$

A degree 0 element $I\in\mathcal{A}[[\hbar]]$ is said to satisfy the *quantum master equation* (QME) if

$$QI + \hbar\Delta I + \frac{1}{2}\{I, I\} = 0.$$

Here $\hbar$ is a formal (perturbative) parameter.

The "second-order" property of Δ implies that QME is equivalent to

$$(Q+\hbar\Delta)e^{I/\hbar} = 0.$$

If we decompose $I = \sum_{g\geq 0} I_g\hbar^g$, then the $\hbar\to 0$ limit of the QME recovers the CME: $QI_0 + \frac{1}{2}\{I_0, I_0\} = 0$.

A solution I_0 of the CME leads to a differential $Q+\{I_0,-\}$, which is usually called the BRST operator in physics.

Definition 16.2.4. Let $(\mathcal{A}, Q, \Delta)$ be a differential BV algebra and $I_0\in\mathcal{A}$ satisfy the CME. Then the *complex of classical observables*, Obs^{cl} is given by

$$Obs^{cl} \stackrel{\text{def}}{=} (\mathcal{A}, Q + \{I_0, -\}).$$

Similarly, a solution I of the QME yields a differential and correspondingly a complex of quantum observables.

Definition 16.2.5. Let $(\mathcal{A}, Q, \Delta)$ be a differential BV algebra and $I \in \mathcal{A}[[\hbar]]$ satisfy the QME. Then the *complex of quantum observables*, Obs^q is given by

$$Obs^q \overset{\text{def}}{=} (\mathcal{A}[[\hbar]], Q + \hbar\Delta + \{I, -\}).$$

Note that Obs^{cl} has a degree 1 Poisson bracket, so following [CG21] we call it a P_0 algebra. Similarly, in *ibid.* the structure on Obs^q is called a BD algebra.

16.2.2 Perturbative BV Quantization

The data of a classical field theory over a manifold M consists of a graded vector bundle E (possibly of infinite rank), whose sections we denote by $\mathcal{E}$, equipped with a -1 symplectic pairing and a local functional $S \in \mathcal{O}_{loc}(\mathcal{E})^4$ expressed as $S(e) = \langle e, Q(e)\rangle + I_0(e)$, where Q is a square zero differential operator of cohomological degree 1, such that

(a) S satisfies the CME, i.e., $\{S, S\} = 0$;

(b) I_0 is at least cubic; and

(c) $(\mathcal{E}, Q)$ is an elliptic complex.

Quantization of a field theory $(\mathcal{E}, S)$ over M consists of two stages:

(a) Build a BV algebra from the data of the pairing on the bundle E; and

(b) Promote the classical action S to a solution of the QME in this BV algebra.

The first difficulty is that the Poisson kernel K dual to the symplectic pairing is nearly always singular, so the naive definition of the BV operator $\Delta_k = \partial_k$ is ill-defined. In [Cos11], Costello uses homotopical ideas (built on the heat kernel) to build a family of well defined (smooth) BV operators Δ_L for $0 < L < \infty$. Consequently, there is a family of differential BV algebras $\{(\mathcal{E}, Q, \Delta_L)\}_{L>0}$. Costello also describes *homotopy renormalization group flow* (HRG) to relate solutions of the QME between algebras in this family.

Definition 16.2.6. Let $(\mathcal{E}, S)$ be a classical field theory over M. A *perturbative quantization* is a family of solutions to the QME, $\{I[L]\}_{L>0}$, linked by the HRG, such that

$$\lim_{L\to 0} I[L] \equiv I_0 \qquad (\text{modulo } \hbar).$$

Remark 16.2.7. Flow via the HRG induces a chain homotopy between quantum observables as we vary within the family of BV algebras $\{(\mathcal{E}, Q, \Delta_L)\}_{L>0}$. Thus, we will supress the dependence on L and abusively refer to these chain homotopic complexes as the *global quantum observables* of our field theory.

[4] As before, $\mathcal{O}(\mathcal{E}) = \widehat{\mathrm{Sym}}(\mathcal{E}^\vee)$, the subspace $\mathcal{O}_{loc}(\mathcal{E}) \subset \mathcal{O}(\mathcal{E})$ consists of those functionals determined by Lagrangian densities.

16.2.3 Scalar Field Theory

As a first example, we consider scalar field theory. Because our space of (classical) fields is simply smooth functions on a manifold $C^\infty(M)$ with no gauge symmetry, we don't need to use the full power of the BV formalism. Nonetheless, we do find this example helpful to illustrate Costello's approach. We will discuss some gauge theory examples in the next lecture.

For simplicity, let us consider $M = \mathbb{R}^n$ and let Δ be the non-negative Laplace operator, i.e., Δ is just the sum of the second partial derivatives in the coordinate directions. Our complex of fields is given by

$$\mathcal{E} : C_c^\infty(\mathbb{R}^n) \xrightarrow{\Delta} C_c^\infty(\mathbb{R}^n)[-1].$$

Our action functional is given by

$$S(\phi) = \int_{\mathbb{R}^n} \phi D \phi + I_0(\phi)$$

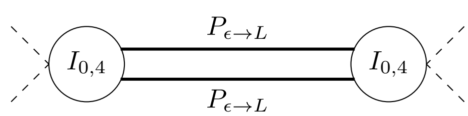
for $\phi \in \mathsf{E}$ and I_0 an interaction term (which is local and at least cubic).

For a specific example, let us consider $n = 4$ and interaction

$$I_0(\phi) = I_{0,4} = \frac{1}{4!} \int_{x \in \mathbb{R}^4} \phi(x)^4,$$

i.e., ϕ^4 theory on $\mathbb{R}^4$. If we try to quantize this theory naively by just using standard Feynman diagrammatic techniques (which is equivalent to running homotopy renormalization group flow), we will run into an issue in the $L \to 0$ limit. Indeed, there are two diagrams/Feynman weights at one-loop which are divergent. Following [Cos11] Section 4, Chapter 4, we introduce *counterterms* to cancel these divergences.

The main idea is that homotopy RG flow/Feynman diagrammatics utilize the propagator of the theory which is determined by the underlying free theory: the complex $(\mathcal{E}, \Delta)$. In this scalar field case, the propagator is simply given by the scalar heat kernel on $\mathbb{R}^4$. These propagators connect functionals at different scales, so for $0 < \epsilon < L$ we have the propagator $P_{\epsilon \to L}$. One diagram/weight that is divergent in the $\epsilon \to 0$ limit is given by the following:



The weight for this diagram, which is a functional on fields, is given explicitly as

$$\omega(\phi) = \frac{1}{2^8 \pi^4} \int_{\ell_1, \ell_2 \in [\epsilon, L]} \int_{x_1, x_2 \in \mathbb{R}^4} \frac{e^{-\|x_1 - x_2\|^2 \left((4\ell_1)^{-1} + (4\ell_2)^{-1} \right)}}{\ell_1^2 \ell_2^2} \phi(x_1)^2 \phi(x_2)^2.$$

In the $\epsilon \to 0$ limit this weight diverges like $\log \epsilon$, therefore we must add another functional to cancel this divergence. The requisite counterterm, which only depends on small ϵ, is given by

$$I_{1,4}^{CT}(\epsilon)(\phi) = -\frac{\log \epsilon}{2^8 \pi^2} \int_{x \in \mathbb{R}^4} \phi(x)^4.$$

For this theory, one other counterterm is needed:

$$I_{1,2}^{CT}(\epsilon)(\phi) = \frac{1}{\epsilon 2^6 \pi^2} \int_{x\in\mathbb{R}^4} \phi(x)^2.$$

The *naive quantization* of our theory is then given by the family of functionals

$$I[L] := \lim_{\epsilon\to 0} W\left(P_{\epsilon\to L}, I_0 + I_{1,4}^{CT}(\epsilon) + I_{1,2}^{CT}(\epsilon)\right),$$

where the operator W is the homotopy RG flow operator. That this family of functionals actually satisfies the QME at all scales follows from simple algebraic techniques Costello sets up in Chapter 5 of [Cos11].

16.3 Exercises

(a) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. Recall that a *central extension* of $\mathfrak{g}$ is a short exact sequence

$$0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{s} \mathfrak{g} \to 0$$

where $\mathfrak{a}$ is an Abelian Lie algebra.

  (a) Prove that $H^2_{\text{Lie}}(\mathfrak{g},\mathbb{R})$ is in bijection with equivalence classes of central extensions of $\mathfrak{g}$.

  (b) Prove that if $\mathfrak{g}$ is finite dimensional and simple, then $H^2_{\text{Lie}}(\mathfrak{g},\mathbb{R}) = 0$. (*Whitehead's Second Lemma* extends this result to say that for any finite dimensional $\mathfrak{g}$-module V, with $\mathfrak{g}$ finite dimensional and semi-simple, $H^2_{\text{Lie}}(\mathfrak{g}, V) = 0$.)

  (c) (Weibel p.235) Let $\mathfrak{n}_3$ be the Lie algebra of strictly upper triangular 3×3 (real) matrices. The commutator $[\mathfrak{n}_3,\mathfrak{n}_3] =: \mathbb{R}e_{13}$ is the Abelian subalgebra of matrices which are zero except for the $(1,3)$ spot. Finally, let $\mathfrak{n}_3^{\text{ab}}$ be the Abelianization of $\mathfrak{n}_3$. Show that the extension

  $$0 \to \mathbb{R}e_{13} \to \mathfrak{n}_3 \to \mathfrak{n}^{\text{ab}} \to 0$$

  does not split. In particular, this shows that the Lie algebra $\mathfrak{n}_3$ is not semi-simple.

(b) Compute $H^*_{\text{Lie}}(sl_2(\mathbb{R}),\mathbb{R})$. Similarly, compute $H^*_{\text{Lie}}(sl_2(\mathbb{R}), sl_2(\mathbb{R}))$ and $H^*_{\text{Lie}}(sl_2(\mathbb{R}), sl_2(\mathbb{R})^\vee)$ for the adjoint and co-adjoint modules.

(c) Consider ϕ^3 theory on $\mathbb{R}^6$. That is, our underlying complex and interaction are given by

$$C_c^\infty(\mathbb{R}^6) \xrightarrow{\Delta} C_c^\infty(\mathbb{R}^6)[-1], \quad I(\phi) = \frac{1}{3!}\int_{x\in\mathbb{R}^6} \phi(x)^3,$$

where Δ is the non-negative Laplacian. There is a one-loop divergence of order $\log \epsilon$, show that the needed (one-loop) counter term (so that the family of functionals $I[L]$ has a well-defined $L \to 0$ limit) is given by

$$I^{CT}_{1,3}(\epsilon) = 2^{-22}\pi^{-3} \log \epsilon \int_{x \in \mathbb{R}^6} \phi(x)^3.$$

(d) This exercise is difficult and considers the Lie algebra cohomology of some infinite dimensional Lie algebras.

(a) ([GS73]) Consider the Lie algebra of formal vector fields in n-dimensions, so

$$\mathfrak{v}_n := \left\{ \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} : f_i \in \mathbb{R}[[x_1, \ldots, x_n]] \right\}.$$

(Guillemin and Shnider 1973.). Prove that $H^i_{\text{Lie}}(\mathfrak{v}_n, \mathbb{R}) = 0$ for $0 < i \leq n$.

(b) ([Fuk86] 2.4.11) Let $Vect(S^1)$ denote the Lie algebra of smooth vector fields. Compute $H^1_{\text{Lie}}(Vect(S^1), C^\infty(S^1))$.

17 Perturbative Gauge Theory in the BV Formalism

A Invariance and Equivariance

Ryan fill in to clarify things that show up repeatedly!!

B Solutions to Selected Exercises

B.1 Section 1

Exercise 1**:** *Use Maxwell's equations to derive the continuity equation.*

Solution:

Taking the divergence of Ampere's Law and applying Gauss's Law yields the desired result:

$$\begin{aligned}\mu_0 \nabla \cdot \vec{J} &= \nabla \cdot (\nabla \times \vec{B} - \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}) \\ &= -\epsilon_0 \mu_0 \nabla \cdot (\frac{\partial \vec{E}}{\partial t}) = -\epsilon_0 \mu_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} \\ &= -\epsilon_0 \mu_0 \frac{\partial}{\partial t} \frac{\rho}{\epsilon_0} = -\mu_0 \frac{\partial \rho}{\partial t}\end{aligned}$$

Thus

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

as desired.

Exercise 2**:** *Let U be an open, contractible subset of $\mathbb{R}^4$. Let $\phi, \phi' \in C^\infty(U)$ be time dependent scalar fields, and let $\vec{A}, \vec{A}' : U \to \mathbb{R}^3$ be C^∞ time dependent vector fields. Show that if $\phi, \vec{A}$ and $\phi', \vec{A}'$ generate the same electric and magnetic field, then there exists a gauge transformation relating $\phi, \vec{A}$ to $\phi', \vec{A}'$.*

Solution:

Since $\nabla \times \vec{A} = \nabla \times \vec{A}'$, then $\nabla \times (\vec{A}' - \vec{A}) = 0$. Since U is contractible, then we know there exists $\chi \in C^\infty(U)$ such that

$$\vec{A}' = \vec{A} + \nabla \chi.$$

Since

$$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t} = -\nabla \phi' - \frac{\partial \vec{A}'}{\partial t} = -\nabla \phi' - \frac{\partial \vec{A}}{\partial t} - \nabla \frac{\partial \chi}{\partial t},$$

then

$$\nabla(\phi' - \phi - \frac{\partial \chi}{\partial t}) = 0$$

on all of U. Therefore $\phi' = \phi - \frac{\partial \chi}{\partial t}$. Hence we conclude $\phi, \vec{A}$ and $\phi', \vec{A}'$ are related be a gauge transformation (in particular a gauge transformation would be χ).

Exercise 3:

(a) Let $\phi \in C^\infty(\mathbb{R}^4)$, and let $\vec{A} : \mathbb{R}^4 \to \mathbb{R}^3$ be a C^∞ vector field. Let

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}.$$

Show the Euler-Lagrange equations from the lagrangian

$$\mathscr{L}(t, \vec{x}, \dot{\vec{x}}) = \frac{1}{2}m(\dot{\vec{x}} \cdot \dot{\vec{x}}) - Q\phi(t, \vec{x}) + Q\dot{\vec{x}} \cdot \vec{A}(t, \vec{x})$$

where Q is a real constant and $\vec{x} = (x^1, x^2, x^3)$ implies the Lorentz Force Law: $\vec{F} = Q(\vec{E} + \dot{\vec{x}} \times \vec{B})$.

Solution:

We simply compute

$$\frac{\partial \mathscr{L}}{\partial x^i} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}^i} = 0 \quad i = 1, 2, 3$$

and show each equation yields a component of $\vec{F} = Q(\vec{E} + \dot{\vec{x}} \times \vec{B})$. Since

$$\frac{\partial \mathscr{L}}{\partial x^i} = -Q\frac{\partial \phi}{\partial x^i}\dot{x}^i + Q\sum_{j=1}^{3} \dot{x}^j \frac{\partial A^j}{\partial x^i}$$

and

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}^i} &= \frac{d}{dt}\left(m\dot{x}^i + QA^i\right) \\ &= m\ddot{x}^i + Q\frac{\partial A^i}{\partial t} + Q\sum_{j=1}^{3}\frac{\partial A^i}{\partial x^j}\dot{x}^j, \end{aligned}$$

then the Euler-Lagrange equation for x^i becomes

$$0 = -Q\frac{\partial \phi}{\partial x^i} + Q\sum_{j=1}^{3}\dot{x}^j\frac{\partial A^j}{\partial x^i} - m\ddot{x}^i - Q\frac{\partial A^i}{\partial t} - Q\sum_{j=1}^{3}\frac{\partial A^i}{\partial x^j}\dot{x}^j$$

Thus

$$m\ddot{x}^i = Q\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A^i}{\partial t} + \sum_{j=1}^{3}\dot{x}^j\left(\frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x^j}\right)\right).$$

Since

$$F^i = m\ddot{x}^i \qquad E^i = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A^i}{\partial t} \qquad (\dot{\vec{x}} \times \vec{B})^i = \sum_{j=1}^{3}\dot{x}^j\left(\frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x^j}\right),$$

then we have

$$F^i = Q(E^i + (\dot{x} \times \vec{B})^i).$$

Since this holds for $i = 1, 2, 3$, then we conclude

$$\vec{F} = Q(\vec{E} + \dot{\vec{x}} \times \vec{B})$$

as desired.

(*b*) *Consider the real lagrangian density*

$$\mathscr{L} = \frac{\hbar}{2m}(\nabla\phi)\cdot(\nabla\phi^*) + V\phi\phi^* - \frac{i\hbar}{2}(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t})$$

where ϕ *and* ϕ^* *are independent of one another and are functions of space-time,* V *is a function of space-time, and* $\hbar$ *is a constant. Show the Euler-Lagrange equations from this lagrangian density implies Schrodinger's Equation:*

$$-\frac{\hbar}{2m}\nabla^2\phi + V\phi = i\hbar\frac{\partial\phi}{\partial t} \qquad -\frac{\hbar}{2m}\nabla^2\phi^* + V\phi^* = -i\hbar\frac{\partial\phi^*}{\partial t}$$

Solution:

We compute the Euler-Lagrange equation with respect to our scalar fields ϕ and ϕ^*. Varying ϕ, we obtain

$$\begin{aligned} 0 &= \frac{\partial\mathscr{L}}{\partial\phi} - \frac{\partial}{\partial t}\frac{\partial\mathscr{L}}{\partial\frac{\partial\phi}{\partial t}} - \sum_{j=1}^{3}\frac{\partial}{\partial x^j}\frac{\partial\mathscr{L}}{\partial\frac{\partial\phi}{\partial x^j}} \\ &= \left(V\phi^* + \frac{i\hbar}{2}\frac{\partial\phi^*}{\partial t}\right) - \left(\frac{-i\hbar}{2}\frac{\partial\phi^*}{\partial t}\right) - \left(\frac{\hbar}{2m}\sum_{i=1}^{3}\frac{\partial^2\phi^*}{\partial(x^i)^2}\right) \\ &= V\phi^* + i\hbar\frac{\partial\phi^*}{\partial t} - \frac{\hbar}{2m}\nabla^2\phi^* \end{aligned}$$

so that

$$-\frac{\hbar}{2m}\nabla^2\phi^* + V\phi^* = -i\hbar\frac{\partial\phi^*}{\partial t}.$$

Varying ϕ^*, we obtain

$$\begin{aligned} 0 &= \frac{\partial\mathscr{L}}{\partial\phi^*} - \frac{\partial}{\partial t}\frac{\partial\mathscr{L}}{\partial\frac{\partial\phi^*}{\partial t}} - \sum_{j=1}^{3}\frac{\partial}{\partial x^j}\frac{\partial\mathscr{L}}{\partial\frac{\partial\phi^*}{\partial x^j}} \\ &= \left(V\phi - \frac{i\hbar}{2}\frac{\partial\phi}{\partial t}\right) - \left(\frac{i\hbar}{2}\frac{\partial\phi}{\partial t}\right) - \left(\frac{\hbar}{2m}\sum_{i=1}^{3}\frac{\partial^2\phi}{\partial(x^i)^2}\right) \\ &= V\phi - i\hbar\frac{\partial\phi}{\partial t} - \frac{\hbar}{2m}\nabla^2\phi \end{aligned}$$

so that

$$-\frac{\hbar}{2m}\nabla^2\phi + V\phi = i\hbar\frac{\partial\phi}{\partial t}.$$

(*c*) *Show the Euler-Lagrange equations from the electrodynamics lagrangian density for the scalar fields from the vector potential implies Ampere's Law.*

Solution:

We compute

$$0 = \frac{\partial \mathscr{L}}{\partial A^i} - \frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \frac{\partial A^i}{\partial t}} - \sum_{j=1}^{3} \frac{\partial}{\partial x^j}\frac{\partial \mathscr{L}}{\partial \frac{\partial A^i}{\partial x^j}} \quad i = 1, 2, 3$$

to obtain the components from Ampere's Law. Note,

$$||\vec{E}||^2 = ||\nabla\phi||^2 + 2\nabla\phi \cdot \frac{\partial \vec{A}}{\partial t} + \left|\left|\frac{\partial \vec{A}}{\partial t}\right|\right|^2$$

and

$$||\vec{B}||^2 = \left(\frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3}\right)^2 + \left(\frac{\partial A^1}{\partial x^3} - \frac{\partial A^3}{\partial x^1}\right)^2 + \left(\frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}\right)^2$$

Since

$$\begin{aligned} 0 &= \frac{\partial \mathscr{L}}{\partial A^i} - \frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \frac{\partial A^i}{\partial t}} - \sum_{j=1}^{3} \frac{\partial}{\partial x^j}\frac{\partial \mathscr{L}}{\partial \frac{\partial A^i}{\partial x^j}} \\ &= J^i - \left(\epsilon_0 \frac{\partial^2 \phi}{\partial t \partial x^i} + \epsilon_0 \frac{\partial^2 \vec{A}^i}{\partial t^2}\right) - \frac{1}{\mu_0}\sum_{j=1}^{3}(-1)^j \frac{\partial}{\partial x^j}\left(\frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x^j}\right) \\ &= J^i + \epsilon_0 \frac{\partial E^i}{\partial t} - \frac{1}{\mu_0}(\nabla \times \vec{B})^i, \end{aligned}$$

then

$$(\nabla \times \vec{B})^i - \mu_0 \epsilon_0 \frac{\partial E^i}{\partial t} = \mu_0 J^i.$$

Since this holds for $i = 1, 2, 3$, then we conclude

$$\nabla \times \vec{B} - \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

which is Ampere's Law.

Exercise 4: *If one starts with equations of motion and determines an appropriate lagrangian density, then it makes sense to ask if the lagrangian density is unique. The answer turns out to be no. Suppose we have a lagrangian density*

$$\mathscr{L} = \mathscr{L}(x_k, \phi_j, \frac{\partial \phi_j}{\partial x_k}) \qquad k = 1, \ldots, n \quad j = 1, \ldots, m$$

where $x_1, \ldots, x_n$ *are the independent variables and* $\phi_1, \ldots, \phi_m$ *are the scalar fields dependent on* $x_1, \ldots, x_n$. *Suppose* $f = (f_1, \ldots, f_n)$ *is a* $\mathbb{R}^n$ *valued-function where* $f_k = f_k(\phi_1, \ldots, \phi_m)$. *Show the lagrangian density*

$$\mathscr{L}' = \mathscr{L} + \sum_{k=1}^{n} \frac{\partial f_k}{\partial x^k}$$

generates the same Euler-Lagrange equations as $\mathscr{L}$.

Solution:

First, note

$$\mathscr{L}' = \mathscr{L} + \sum_{k=1}^{n} \frac{\partial f_k}{\partial x^k} = \mathscr{L} + \sum_{k=1}^{n} \sum_{l=1}^{m} \frac{\partial f_k}{\partial \phi^l} \frac{\partial \phi^l}{\partial x^k}$$

Since for $j \in \{1, \ldots, m\}$,

$$\frac{\partial \mathscr{L}'}{\partial \phi^j} = \frac{\partial \mathscr{L}}{\partial \phi^j} + \sum_{k=1}^{n} \frac{\partial^2 f_k}{\partial \phi^j \partial x^k}$$

and

$$\begin{aligned}
\sum_{i=1}^{n} \frac{\partial}{\partial x^i} \frac{\partial \mathscr{L}'}{\partial \frac{\partial \phi^j}{\partial x^i}} &= \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \frac{\partial \mathscr{L}}{\partial \frac{\partial \phi^j}{\partial x^i}} + \sum_{i=1}^{n} \frac{1}{\partial x^i} \frac{1}{\partial \frac{\partial \phi^j}{\partial x^i}} \sum_{k=1}^{n} \sum_{l=1}^{m} \frac{\partial f_k}{\partial \phi^l} \frac{\partial \phi^l}{\partial x^k} \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \frac{\partial \mathscr{L}}{\partial \frac{\partial \phi^j}{\partial x^i}} + \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \frac{\partial f_i}{\partial \phi^j} \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \frac{\partial \mathscr{L}}{\partial \frac{\partial \phi^j}{\partial x^i}} + \sum_{i=1}^{n} \sum_{l=1}^{m} \frac{\partial^2 f_i}{\partial \phi^l \partial \phi^j} \frac{\partial \phi^l}{\partial x^i} \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \frac{\partial \mathscr{L}}{\partial \frac{\partial \phi^j}{\partial x^i}} + \sum_{i=1}^{n} \sum_{l=1}^{m} \frac{\partial^2 f_i}{\partial \phi^j \partial \phi^l} \frac{\partial \phi^l}{\partial x^i} \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \frac{\partial \mathscr{L}}{\partial \frac{\partial \phi^j}{\partial x^i}} + \sum_{k=1}^{n} \frac{\partial^2 f_k}{\partial \phi^j \partial x^k}
\end{aligned}$$

then

$$\begin{aligned}
\frac{\partial \mathscr{L}'}{\partial \phi^j} - \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \frac{\partial \mathscr{L}'}{\partial \frac{\partial \phi^j}{\partial x^i}} &= \frac{\partial \mathscr{L}}{\partial \phi^j} + \sum_{k=1}^{n} \frac{\partial^2 f_k}{\partial \phi^j \partial x^k} - \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \frac{\partial \mathscr{L}}{\partial \frac{\partial \phi^j}{\partial x^i}} - \sum_{k=1}^{n} \frac{\partial^2 f_k}{\partial \phi^j \partial x^k} \\
&= \frac{\partial \mathscr{L}}{\partial \phi^j} - \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \frac{\partial \mathscr{L}}{\partial \frac{\partial \phi^j}{\partial x^i}}
\end{aligned}$$

proving the claim.

B.2 Section 2

Exercise 1: *Let* $p \in \mathbb{R}$.

(a) *Verify the Dirac Delta function at p, denoted as* δ_p*, is a distribution on* $\mathbb{R}$.

Solution:

Clearly $\delta_p : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ Is a linear map. To see δ_p is continuous, let K be a compact subset of $\mathbb{R}$. Since for each $\phi \in \mathcal{D}(\mathbb{R})$ with $\text{supp}(\phi) \subset K$,

$$|\delta_p \phi| = |\phi(p)| \leq ||\phi||_\infty$$

then δ_p is continuous. Therefore is a distribution on $\mathbb{R}$.

(b) *Verify the heavy side function at p, denoted as* H_p*, is locally integrable on* $\mathbb{R}$.

Solution:

Since $H_p = \chi_{(p,\infty)}$ and (p, ∞) is a Borel measureable set, then H_p is indeed a measurable function. Let K be a compact subset of $\mathbb{R}$, then, using the Heine-Borel Theorem, $K \subset [-a, a]$ for some $a \in \mathbb{R}$. Let λ denote the Lebesgue measure on $\mathbb{R}$, then

$$\int_K |H_p| d\lambda \leq \int_{[-a,a]} |H_p| d\lambda \leq \int_{[-a,a]} c_1 d\lambda = 2a < \infty$$

where $c_1(x) = 1$. Therefore H_p is indeed a locally integrable function on $\mathbb{R}$.

(c) *Verify* $\frac{d}{dx}\Lambda_{H_p} = \delta_p$.

Solution:

Let $\phi \in \mathcal{D}(\mathbb{R})$. Since $\text{supp}(\phi)$ is compact, then $\text{supp}(\phi) \subset [-a, a]$ where $a > 0$. In particular, we can pick $a > 0$ such that $\phi(a) = 0$ and $a > p$. Therefore

$$\begin{aligned} \frac{d}{dx}\Lambda_{H_p}\phi &= -\int_{-\infty}^{\infty} H_p(x)\phi'(x)dx = -\int_p^{\infty} \phi'(x)dx = -\int_p^a \phi'(x)dx \\ &= \phi(p) - \phi(a) = \phi(p) = \delta_p\phi \end{aligned}$$

Since ϕ was arbitrary, we conclude $\frac{d}{dx}\Lambda_{H_p} = \delta_p$.

Exercise 2: *Verify that if* G *is the Green's function for a continuous partial differential operator* L*, then a solution to* $Lf = g$ *is indeed* $f(x) = \int G(x, x')g(x')dx'$.

Solution:

Since L is continuous and differentiates in x while the f is define as an integral in x', then

$$(Lf)(x) = L\int G(x,x')g(x')dx' = \int LG(x,x')g(x')dx = \int \delta(x'-x)g(x')dx' = g(x).$$

Hence $Lf = g$.

Exercise 3: *Verify taking $\vec{X} = \phi\nabla G + G\nabla\phi$ and $f = \phi$ in the Divergence Theorem does in fact yield the equation for ϕ in terms of the the Green's function G, charge density ρ, and boundary condition ψ.*

Solution:

Since

$$\begin{aligned}\nabla\cdot(G\nabla\phi - \phi\nabla G) &= \nabla G\cdot\nabla\phi + G\nabla^2\phi - \nabla\phi\cdot\nabla G - \phi\nabla^2 G\\ &= -G\frac{\rho}{\epsilon_0} - \phi\delta(x'-x)\end{aligned}$$

then

$$\int_U \nabla\cdot(G(x,x')\nabla\phi - \phi(x')\nabla G)dV = -\int_U G(x,x')\frac{\rho(x')}{\epsilon_0}dV - \phi(x)$$

Using the Divergence Theorem, we also know

$$\int_U \nabla\cdot(G\nabla\phi - \phi\nabla G)dV = \int_{\partial U}(G\nabla\phi - \phi\nabla G)\cdot\hat{n}dA$$

Since

$$D_{\hat{n}}\phi = \nabla\phi\cdot\hat{n} \qquad D_{\hat{n}}G = \nabla G\cdot\hat{n},$$

then

$$\int_{\partial U}(G(x,x')\nabla\phi - \phi(x')\nabla G)\cdot\hat{n}dA = \int_{\partial U} G(x,x')D_{\hat{n}}\phi - \phi)x')D_{\hat{n}}GdA$$

so that

$$\phi(x) = \frac{-1}{\epsilon_0}\int_U G(x,x')\rho(x')dV + \int_{\partial U}\phi(x')D_{\hat{n}}G - G(x,x')D_{\hat{n}}\phi dA.$$

Since $\phi|_{\partial U} = \psi$, then

$$\phi(x) = \frac{-1}{\epsilon_0}\int_U G(x,x')\rho(x')dV + \int_{\partial U}\psi(x')D_{\hat{n}}G - G(x,x')D_{\hat{n}}\phi dA.$$

as desired.

Exercise 4**:** *The following problem will have you prove the uniqueness for solutions to the Dirichlet Boundary Value Problem. Let $U \subset \mathbb{R}^n$ be open such that $\overline{U}$ is compact. Fix a continuous functions $f : \partial U \to \mathbb{R}$ and $g : U \to \mathbb{R}$. Let $\phi : \overline{U} \to \mathbb{R}$ be a function such that ϕ is C^2 on U and ϕ is continuous on ∂U. Show that if $\phi|_{\partial U} = f$ and $\nabla^2\phi = g$ on U, then ϕ is unique. (Hint: consider an approach similar to showing Poisson's equation has unique solutions using the Divergence Theorem)*

Solution:

Suppose ϕ_1 and ϕ_2 are two solutions, then $\psi = \phi_1 - \phi_2$ is a solution to $\nabla^2\psi = 0$ on U with $\psi|_{\partial U} = 0$. Therefore

$$0 = \psi(\nabla^2\psi) = \nabla \cdot (\psi\nabla\psi) - ||\nabla\psi||^2$$

so that

$$0 = \int_U \psi\nabla^2\psi dV = \int_U \nabla \cdot (\psi\nabla\psi) - ||\nabla\psi||^2 dV$$

Let $\hat{n}$ be an outward point normal vector field along ∂U, then the Divergence Theorem tells us

$$\int_U \nabla \cdot \psi\nabla\psi dV = \int_{\partial U} \psi\nabla\psi \cdot \hat{n} dA$$

Since $\psi|_{\partial U} = 0$, then

$$\int_U \nabla \cdot \psi\nabla\psi dV = \int_{\partial U} \psi\nabla\psi \cdot \hat{n} dA = \int_{\partial U} \psi|_{\partial U}\nabla\psi \cdot \hat{n} dA = 0.$$

Thus

$$0 = \int_U \psi\nabla^2\psi dV = \int_U \nabla \cdot (\psi\nabla\psi) - ||\nabla\psi||^2 dV = -\int_U ||\nabla\psi||^2 dV$$

Therefore $||\nabla\psi|| = 0$ on U which implies $\nabla\psi = 0$ on U. Therefore ψ is constant on U. Since ψ is continuous on the boundary of U and is zero on the boundary, then $\psi = 0$ on U. Therefore $\phi_1 = \phi_2$. Hence a solution to the Dirichlet Boundary Problem is indeed unique.

B.3 Section 3: Electrodynamics

1. Prove following identities:

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$$
$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{2}\nabla|\mathbf{E}|^2 - (\mathbf{E} \cdot \nabla) \cdot \mathbf{E}.$$

Solution We use the Levi-Civita symbol to compute

$$\begin{aligned}\nabla \cdot (\mathbf{E} \times \mathbf{B}) &= \partial_i \epsilon_{ijk} E_j B_k \\ &= \epsilon_{ijk}(\partial_i E_j)B_k + \epsilon_{ijk}E_j(\partial_i B_k) \\ &= \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})\end{aligned}$$

and we also use the identity, $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, to have

$$\begin{aligned}
[\mathbf{E} \times (\nabla \times \mathbf{E})]_i &= \epsilon_{ijk} E_j \epsilon_{klm} \partial_l E_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) E_j \partial_l E_m \\
&= E_j \partial_i E_j - E_j \partial_j E_i \\
&= \left[\frac{1}{2}\nabla|\mathbf{E}|^2 - (\mathbf{E}\cdot\nabla)\cdot\mathbf{E}\right]_i .
\end{aligned}$$

2. *Conservation of the total angular momentum of the electromagnetic fields* [Wald, Problem 5.2] The angular momentum density of the electromagnetic field is given by

$$\mathbf{l} = \mathbf{x} \times \mathcal{P} = \epsilon_0 \mathbf{x} \times (\mathbf{E} \times \mathbf{B}).$$

Consider a source-free ($\rho = 0, \mathbf{J} = \mathbf{0}$) solution to Maxwell's equations with $\mathbf{E}$ and $\mathbf{B}$ vanishing rapidly as $|\mathbf{x}| \to \infty$, so the total momentum

$$\mathbf{L} = \int \mathbf{l}\, d^3x$$

is well defined. Show that $\mathbf{L}$ is conserved (i.e., independent of time).

Solution In the source free case we have $\partial \mathcal{P}_i/\partial t = \partial_j \Theta_{ij}$. Hence, assuming that $|\mathbf{E}|$ and $|\mathbf{B}|$ decrease to zero fast as $|\mathbf{x}|$ tends to infinity so that the same holds for $x_j\Theta_{kl}$, we get

$$\begin{aligned}
\frac{dL_i}{dt} &= \int \frac{\partial l_i}{\partial t}\, d^3x \\
&= \int \left[\mathbf{x} \times \frac{\partial \mathcal{P}}{\partial t}\right]_i d^3x \\
&= \int \epsilon_{ijk} x_j \frac{\partial \mathcal{P}_k}{\partial t} d^3x \\
&= \int \epsilon_{ijk} x_j \partial_l \Theta_{kl} d^3x \\
&= \int \left[\partial_l(\epsilon_{ijk} x_j \Theta_{kl}) - \epsilon_{ijk}\delta_{jl}\Theta_{kl}\right] d^3x \\
&= -\int \epsilon_{ilk}\Theta_{kl}\, d^3x \\
&= 0,
\end{aligned}$$

where we observed that $\epsilon_{ilk} = -\epsilon_{ikl}$ and $\Theta_{kl} = \Theta_{lk}$, hence $\epsilon_{ilk}\Theta_{kl} = 0$.

3. *Force on a charge from a circularly moving charge* [Wald, Problem 5.3] A particle of charge q_1 moves with velocity v in a circular orbit of radius R about the origin in the x-y plane, such that its ϕ coordinate varies as $\phi = \omega t$, with $\omega = v/R$. Assume that $v \ll c$. Another particle of charge q_2 is at rest at point $\mathbf{x}$, where $|\mathbf{x}| \gg R$. To order $1/|\mathbf{x}|$, find the force $\mathbf{F}$ on the particle of charge q_2 at time t.

Solution We are in the radiation zone approximation. With $\rho(t, \mathbf{x}) = q_1\delta(\mathbf{x} - Re^{i\omega t})$ for the moving charge we get the electric dipole moment $\mathbf{p}(t) = \int \mathbf{x}\rho(t, \mathbf{x})\, d^3x = Rq_1 e^{i\omega t}$. Here $e^{i\omega t} = (\cos\omega t, \sin\omega t, 0)$. Thus,

$$\begin{aligned}\mathbf{E}(t, \mathbf{x}) &\approx \frac{\mu_0}{4\pi x}\hat{\mathbf{x}} \times (\hat{\mathbf{x}} \times \ddot{\mathbf{p}}(t - x/c)) \\ &= -\frac{\mu_0}{4\pi x}\omega^2 Rq_1 \hat{\mathbf{x}} \times (\hat{\mathbf{x}} \times \hat{\mathbf{p}}(t - x/c)).\end{aligned}$$

The force on the charge q_2 from oscillating charge q_1 is

$$\mathbf{F}(t, \mathbf{x}) = q_2\mathbf{E}(t, \mathbf{x}) \approx -\frac{\mu_0}{4\pi}\frac{q_1 q_2}{x}\omega^2 R \sin(\theta(t - x/c))\hat{\mathbf{n}}(t - x/c),$$

where $\theta(t - x/c)$ is the angle between $\mathbf{x}$ and $\mathbf{p}(t - x/c)$, and $\hat{\mathbf{n}}(t - x/c)$ is the unit vector on the plane containing $\mathbf{x}$ and $\mathbf{p}(t - x/c)$ obtained by rotating $\hat{\mathbf{x}}$ by 90 degree away from $\hat{\mathbf{p}}$.

4. *Radiation of electromagnetic energy from an oscillating charge* [Wald, Problem 5.6] A point charge of charge q and mass m is placed on the end of a spring with spring constant k. The charge is displaced in the z-direction by an amount α away from its equilibrium position and is then released to oscillate. Assume that the resulting motion is nonrelativistic, $v \ll c$.

(a) Assume that the charge oscillates harmonically with amplitude α. To order $1/r$ in distance from the charge and to leading order in v/c, what are the resulting electromagnetic potential $\phi, \mathbf{A}$?

(b) What is the radiated power?

(c) As a result of the radiation of electromagnetic energy, the maximum amplitude of oscillation, α, will, in fact, slowly decay with time. Find $\alpha(t)$.

Solution (a) From $\mathbf{p}(t) = \hat{z}q\alpha(t)\cos\omega t$ we get

$$\dot{\mathbf{p}}(t) = q(\dot{\alpha}(t)\cos\omega t - \omega\alpha(t)\sin\omega t)\hat{z}, \qquad \ddot{\mathbf{p}}(t) = q(\ddot{\alpha}(t)\cos\omega t - 2\omega\dot{\alpha}(t)\sin\omega t - \omega^2\alpha(t)\cos\omega t)\hat{z}.$$

We have

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x' \approx \frac{q}{4\pi r}$$

and

$$\begin{aligned}\mathbf{A}(t, \mathbf{x}) &\approx \frac{\mu_0}{4\pi r}\dot{\mathbf{p}}\left(t - \frac{r}{c}\right) = \frac{\mu_0}{4\pi r}q\left[\dot{\alpha}(t)\cos\omega\left(t - \frac{r}{c}\right) - \omega\alpha(t)\sin\omega\left(t - \frac{r}{c}\right)\right]\hat{z} \\ &\approx -\hat{z}\frac{\mu_0}{4\pi r}q\omega\alpha(t - r/c)\sin\omega\left(t - \frac{r}{c}\right)\end{aligned}$$

if we assume $|\dot{\alpha}(t)/\alpha(t)| \ll \omega$.

(b) If θ is the angle between z-axis and $\mathbf{x}$, then assuming $|\dot{\alpha}(t)/\alpha(t)|, |\ddot{\alpha}(t)/\alpha(t)|^{1/2} \ll \omega$, we have

$$\mathcal{S}(t, \mathbf{x}) \approx \frac{\mu_0}{4\pi^2 r^2 c}|\hat{\mathbf{x}} \times \ddot{\mathbf{p}}(t - r/c)|^2\hat{\mathbf{x}} \approx \frac{\mu_0}{4\pi^2 r^2 c}\left[q\omega^2\alpha(t - r/c)\cos\omega(t - r/c)\right]^2 \sin^2\theta\; \hat{\mathbf{x}}$$

and the radiated power is

$$P(t) \approx \frac{\mu_0}{4\pi^2 d^2 c}(q\omega^2\alpha(t-d/c))^2\cos^2(\omega(t-d/c))\int_{dS^2}\sin^2\theta dS$$
$$= \frac{\mu_0}{6\pi c}(q\omega^2\alpha(t-d/c))^2\cos^2(\omega(t-d/c))$$
$$= \frac{\mu_0}{12\pi c}q\omega^4\alpha^2(t),$$

where we assumed that $\alpha < d \ll ct$, hence $\cos^2(\omega t)$ was averaged over time.

(c) Under the same assumption about α as above we have the total energy of the charge

$$E_{\text{charge}} = \frac{m}{2}\dot{z}^2 + \frac{k}{2}z^2 = \frac{m}{2}[(\dot{\alpha}(t)\cos\omega t - \omega\alpha(t)\sin\omega t)^2 + \omega^2\alpha^2(t)\cos^2\omega t]$$
$$\approx \frac{m}{2}\omega^2\alpha^2(t).$$

Thus, from $dE_{\text{charge}}/dt = -P$, we get

$$\frac{d}{dt}\left(\frac{m}{2}\omega^2\alpha^2(t)\right) = -\frac{\mu_0}{12\pi c}q^2\omega^4\alpha^2(t), \text{ or } \frac{d}{dt}\alpha^2(t) = -\frac{\mu_0 q^2\omega^2}{6\pi mc}\alpha^2(t),$$

which implies

$$\alpha(t) = \alpha\exp\left(-\frac{\mu_0 q^2\omega^2}{12\pi mc}t\right).$$

This solution satisfies the condition $|\dot{\alpha}(t)/\alpha(t)|, |\ddot{\alpha}(t)/\alpha(t)|^{1/2} \ll \omega$, since we have a non-relativistic motion.

5. *Schwartz space and tempered distributions*

(a) Show that the Fourier transform is a bijection on the Schwartz space $\mathcal{S}(\mathbb{R}^4)$. Is it continuous?

(b) Show that any tempered distribution is a distribution. Is the delta function $\delta(\mathbf{x})$ on $\mathbb{R}^4$ a tempered distribution?

(c) Show that the Fourier inverse transform of the Fourier transform of a tempered distribution T is T itself.

Solution (a) Let $f \in \mathcal{S}(\mathbb{R}^4)$ and α, β be multi-indices. From the definition of the Schwartz space we have $\partial_i f(\mathbf{x}), x^i f(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^4)$ for each i, hence $g(\mathbf{x}) := D^\beta(i\mathbf{x})^\alpha f(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^4)$. Thus, there is $C > 0$ such that $|g(\mathbf{x})| \le C/|\mathbf{x}|^5$, hence $\hat{g}(\mathbf{k})$ is bounded. Using

$$\hat{f}(\mathbf{k}) = \frac{1}{4\pi^2}\int f(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{k}}\,d^4x$$

and integration by parts, we get

$$(i\mathbf{k})^\beta D^\alpha \hat{f}(\mathbf{k}) = \frac{1}{4\pi^2}\int D^\beta(i\mathbf{x})^\alpha f(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{k}}\,d^4x = \hat{g}(\mathbf{k}),$$

hence $1\mathbf{k}^\beta D^\alpha \hat{f}(\mathbf{k})$ is bounded. This shows that $\hat{f} \in \mathcal{S}(\mathbb{R}^4)$.

(b) Clearly $\mathcal{D}(\mathbb{R}^4) \subset \mathcal{S}(\mathbb{R}^4)$. Also, if $\phi_n \to 0$ in $\mathcal{D}(\mathbb{R}^4)$, then the same holds in $\mathcal{S}(\mathbb{R}^4)$. Thus, if T is a tempered distribution, then whenever $\phi_n \to 0$ in $\mathcal{D}(\mathbb{R}^4)$, we have $\langle T, \phi_n\rangle \to 0$, that is, T is a distribution. Suppose that $\phi_n \to 0$ in $\mathcal{S}(\mathbb{R}^4)$. Then, $\langle \delta(\mathbf{x}), \phi_n \rangle = \phi_n(0) \to 0$, showing that the Dirac delta function is a tempered distribution.

(c) The inverse Fourier transform is given by

$$\check{f}(\mathbf{k}) = \frac{1}{4\pi^2} \int f(\mathbf{x}) e^{i\mathbf{x}\cdot\mathbf{k}}\, d^4x.$$

The Fourier inversion theorem and the change of variable yields that $\check{\hat{\phi}} = \phi$ for all $\phi \in \mathcal{S}(\mathbb{R}^4)$. Thus, if T is a tempered distribution, then, by the definition of the inverse Fourier transformation we have $\langle \check{\hat{T}}, \phi \rangle = \langle \hat{T}, \check{\phi} = \langle T, \check{\hat{\phi}} \rangle = \langle T, \phi \rangle$.

B.4 Solutions to Section 4 Exercises

(a) The equation of propagation from the WKB approximation is

$$\frac{d^2\boldsymbol{x}}{d\tau^2} = \frac{\omega^2}{c^2} n(\boldsymbol{x}) \nabla n(\boldsymbol{x})$$

and the action integral is given by

$$\mathcal{S} = \int n(\boldsymbol{x}) \sqrt{\left\| \frac{d\boldsymbol{x}}{d\lambda} \right\|^2}\, d\lambda = \int L(\boldsymbol{x}, \dot{\boldsymbol{x}})\, d\lambda$$

Here $\dot{\boldsymbol{x}} = d\boldsymbol{x}/d\lambda$.

Let's find the Euler-Lagrange equation of this action

$$\frac{\partial L}{\partial \dot{\boldsymbol{x}}} = n(\boldsymbol{x}) \frac{\dot{\boldsymbol{x}}}{\sqrt{\|\dot{\boldsymbol{x}}\|^2}}$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\right) = \frac{d}{d\lambda}\left(n(\boldsymbol{x}) \frac{\dot{\boldsymbol{x}}}{\sqrt{\|\dot{\boldsymbol{x}}\|^2}} \right) = \frac{dn(\boldsymbol{x})}{d\lambda} \frac{\dot{\boldsymbol{x}}}{\|\dot{\boldsymbol{x}}\|} + \frac{n(\boldsymbol{x})}{\|\dot{\boldsymbol{x}}\|} \ddot{\boldsymbol{x}} = \frac{n(\boldsymbol{x})}{\|\dot{\boldsymbol{x}}\|} \ddot{\boldsymbol{x}}$$

Since $dn/d\lambda = 0$.

$$\frac{\partial L}{\partial \boldsymbol{x}} = \nabla n(\boldsymbol{x}) \sqrt{\|\dot{\boldsymbol{x}}\|^2}$$

Putting these together

$$\frac{n(\boldsymbol{x})}{\|\dot{\boldsymbol{x}}\|} \ddot{\boldsymbol{x}} = \nabla n(\boldsymbol{x}) \|\dot{\boldsymbol{x}}\|$$

or

$$\frac{d^2\boldsymbol{x}}{d\lambda^2} = \frac{1}{n(\boldsymbol{x})} \left\| \frac{d\boldsymbol{x}}{d\lambda} \right\|^2 \nabla n(\boldsymbol{x})$$

Then if we reparameterize to τ, in which $\|\frac{d\boldsymbol{x}}{d\tau}\|^2 = n^2 \frac{\omega^2}{c^2}$ we recover

$$\frac{d^2\boldsymbol{x}}{d\tau^2} = \frac{\omega^2}{c^2} n(\boldsymbol{x}) \nabla n(\boldsymbol{x})$$

(b) Noting that

$$n(\boldsymbol{x}) = \sqrt{\epsilon/\epsilon_0} = \sqrt{a - b(x^2 + y^2)}$$

the gradient of n can be calculated

$$\nabla n(\boldsymbol{x}) = \left(\frac{-bx}{n(\boldsymbol{x})}, \frac{-by}{n(\boldsymbol{x})} \right)$$

The equation of propagation then becomes

$$\frac{d^2\boldsymbol{x}}{d\tau^2} = \frac{\omega^2}{c^2} n(\boldsymbol{x}) \nabla n(\boldsymbol{x}) = \frac{\omega^2}{c^2}(-bx, -by) = \frac{-b\omega^2}{c^2}\boldsymbol{x} = 0$$

This is a decoupled system of undamped harmonic oscillators. Hence, the solution is

$$\boldsymbol{x}(\tau) = \boldsymbol{x}(0)\cos(\kappa t) + \frac{\dot{\boldsymbol{x}}(0)}{\kappa}\sin(\kappa\tau)$$

where $\kappa = \frac{\sqrt{b}\omega}{c}$. If the initial conditions are sufficiently small the solution will oscillate around the axis for all time.

(c) Note that each side of the xy-plane has a homogeneous index of refraction so we have a straight line from $\boldsymbol{x}_1$ to some point $\boldsymbol{p}$ in the xy-plane and a straight line from $\boldsymbol{p}$ to $\boldsymbol{x}_2$. This means the total time of travel is the sum of times for each straight segment, i.e.

$$T(\boldsymbol{p}) = T_1 + T_2 = \frac{1}{c}\|\boldsymbol{p} - \boldsymbol{x}_1\| + \frac{n_2}{c}\|\boldsymbol{x}_2 - \boldsymbol{p}\|$$

Now we want to minimize time by varying $\boldsymbol{p}$, that is we want

$$\frac{dT}{dp_i} = \frac{1}{c}\frac{|p_i - x_{1i}|}{\|\boldsymbol{p} - \boldsymbol{x}_1\|} - \frac{n_2}{c}\frac{|x_{2i} - p_i|}{\|\boldsymbol{x}_2 - \boldsymbol{p}\|} = 0$$

Using basic trigonometry we have for each i

$$\frac{|p_i - x_{1i}|}{\|\boldsymbol{p} - \boldsymbol{x}_1\|} = \sin\theta_{1i} \quad \text{and} \quad \frac{|x_{2i} - p_i|}{\|\boldsymbol{x}_2 - \boldsymbol{p}\|} = \sin\theta_{2i}$$

Which is exactly Snell's law

$$\frac{\sin\theta_1}{c} = \sin\theta_2 \frac{n_2}{c}$$

B.5 Section 5

Exercise 1**:** *Let* V *be a finite dimensional vector space with symmetric, nondegenerate bilinear form* ω *on* V*. For a vector subspace* $S \subset V$*, define* $S^{\perp} = \{u \in V : \forall v \in S, \omega(u, v) = 0\}$*.*

(a) *Show* $\dim(S^{\perp}) + \dim(S) = \dim(V)$*.*

Proof.

Define $f : V \to S^{\vee}$ (the dual of S) where $f(v) : S \to \mathbb{R}$ given by $f(v)s = \omega(s, v)$. Since ω is bilinear, then $f(v)$ is a linear map. Therefore f is well defined. Using the bilinearity of ω again, we know f is in fact bilinear. We claim $\ker(f) = S^{\perp}$ and $\text{Image}(f) = S^{\vee}$ so that

$$\dim(V) = \dim(\ker(f)) + \dim(\text{Image}(f)) = \dim(S^{\perp}) + \dim(S)$$

To see $\ker(f) = S^{\perp}$, suppose $v \in S^{\perp}$, then for all $s \in S$, $\omega(s, v) = 0$. Hence $f(v) = 0$ so that $v \in \ker(f)$. Now suppose $v \in \ker(f)$, then for all $s \in S$, $0 = f(v)s = \omega(s, v)$. Hence $v \in S^{\perp}$. Thus we conclude $\ker(f) = S^{\perp}$. To see $\text{Image}(f) = S^{\vee}$, note that the inclusion map $\text{inc} : S \to V$ induces a linear map $r : V^{\vee} \to S^{\vee}$ where $r(\phi) = \phi|_S$. Since V is finite dimensional, then we know r is surjective. Furthermore, $f = r \circ \omega^{\flat}$. Thus f is surjective proving $\text{Image}(f) = S^{\vee}$.

□

(b) *Show* $(S^{\perp})^{\perp} = S$*.*

Proof.

Note,

$$S^{\perp} = \{u \in V : \forall v \in S, \omega(u, v) = 0\}$$

and

$$(S^{\perp})^{\perp} = \{v \in V : \forall s \in S^{\perp}, \omega(v, s) = 0\}$$

Thus we clearly have $S \subset (S^{\perp})^{\perp}$. Using (a), we know $\dim(S) = \dim((S^{\perp})^{\perp})$. Therefore we conclude $S = (S^{\perp})^{\perp}$.

□

Exercise 2: *Let V be a finite dimensional vector space with symmetric, nondegenerate bilinear form ω on V. A vector subspace $S \subset V$ is nondegenerate if $\omega|_{S\times S}$ is nondegenerate. Show the following are equivalent:*

(a) *S is nondegenerate.*

(b) *$S^\perp$ is nondegenerate.*

(c) *$S^\perp \cap S = \{0\}$.*

(d) *$V = S \oplus S^\perp$.*

Proof.

First, suppose S is nondegenerate. For the sake of contradiction, suppose $S^\perp$ is not nondegenerate, then there exists a non-zero $v \in S^\perp$ such that for all $w \in S^\perp$, $\omega(v, w) = 0$. Thus $v \in (S^\perp)^\perp = S$. However, for all $s \in S$, $\omega(v, s) = 0$ as $v \in S^\perp$. Therefore S is not nondegenerate which is a contradiction. Hence $S^\perp$ is indeed nondegenerate. Therefore (1) implies (2).

Now suppose $S^\perp$ is nondegenerate. Let $v \in S^\perp \cap S$. Since $v \in S$, then for all $s \in S^\perp$, $\omega(v, s) = 0$. Since $v \in S^\perp$ and $\omega|_{S^\perp\times S^\perp}$ is nondegenerate, then $v = 0$. Hence $S^\perp \cap S = \{0\}$. Thus (2) implies (3).

Now suppose $S^\perp \cap S = \{0\}$, then, using Exercise 1.a, we know $\dim(S^\perp + S) = \dim(V)$. Hence $S^\perp \oplus S = V$. Therefore (3) implies (4).

Finally, suppose $V = S \oplus S^\perp$. Let $s \in S$, then there exists $v \in V$ for which $\omega(s, v) \neq 0$. Since $V = S \oplus S^\perp$, then we can uniquely write $v = a + b$ where $a \in S$ and $b \in S^\perp$. Therefore

$$0 \neq \omega(s, v) = \omega(s, a + b) = \omega(s, a) + \omega(s, b) = \omega(s, a)$$

Therefore $\omega|_{S\times S}$ is indeed nondegenerate. Hence (4) implies (1).

□

Exercise 3: *Let V be a finite dimensional vector space with symmetric, nondegenerate bilinear form ω on V.*

(a) *Show that if S is nondegenerate and $S \neq 0$, then there exits $v \in S$ such that $\omega(v, v) \neq 0$.*

Proof.

Suppose S is nondegenerate with $S \neq 0$, but for all $v \in V$, $\omega(v, v) = 0$. Then ω is anti-symmetric on S. Therefore $\omega|_{S \times S} = 0$ which contradicts S being nondegenerate. Thus there indeed exists $v \in S$ such that $\omega(v, v) \neq 0$.

□

(b) *A linearly independent list of vectors $v_1, \ldots, v_k \in V$ are nondegenerate provided $\omega(v_j, v_j) \neq 0$ for each $j \in \{1, \ldots, k\}$. Suppose $\dim(V) = n$ and $k < n$. Show there exists $v_{k+1}, \ldots, v_n \in V$ such that $v_1, \ldots, v_n$ is a nondegenerate basis for V.*

Proof.

Suppose $v_1, \ldots, v_k \in V$ are nondegenerate. Therefore $S = \mathrm{Span}(v_1, \ldots, v_k)$ is a nondegenerate subspace of V. Therefore $S^{\perp}$ is a nondegenerate subspace of V. Since $k < n$, then $S^{\perp} \neq 0$. Hence, by (a), there exists $v_{k+1} \in S^{\perp}$ with $\omega(v_{k+1}, v_{k+1}) \neq 0$. Thus $v_1, \ldots, v_{k+1}$ is nondegenerate. Applying the same argument to $v_1, \ldots, v_{k+1}$, we can continue to find such vectors $v_{k+2}, \ldots, v_n \in V$ such that $v_1, \ldots, v_n$ is nondegenerate. Since we have n-linearly independent vectors, then $v_1, \ldots, v_n$ is a basis for V.

□

(c) *Show that for V has a nondegenerate basis.*

Proof.

Since V is nondegenerate and $V \neq 0$, then there exists $v \in V$ such that $\omega(v, v) \neq 0$. Thus, by (b), we know v can be extended to a nondegenerate basis for V. Hence V has a nondegenerate basis.

□

(d) *Show that V has a basis $e_1, \ldots, e_n$ for such that $\omega(e_i, e_j) = \delta_{ij}$ or $\omega(e_i, e_j) = -\delta_{ij}$.*

Proof.

Let $v_1, \ldots, v_n$ be a nondegenerate basis of V which exists by (c). Applying Graham-Schmidt to $v_1, \ldots, v_n$ yields the desired basis for V.

□

Exercise 4**:** *The following exercise investigates some of the topological features of* $O(r, s)$.

(*a*) *Assume* $r, s > 0$. *Show that* $O(r, s)$ *has at least four connected components.*

(*b*) *Assume* $r, s > 0$. *Show that* $O(r, s)$ *is not compact.*

(*c*) *Show for all* $r, s \in \mathbb{N}_0$, $O(r, s)$ *and* $O(s, r)$ *are isomorphic as Lie groups (recall that bijective Lie group homomorphisms are automatically Lie group isomorphisms)*

Exercise 5: *Verify* $*d*\mathfrak{F} = \mathfrak{J}$

Answer (not fully complete yet need to check signs):

First, observe

k-form	Hodge star of the k-form
1	$dt \wedge dx \wedge dy \wedge dz$
dt	$dx \wedge dy \wedge dz$
dx	$dt \wedge dy \wedge dz$
dy	$-dt \wedge dx \wedge dz$
dz	$dt \wedge dx \wedge dy$
$dt \wedge dx$	$dy \wedge dz$
$dt \wedge dy$	$-dx \wedge dz$
$dt \wedge dz$	$dx \wedge dy$
$dx \wedge dy$	$-dt \wedge dz$
$dx \wedge dz$	$dt \wedge dy$
$dy \wedge dz$	$dt \wedge dx$
$dt \wedge dx \wedge dy$	$-dz$
$dt \wedge dx \wedge dz$	dy
$dt \wedge dy \wedge dz$	$-dx$
$dx \wedge dy \wedge dz$	$-dt$
$dt \wedge dx \wedge dy \wedge dz$	-1

Using the table, we compute $\star d(\star F)$ explicitly. We start by determining $\star F$:

$$\star F = B_x dt \wedge dx + B_y dt \wedge dy + B_z dt \wedge dz + E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy$$

With $\star F$ in hand, we compute $d(\star F)$:

$$\begin{aligned} d(\star F) =& \frac{\partial B_x}{\partial y} dy \wedge dt \wedge dx + \frac{\partial B_x}{\partial z} dz \wedge dt \wedge dx + \frac{\partial B_y}{\partial x} dx \wedge dt \wedge dy + \frac{\partial B_y}{\partial z} dz \wedge dt \wedge dy \\ &+ \frac{\partial B_z}{\partial x} dx \wedge dz \wedge dt + \frac{\partial B_z}{\partial y} dy \wedge dt \wedge tz + (\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}) dx \wedge dy \wedge dz \\ &- \frac{\partial E_x}{\partial t} dt \wedge dy \wedge dz - \frac{\partial E_y}{\partial t} dt \wedge dz \wedge dx - \frac{\partial E_z}{\partial t} dt \wedge dx \wedge dy \\ &= (\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} - \frac{\partial E_x}{\partial t}) dt \wedge dy \wedge dz + (\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} + \frac{\partial E_y}{\partial t}) dt \wedge dx \wedge dz \\ &+ (\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} - \frac{\partial E_z}{\partial t}) dt \wedge dx \wedge dz + (\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}) dx \wedge dy \wedge z \\ &= \rho dx \wedge dy \wedge dz - J_x dt \wedge dy \wedge dz + J_y dt \wedge dx \wedge dz - J_z dt \wedge dx \wedge dy \end{aligned}$$

where the last equality is equivalent to Ampere's Law and Gauss's Law. Finally, we compute

$\star d(\star F)$:

$$\star d(\star F) = -\rho dt + J_x dx + J_y dy + J_z dz = \mathcal{J}$$

B.6 Section 6

Exercise 1: *Show that if we make the transformation $A_\mu \mapsto A_\mu + \partial_\mu \chi$, then Equation 5 is not necessarily preserved.*
Solution:

Exercise 2: *Show that if we assume that the inverse Compton wavelength is much larger than the space derivatives of Φ, then we can recover Equation 3 from Equation 5.*
Solution:

Exercise 3: *Show that a G-bundle over X is isomorphic to the trivial bundle $X \times G$ if and only if it admits a global section.*
Solution: (sketch) Let $\mathcal{P} : P \to X$ be a principal G-bundle with global section $s : X \to P$. For $p \in P$, let $c(p) \in G$ be the (unique) element of G where $p = s(\mathcal{P}(p)) \cdot c(p)$. Then define a morphism $\Psi : P \to X \times G$ where $\Psi(p) = (\mathcal{P}(p), g(p))$. Then Ψ is a isomorphism with inverse $(x, g) \mapsto s(x) \cdot g$. Conversely, suppose that we have an isomorphism $\Psi : P \to X \times G$. Then we can define a global section $s : X \to P$ where $s(x)$ is the first projection of $\Psi^{-1}(x, e)$.

B.7 Section 8

(a) The group $SO(n)$ consists of $n \times n$ real orthogonal matrices with determinant 1. These matrices represent rotations in n-dimensional Euclidean space:

$$SO(n) = \{R \in \mathbb{R}^{n\times n} \mid R^\top R = I, \det(R) = 1\}.$$

The spin group Spin(n) is defined as a subgroup of the even Clifford algebra $\mathcal{C}l_n^0$, generated by products of even numbers of elements from the Clifford algebra $\mathcal{C}l_n$. Let $\mathcal{C}l_n$ be the Clifford algebra generated by a set of elements $\{e_1, e_2, \ldots, e_n\}$ with the relation

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad \text{for } i, j = 1, 2, \ldots, n.$$

We now define the covering map $\rho : \text{Spin}(n) \to SO(n)$. Given an element $g \in \text{Spin}(n)$ and a vector $v \in \mathbb{R}^n$, the action of g on v is given by

$$v \mapsto gvg^{-1}.$$

For each element $R \in SO(n)$, there are exactly two elements in Spin(n) that map to R under ρ. This is because the kernel of ρ consists of two elements: $\{1, -1\}$. Therefore, the covering map is 2-to-1.

(b) We will use the description of Spin(4) as the universal (double-cover) of $SO(4)$ so we will explicitly produce a smooth 2 : 1 group morphism $SU(2) \times SU(2) \to SO(4)$. Again we think of $SU(2)$ as the group of unit quaternions. Thus each pair $(q_1, q_2) \in SU(2)\times SU(2)$ defines a real linear map

$$T_{q_1,q_2} : \mathbb{H} \to \mathbb{H}, \quad x \mapsto T_{q_1,q_2}x = q_1 x \bar{q}_2$$

Clearly $|x| = |q_1| \cdot |x| \cdot |\overline{q_2}| = |T_{q_1,q_2}x| \, \forall x \in \mathbb{H}$, so that each T_{q_1,q_2} is an orthogonal transformation of $\mathbb{H}$. Since $SU(2) \times SU(2)$ is connected, all the operators T_{q_1,q_2} belong to the component of $O(4)$ containing $\nVdash$, i.e,. T defines an (obviously smooth) group morphism

$$T : SU(2) \times SU(2) \to SO(4)$$

Note that $\ker T = \{1, -1\}$, so that T is 2 : 1. In order to prove T is a double cover it suffices to show it is onto. This follows easily by noticing T is a submersion, so that its range must contain an entire neighborhood of $\nVdash \in SO(4)$. Since the range of T is closed we conclude that T must be onto because the closure of the subgroup (algebraically) generated by an open set in a connected Lie group coincides with the group itself.

(c) The first Stiefel-Whitney class $w_1(M) \in H^1(M, \mathbb{Z}/2)$ represents the orientability of a manifold M. To find a representative for $w_1(M)$ using Čech cocycles built from transition functions, we begin by choosing an open cover $\{U_i\}$ of M such that the tangent bundle TM is trivialized over each open set U_i. Over each overlap $U_i \cap U_j$, the difference in trivializations is captured by the transition functions $g_{ij} : U_i \cap U_j \to GL(n, \mathbb{R})$, where n is the dimension of M. The orientation-preserving or orientation-reversing nature of these transition functions is determined by their determinant. Define $s_{ij} : U_i \cap U_j \to \mathbb{Z}/2$ by $s_{ij} = 0$ if $\det(g_{ij}) > 0$, and $s_{ij} = 1$ if $\det(g_{ij}) < 0$. The functions s_{ij} form a Čech 1-cocycle, as they satisfy the cocycle condition $s_{ij} + s_{jk} + s_{ki} = 0 \mod 2$ on triple overlaps $U_i \cap U_j \cap U_k$. This ensures that the transition functions are consistent across overlapping charts, and the cocycle $\{s_{ij}\}$ defines a class in $H^1(M, \mathbb{Z}/2)$, which represents $w_1(M)$.

B.8 Section 9: Dirac Operators and Indices

(a) Let us consider a simplified equation of the form $c^\mu \partial_\mu \phi(\mathbf{x}) = 0$. If $A \in O(1, 3)$, i.e., a transformation $x^\mu \mapsto A^\mu_\nu x^\nu$, then

$$\partial_\mu \phi(\mathbf{x}) \mapsto \partial_\mu(\phi(A^{-1}\mathbf{x})) = (A^{-1})^\nu_\mu (\partial_\nu \phi)(A^{-1}\mathbf{x}).$$

Now if $c^\mu \in \mathbb{C}$ is a scalar, then they assemble to a vector $\mathbf{c} \in \mathbb{C}^4$. In order, for the our (simplified) equation to be Lorentz invariant, we need the vector $\mathbf{c}$ to transform under A as well, but as c^μ are simply scalars they commute with matrix multiplication and invariance fails.

(b) (a) If our new basis is written as τ^μ it is an explicit verification that the anti-commutation $\{\tau^\mu, \tau^\nu\} = 2\eta^{\mu\nu}$ holds, so we obtain a different (faithful) matrix representation of $\mathrm{Cl}_{1,3}$ (b) $C = i\gamma^2\gamma^0$. (c) plug and chug.

(c) The kernel is one-dimensional, while the operator is surjective, and hence has trivial cokernel.

(d) We have that

$$\dim\ker S \circ T \leq \dim\ker S + \dim\ker T$$

and

$$\dim\operatorname{coker} S \circ T \leq \dim\operatorname{coker} S + \dim\operatorname{coker} T.$$

As the closed range condition is actually redundant, it follows that $S \circ T$ is indeed Fredholm.

To compute the index, we take advantage of the local constancy of the index. For $0 \leq t \leq \pi$ define $C_t\colon Y \oplus X \to Z \oplus X$ by

$$C_t = \begin{bmatrix} S\cos t & -(S\circ T)\sin t \\ I\sin t & T\cos t \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} (\mathrm{Rot}_t \otimes I) \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}.$$

For each t, C_t is a composition of Fredholm operators and hence defines a (continuous) path in the space of such operators. Finally,

$$\operatorname{Ind} C_0 = \operatorname{Ind} T + \operatorname{Ind} S, \qquad \text{while} \qquad \operatorname{Ind} C_{\frac{\pi}{2}} = \operatorname{Ind} S \circ T.$$

(e) The key idea is that if $\lambda \neq 0$, then the composition

$$H_\lambda^+ \xrightarrow{D^+} H_\lambda^- \xrightarrow{\lambda^- D^-} H_\lambda^+$$

is the identity, so $n_\lambda^+ = n_\lambda^-$ for $\lambda > 0$.

(a) Fix a basis $\{[A_1], \ldots, [A_n]\}$ for $H_2(M, \mathbb{Z})/\mathsf{Tor}$. The dual basis for $(H_2(M, \mathbb{Z})/\mathsf{Tor})^\vee \cong H^2(M, \mathbb{Z})/\mathsf{Tor}$ is $\{\alpha_1, \ldots, \alpha_n\}$ which has the defining property $\alpha_i([A_j]) = \delta_{ij}$. Take the Poincaré dual of the dual basis to get $\{\alpha_1^*, \ldots, \alpha_n^*\}$, a different basis of $H_2(M, \mathbb{Z})/\mathsf{Tor}$. This basis has the property that $Q([A_i], \alpha_j^*) = \delta_{ij}$. Hence the matrix representing Q in the basis $\{[A_1], \ldots, [A_n]\}$ is the same as the change of basis matrix from $\{[A_1], \ldots, [A_n]\} \mapsto \{\alpha_1^*, \ldots, \alpha_n^*\}$ which is invertible over $\mathbb{Z}$ so must have det $=\pm 1$.

(b) Let M, N be two connected, closed, oriented 4-manifolds. The connected sum only alters 4-cells so the homology in dimension 2 should be unaffacted. Explicitly, apply Mayer-Vietoris to the triple $(M \# N, M - \mathbb{D}^4, N - \mathbb{D}^4)$ where $\mathbb{D}^4$ is the open 4-disk. This yields the result.

(c) $\mathbb{CP}^2$ has $H_2(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}$ with generator $[\mathbb{CP}^1]$. Since any projective line intersects itself once we have

$$Q_{\mathbb{CP}^2} = [1]$$

. The opposite orientation $\overline{\mathbb{CP}}^2$ has intersection form

$$Q_{\overline{\mathbb{CP}}^2} = [-1]$$

. Using the result of exercise 2 we have

$$Q_{\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(d) We know from the Künneth formula that

$$H_2(\mathbb{S}_a^2 \times \mathbb{S}_b^2) \cong \mathbb{Z} \oplus \mathbb{Z}$$

Fix points $p \in \mathbb{S}_a^2$ and $q \in \mathbb{S}_b^2$ so that the generators for the homology group are $\alpha = [\mathbb{S}_a^2 \times \{q\}]$ and $\beta = [\{p\} \times \mathbb{S}_b^2]$. Then we can compute the intersections

$$\alpha \cdot \beta = 1 = \beta \cdot \alpha$$

since these intersect at the point $(p, q) \in \mathbb{S}_a^2 \times \mathbb{S}_b^2$ and the self intersections are both zero as we can perturb the spheres so that that do not intersect at all.

For the twisted bundle we can think of the first generator of the homology as a section of the bundle $\mathbb{S}^2 \to \mathbb{S}^2$ which due to the twist at the equator there is no way to perturb this sphere out of self intersection. This gives the intersection form

$$Q_{\mathbb{S}^2 \tilde{\times} \mathbb{S}^2} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

If we change basis $\alpha \mapsto A$ and $\alpha - \beta \mapsto B$ then the matrix becomes

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which is the same intersection form of $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. While this does not imply that the two manifolds are diffeomorphic or even homeomorphic it can be shown that in fact $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2 \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ as smooth manifolds.

B.9 Section 14: Gravity as a Gauge Theory

1. We define $R^{\alpha}{}_{\beta\mu\nu}$ by the equation

$$[\nabla_\nu, \nabla_\mu] w_\beta = w_\alpha R^{\alpha}{}_{\beta\mu\nu},$$

which holds for every tensor w_α.

(a) Show that $R^{\alpha}{}_{\beta\mu\nu}$ is a tensor.

(b) Using the definition of covariant derivative derive that

$$R^{\alpha}{}_{\beta\mu\nu} = (\partial_{\mu}\Gamma^{\alpha}{}_{\nu\beta} + \Gamma^{\alpha}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\beta}) - (\mu \leftrightarrow \nu).$$

Proof of (a) Put $T_{\beta\mu\nu} = [\nabla_{\nu}, \nabla_{\mu}]w_{\beta}$. We know that $T_{\beta\mu\nu}$ is a tensor. Thus we have

$$\begin{aligned} w_{\alpha'} R^{\alpha'}{}_{\beta'\mu'\nu'} = T_{\beta'\mu'\nu'} &= \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} T_{\beta\mu\nu} = \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \omega_{\alpha} R^{\alpha}{}_{\beta\mu\nu} \\ &= w_{\alpha'} \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} R^{\alpha}{}_{\beta\mu\nu} \end{aligned}$$

for all tensor $w_{\alpha'}$, which yields that

$$R^{\alpha'}{}_{\beta'\mu'\nu'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} R^{\alpha}{}_{\beta\mu\nu},$$

proving (a).

Proof of (b) We compute

$$\begin{aligned} \nabla_{\nu}\nabla_{\mu}\omega_{\beta} &= \partial_{\nu}(\nabla_{\mu}\omega_{\beta}) - \Gamma^{\sigma}{}_{\nu\mu}\nabla_{\sigma}\omega_{\beta} - \Gamma^{\sigma}{}_{\nu\beta}\nabla_{\mu}\omega_{\sigma} \\ &= \partial_{\nu}(\partial_{\mu}\omega_{\beta} - \Gamma^{\alpha}{}_{\mu\beta}\omega_{\alpha}) - \Gamma^{\sigma}{}_{\nu\mu}(\partial_{\sigma}\omega_{\beta} - \Gamma^{\alpha}{}_{\sigma\beta}\omega_{\alpha}) - \Gamma^{\sigma}{}_{\nu\beta}(\partial_{\mu}\omega_{\sigma} - \Gamma^{\alpha}{}_{\sigma\mu}\omega_{\alpha}) \\ &= \partial_{\nu}\partial_{\mu}\omega_{\beta} - \Gamma^{\sigma}{}_{\nu\mu}(\partial_{\sigma}\omega_{\beta} - \Gamma^{\alpha}{}_{\sigma\beta}\omega_{\alpha}) - (\Gamma^{\alpha}{}_{\mu\beta}\partial_{\nu}\omega_{\alpha} + \Gamma^{\alpha}{}_{\nu\beta}\partial_{\mu}\omega_{\alpha}) + (-\partial_{\nu}\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\sigma}{}_{\nu\beta}\Gamma^{\alpha}{}_{\mu\sigma})\omega_{\alpha}. \end{aligned}$$

Here, the first three terms are symmetric in μ and ν, hence

$$\begin{aligned} w_{\alpha}R^{\alpha}{}_{\beta\mu\nu} = [\nabla_{\nu}, \nabla_{\mu}]w_{\beta} &= ([\nabla_{\nu}, \nabla_{\mu}]\omega_{\beta}) - (\mu \leftrightarrow \nu) \\ &= ((-\partial_{\nu}\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\sigma}{}_{\nu\beta}\Gamma^{\alpha}{}_{\mu\sigma})\omega_{\alpha}) - (\mu \leftrightarrow \nu) \\ &= \omega_{\alpha}((\partial_{\mu}\Gamma^{\alpha}{}_{\nu\beta} + \Gamma^{\alpha}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\beta}) - (\mu \leftrightarrow \nu)) \end{aligned}$$

for all w_{α}, which proves (b).

2. (a) Suppose that we are falling in a gravitational field. If we use the frame of our path, then we will find that all objects near us are falling at the same rate. This follows from

$$\mathbf{F} = m_I\mathbf{a} = -m_G\nabla\Phi \quad \text{or} \quad \frac{d^2\mathbf{r}}{dt^2} = -\frac{m_G}{m_I}\nabla\Phi$$

and the experimental fact that m_G/m_I is the same for every object. Here m_I is the inertia mass and m_G is the gravitational mass. So, there is no acceleration in the relative motion between objects around us, that is, we are in a flat spacetime if we use the freely falling coordinate. Mathematically, in a Lorentzian manifold $(\mathbb{R}^4, g_{\mu\nu})$ show the following: for each event P there is a coordinate $x^{\mu'}$ near P such that

$$g_{\mu'\nu'} = \eta_{\mu'\nu'} \quad \text{and} \quad \Gamma^{\alpha'}{}_{\mu'\nu'} = 0$$

at P. This coordinate is called a normal coordinate at P. [Hint. Start with any coordinate x^μ near P and define a new coordinate $x^{\alpha'}$ by

$$x^\mu(x^{\mu'}) = x^\mu_P + \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)_P (x^{\mu'} - x^{\mu'}_P) + \frac{1}{2}\left(\frac{\partial^2 x^\mu}{\partial x^{\mu'}\partial x^{\nu'}}\right)_P (x^{\mu'} - x^{\mu'}_P)(x^{\nu'} - x^{\nu'}_P)$$

and count the number of coefficients in the above so that we can impose $g_{\mu'\nu'} = \eta_{\mu'\nu'}$ and $\partial_{\alpha'} g_{\mu'\nu'} = 0$ at P.]

(b) Prove that

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}, \quad R_{\alpha[\beta\mu\nu]} = 0.$$

(c) Prove the Bianchi identity

$$\nabla_{[\gamma} R_{\alpha\beta]\mu\nu} = 0.$$

(d) Show that $\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$.

Proof of (a) Two equations in the hint are written as

$$(g_{\mu'\nu'})_P = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)_P \left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)_P (g_{\mu\nu})_P = \eta_{\mu'\nu'}$$

and

$$\begin{aligned}(\partial_{\alpha'} g_{\mu'\nu'})_P = &\left[\left(\frac{\partial^2 x^\mu}{\partial x^{\alpha'}\partial x^{\mu'}}\right)_P \left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)_P + \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)_P \left(\frac{\partial^2 x^\nu}{\partial x^{\alpha'}\partial x^{\nu'}}\right)_P\right] (g_{\mu\nu})_P \\ &+ \left(\frac{\partial x^\alpha}{\partial x^{\alpha'}}\right)_P \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)_P \left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)_P (\partial_\alpha g_{\mu\nu})_P \\ = &\ 0.\end{aligned}$$

Because of the symmetry of the metric the first one is 10 equations for which we have 16 variables $\frac{\partial x^\mu}{\partial x^{\mu'}}$ available and the second one is 40 equations for which we also have 40 variables $\left(\frac{\partial^2 x^\mu}{\partial x^{\mu'}\partial x^{\nu'}}\right)_P$ available. Hence we have solutions of such partial derivative at P, which defines the new coordinate $x^{\alpha'}$. From the definition of the Levi-Civita connection $\Gamma^{\alpha'}_{\mu'\nu'} = 0$ at P follows from $\partial_{\alpha'} g_{\mu'\nu'} = 0$ at P.

Proof of (b) We have all tensor equations. Thus, it is enough to check them in one specific coordinate. Let P be an arbitrary event. Then by (a) we have a normal coordinate x^μ about P. Thus, at P we have $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\alpha_{\mu\nu} = 0$. Then, at P we use the metric compatibility $\nabla_\mu g^{\eta\delta} = \partial_\mu g^{\eta\delta} = 0$ to get

$$\begin{aligned}2R_{\alpha\beta\mu\nu} &= 2g_{\alpha\eta} R^\eta{}_{\beta\mu\nu} \\ &= 2g_{\alpha\eta}\partial_\mu \Gamma^\eta_{\nu\beta} - (\mu \leftrightarrow \nu) \\ &= g_{\alpha\eta}\partial_\mu[g^{\eta\delta}(\partial_\nu g_{\delta\beta} + \partial_\beta g_{\nu\delta} - \partial_\delta g_{\nu\beta})] - (\mu \leftrightarrow \nu) \\ &= g_{\alpha\eta} g^{\eta\delta}(\partial_\mu\partial_\nu g_{\delta\beta} + \partial_\mu\partial_\beta g_{\nu\delta} - \partial_\mu\partial_\delta g_{\nu\beta})] - (\mu \leftrightarrow \nu) \\ &= (\partial_\mu\partial_\nu g_{\alpha\beta} + \partial_\mu\partial_\beta g_{\nu\alpha} - \partial_\mu\partial_\alpha g_{\nu\beta}) - (\mu \leftrightarrow \nu) \\ &= (\partial_\mu\partial_\beta g_{\nu\alpha} - \partial_\mu\partial_\alpha g_{\nu\beta}) - (\mu \leftrightarrow \nu).\end{aligned}$$

Now from the above all four equations in (b) follows immediately.

Proof of (c) Let x^μ be a normal coordinates about P. At P we have

$$2R_{\alpha\beta\mu\nu} = 2R_{\mu\nu\alpha\beta} = (\partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\alpha\partial_\mu g_{\beta\nu}) - (\alpha \leftrightarrow \beta).$$

Hence,

$$\begin{aligned}
2\nabla_{[\gamma}R_{\alpha\beta]\mu\nu} &= 2\nabla_\gamma R_{\alpha\beta\mu\nu} + 2\nabla_\alpha R_{\beta\gamma\mu\nu} + 2\nabla_\beta R_{\gamma\alpha\mu\nu} \\
&= (\partial_\gamma\partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\gamma\partial_\alpha\partial_\mu g_{\beta\nu} + \partial_\alpha\partial_\beta\partial_\nu g_{\gamma\mu} - \partial_\alpha\partial_\beta\partial_\mu g_{\gamma\nu} \\
&\quad + \partial_\beta\partial_\gamma\partial_\nu g_{\alpha\mu} - \partial_\beta\partial_\gamma\partial_\mu g_{\alpha\nu}) - (\alpha \leftrightarrow \beta) \\
&= 0,
\end{aligned}$$

since the first term is symmetric in α and β.

Proof of (d) In the Bianchi identity in (c)

$$\nabla_\gamma R_{\alpha\beta\mu\nu} + \nabla_\alpha R_{\beta\gamma\mu\nu} + \nabla_\beta R_{\gamma\alpha\mu\nu} = 0$$

we multiply both sides by $g^{\alpha\mu}$ and use the metric compatibility to have

$$\nabla_\gamma R_{\beta\nu} + \nabla^\alpha R_{\beta\gamma\alpha\nu} - \nabla_\beta R_{\gamma\nu} = 0.$$

This time we multiply $g^{\gamma\nu}$ to get

$$\nabla^\nu R_{\beta\nu} + \nabla^\alpha R_{\beta\alpha} - \nabla_\beta R = 0.$$

On the other hand multiplying the first equation in (b) by $g^{\alpha\mu}$ we get $R_{\beta\nu} = R_{\nu\beta}$, that is, $R_{\mu\nu}$ is symmetric. Thus we have $\nabla^\nu R_{\nu\beta} = \frac{1}{2}\nabla_\beta R$, which proves (d).

3. Consider the Einstein equation with cosmological constant

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}.$$

(a) Show that the above equation is consistent with the energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$. Show that $-R + 4\Lambda = \kappa T$.

(b) In the weak-field Newtonian approximation show that the above equation becomes

$$\nabla^2\Phi = 4\pi G\rho - \Lambda c^2.$$

Also, with $\rho(\mathbf{r}) = M\delta(\mathbf{r})$ show that

$$-\nabla\Phi = -\frac{GM}{r^3}\mathbf{r} + \frac{c^2\Lambda}{3}\mathbf{r}.$$

(c) Write the general metric for a spacetime with homogeneous and isotropic spatial structure with a constant curvature K. Using a perfect fluid distribution, impose the Einstein equation with cosmological constant to get Friedmann equations. When is this universe expanding?

Proof of (a) By the metric compatibility we have $\nabla^\alpha g_{\mu\nu} = g^{\alpha\beta}\nabla_\beta g_{\mu\nu} = 0$, hence

$$\nabla^\mu\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right) = \nabla^\mu R_{\mu\nu} - \frac{1}{2}(\nabla^\mu R)g_{\mu\nu} + \left(-\frac{1}{2}R + \Lambda\right)\nabla^\mu g_{\mu\nu}$$
$$= \nabla^\mu R_{\mu\nu} - \frac{1}{2}\nabla_\nu R = 0,$$

which follows from 2(d). Since $g^{\mu\nu}g_{\mu\nu} = 4$ it follows that

$$\kappa T = g^{\mu\nu}\kappa T_{\mu\nu} = g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right) = R - \frac{1}{2}4R + 4\Lambda = -R + 4\Lambda.$$

Proof of (b) Using the result from (a) we write the equation as

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) + \Lambda g_{\mu\nu}.$$

In the weak-field Newtonian approximation we had $R_{00} = -\frac{1}{2}\nabla^2 h_{00}$, $T_{00} = -T = \rho c^2$, $g_{00} = -1$ and $h_{00} = -2\Phi/c^2$. Thus,

$$\frac{1}{c^2}\nabla^2\Phi = R_{00} = \kappa\left(T_{00} - \frac{1}{2}Tg_{00}\right) + \Lambda g_{00} = \frac{1}{2}\kappa\rho c^2 - \Lambda,$$

or

$$\nabla^2\Phi = 4\pi G\rho - \Lambda c^2.$$

Now, with $\rho(\mathbf{r}) = M\delta(\mathbf{r})$ we have the acceleration

$$-\nabla\Phi = -\frac{GM}{r^3}\mathbf{r} + \frac{c^2\Lambda}{3}\mathbf{r},$$

where the last term explains the expanding acceleration if the cosmological constant is positive.

Answer of (c) It turns out that the general metric for a spacetime with homogeneous and isotropic spatial structure with constant curvature K is

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2,\right]$$

where $a(t)$ is the scale factor describing the expansion of the universe. Indeed, when $K > 0$ we can write the metric as $ds^2 = -c^2\,dt^2 + a^2(t)d\sigma^2$, where the 3-space is identified with the hyperspace $w^2 + x^2 + y^2 + z^2 = 1/K$ with the induced metric

$$d\sigma^2 = dw^2 + dx^2 + dy^2 + dz^2.$$

Using the spherical coordinates, $w^2 + r^2 = 1/K$ and $wdw + rdr = 0$, we get

$$d\sigma^2 = \frac{r^2}{w^2}dr^2 + dr^2 + r^2d\Omega^2 = \frac{dr^2}{1 - Kr^2} + r^2d\Omega^2.$$

The cases $K = 0$ and $K < 0$ can be handled similarly.

With a perfect fluid

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u_\mu u_\nu + pg_{\mu\nu}$$

we impose the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}.$$

After some computation involving the Riemann tensor we get Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{Kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}.$$

Here the first equation relates the rate of the expansion to the energy density, the curvature and the cosmological constant and the second describes the acceleration or deceleration of the expansion. Thus, Λ drives the overall dynamics of the universe's expansion (accelerated or decelerated), while K influences the spatial geometry (flat, open, or closed).

4. Let $g_{\mu\nu}$ be a Lorentzian metric on a manifold M.

(a) Show that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.$$

(b) Show that there is a tensor $V^\alpha = V^\alpha(g_{\mu\nu})$ such that $(\delta R_{\mu\nu})g^{\mu\nu} = \nabla_\alpha V^\alpha$.

Proof of (a) For fixed μ, ν we write

$$g = \det(g_{\alpha\beta}) = \sum_\beta a_{\mu\beta}g_{\mu\beta},$$

where $(a_{\mu\beta})$ is the adjugate matrix of $(g_{\alpha\beta})$. Notice that in the above we are not using the Einstein summation convention. Observe that the term $a_{\mu\beta}$ is independent of $g_{\mu\nu}$, hence

$$\frac{\partial g}{\partial g_{\mu\nu}} = \sum_\beta a_{\mu\beta}\frac{\partial g_{\mu\beta}}{\partial g_{\mu\nu}} = \sum_\beta a_{\mu\beta}\delta_{\beta\nu} = a_{\mu\nu}.$$

Recall that the matrix $(a_{\mu\beta})$ is g times the inverse matrix of $(g_{\mu\nu})$. Thus, coming back to summation convention, we have

$$\delta g = \frac{\partial g}{\partial g_{\mu\nu}}\delta g_{\mu\nu} = gg^{\mu\nu}\delta g_{\mu\nu}.$$

On the other hand we have

$$0 = \delta 4 = \delta(g^{\mu\nu}g_{\mu\nu}) = (\delta g^{\mu\nu})g_{\mu\nu} + g^{\mu\nu}\delta g_{\mu\nu},$$

hence

$$\delta g = -gg_{\mu\nu}\delta g^{\mu\nu}.$$

Thus,

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta g = \left(-\frac{1}{2\sqrt{-g}}\right)(-gg_{\mu\nu}\delta g^{\mu\nu}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.$$

Proof of (b) First we notice that

$$\begin{aligned}\delta R_{\mu\nu} = \delta R^\alpha_{\ \mu\alpha\nu} &= \delta(\partial_\alpha\Gamma^\alpha_{\ \nu\mu} + \Gamma^\alpha_{\ \alpha\beta}\Gamma^\beta_{\ \nu\mu}) - (\alpha(\text{lower}) \leftrightarrow \nu)\\ &= \partial_\alpha\delta\Gamma^\alpha_{\ \nu\mu} + (\delta\Gamma^\alpha_{\ \alpha\beta})\Gamma^\beta_{\ \nu\mu} + \Gamma^\alpha_{\ \alpha\beta}\delta\Gamma^\beta_{\ \nu\mu} - \partial_\nu\delta\Gamma^\alpha_{\ \alpha\mu} - (\delta\Gamma^\alpha_{\ \nu\beta})\Gamma^\beta_{\ \alpha\mu} - \Gamma^\alpha_{\ \nu\beta}\delta\Gamma^\beta_{\ \alpha\mu}.\end{aligned}$$

Since

$$\Gamma^{\alpha'}_{\mu'\nu'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \Gamma^{\alpha}_{\mu\nu} + \frac{\partial x^{\alpha'}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial x^{\mu'} \partial x^{\nu'}},$$

$\Gamma^{\alpha}_{\mu\nu}$ is not a tensor, but $\delta\Gamma^{\alpha}_{\mu\nu}$ is a tensor and we can consider its covariant derivative. Rearranging some terms in the previous equation for $\delta R_{\mu\nu}$ and using the metric compatibility, we arrive at

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\left(\nabla_\alpha \delta\Gamma^{\alpha}_{\nu\mu} - \nabla_\nu \delta\Gamma^{\alpha}_{\alpha\mu}\right) = \nabla_\alpha V^\alpha,$$

where

$$V^\alpha = g^{\mu\nu}\delta\Gamma^{\alpha}_{\nu\mu} - g^{\mu\alpha}\delta\Gamma^{\nu}_{\nu\mu}$$

is a tensor in term of $g^{\mu\nu}$ and $\delta g^{\mu\nu}$. Observe that $V^\alpha = 0$ if $\delta g^{\mu\nu} = 0$ and $\partial_\alpha \delta g^{\mu\nu} = 0$.

B.10 Section 15: Gravity (cont): Palatini–Cartan

1. (a) Show that if ${e_\mu}^a$ is related to $g_{\mu\nu}$, then so is ${e_\mu}^{a'} = {\Lambda^{a'}}_a {e_\mu}^a$.

(b) Assuming that ${e_\mu}^a$ is related to $g_{\mu\nu}$, show that $\det({e^\mu}_a) = \sqrt{-g}$.

(c) If V^μ is a vector in the curved spacetime, show that V^a is a tensor in flat Lorentzian spacetime, but a scalar in the curved spacetime.

Proof of (a) Since ${e_\mu}^a$ is related to $g_{\mu\nu}$, we have

$$g_{\mu\nu} = {e_\mu}^a {e_\nu}^b \eta_{ab} = {e_\mu}^a {e_\nu}^b \Lambda^{a'}_a \Lambda^{b'}_b \eta_{a'b'} = {e_\mu}^{a'} {e_\nu}^{b'} \eta_{a'b'},$$

that is, ${e_\mu}^{a'}$ is also related to $g_{\mu\nu}$

Proof of (b) We have

$$-g = -\det(g_{\mu\nu}) = -\det({e_\mu}^a {e_\nu}^b \eta_{ab}) = -\det{}^2({e_\mu}^a)\det(\eta_{ab}) = \det{}^2({e_\mu}^a),$$

which proves the formula upon our convention that $\det({e_\mu}^a) > 0$.

Proof of (c) This follows from

$$V^{a'} = {e_\mu}^{a'} V^\mu = \Lambda^{a'}_a {e_\mu}^a V^\mu = \Lambda^{a'}_a V^a$$

and

$$V^a = {e_{\mu'}}^a V^{\mu'} = {e_\mu}^a \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\mu'}}{\partial x^\nu} V^\nu = {e_\mu}^a \delta^\mu_\nu V^\nu = {e_\mu}^a V^\mu = V^a.$$

2. Show that

$$R^{\alpha}_{\beta\mu\nu} = {e^\alpha}_a {e_\beta}^b \left[\partial_\mu {\omega_\nu}^a{}_b - {\omega_\mu}^c{}_b {\omega_\nu}^a{}_c\right] - (\mu \leftrightarrow \nu).$$

Proof We use the definition of the Riemann curvature tensor and the tetrad postulate to get

$$\begin{aligned}
A_\alpha R^\alpha{}_{\beta\mu\nu} &= [\nabla_\nu, \nabla_\mu] A_\beta = \nabla_\nu \nabla_\mu (e_\beta{}^b A_b) - (\mu \leftrightarrow \nu) \\
&= e_\beta{}^b \nabla_\nu \nabla_\mu A_b - (\mu \leftrightarrow \nu) \\
&= e_\beta{}^b \left[\partial_\nu \nabla_\mu A_b - \Gamma^\alpha_{\ \nu\mu} \nabla_\alpha A_b - \omega_\nu{}^c{}_b \nabla_\mu A_c\right] - (\mu \leftrightarrow \nu) \\
&= e_\beta{}^b \left[\partial_\nu (\partial_\mu A_b - \omega_\mu{}^a{}_b A_a) - \Gamma^\alpha_{\ \nu\mu} \nabla_\alpha A_b - \omega_\nu{}^c{}_b (\partial_\mu A_c - \omega_\mu{}^a{}_c A_a)\right] - (\mu \leftrightarrow \nu) \\
&= e_\beta{}^b \left[-\partial_\nu \omega_\mu{}^a{}_b + \omega_\nu{}^c{}_b \omega_\mu{}^a{}_c\right] A_a - (\mu \leftrightarrow \nu) \\
&= e^\alpha{}_a e_\beta{}^b \left[\partial_\mu \omega_\nu{}^a{}_b - \omega_\mu{}^c{}_b \omega_\nu{}^a{}_c\right] A_\alpha - (\mu \leftrightarrow \nu),
\end{aligned}$$

from which the equation follows when we read off the coefficients of A_α.

3. (a) Show

$$\frac{1}{2} R_{\mu\nu}{}^{ab} dx^\mu \wedge dx^\nu = d\omega^{ab} + \omega^a{}_c \wedge \omega^c{}_b.$$

(b) Prove two Bianchi identities $DT^a = R^a{}_b \wedge e^b$ and $DR^{ab} = 0$.

Proof of (a) We use the symmetry of Riemann tensor $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$ and the result from Exercise 2 to have

$$\begin{aligned}
R_{\mu\nu}{}^{ab} dx^\mu \wedge dx^\nu &= e_\alpha{}^a e_\beta{}^b R_{\mu\nu}{}^{\alpha\beta}\, dx^\mu \wedge dx^\nu \\
&= e_\alpha{}^a e_\beta{}^b R^{\alpha\beta}{}_{\mu\nu}\, dx^\mu \wedge dx^\nu \\
&= e_\alpha{}^a e_\beta{}^b g^{\beta\beta'} R^\alpha{}_{\beta'\mu\nu}\, dx^\mu \wedge dx^\nu \\
&= \left(e_\alpha{}^a e_\beta{}^b g^{\beta\beta'} e^\alpha{}_{a'} e_{\beta'}{}^{b'} \left(\partial_\mu \omega_\nu{}^{a'}{}_{b'} - \omega_\mu{}^{c'}{}_{b'} \omega_\nu{}^{a'}{}_{c'}\right) - (\mu \leftrightarrow \nu)\right) dx^\mu \wedge dx^\nu \\
&= \left(\partial_\mu \omega_\nu{}^a{}_b - \omega_\mu{}^c{}_b \omega_\nu{}^a{}_c - (\mu \leftrightarrow \nu)\right) dx^\mu \wedge dx^\nu \\
&= 2(d\omega^{ab} + \omega^a{}_c \wedge \omega^c{}_b).
\end{aligned}$$

Proof of (b) We have

$$\begin{aligned}
DT^a &= dT^a + \omega^a{}_b \wedge T^b = d(de^a + \omega^a{}_b \wedge e^b) + \omega^a{}_b \wedge (de^b + \omega^b{}_c \wedge e^c) \\
&= d\omega^a{}_b \wedge e^b - \omega^a{}_b \wedge de^b + \omega^a{}_b \wedge de^b + \omega^a{}_c \wedge \omega^c{}_b \wedge e^b \\
&= R^a{}_b \wedge e^b.
\end{aligned}$$

Also, we have

$$\begin{aligned}
DR^{ab} &= dR^{ab} + \omega^a{}_c \wedge R^{cb} + \omega^b{}_c \wedge R^{ac} \\
&= d(d\omega^{ab} + \omega^a{}_c \omega^{cb}) + \omega^a{}_c (d\omega^{cb} + \omega^c{}_d \omega^{db}) + \omega^b{}_c (d\omega^{ac} + \omega^a{}_d \omega^{dc}) \\
&= d\omega^a{}_c \wedge \omega^{cb} - \omega^a{}_c \wedge d\omega^{cb} + \omega^a{}_c \wedge d\omega^{cb} + \omega^a{}_c \wedge \omega^c{}_d \wedge \omega^{db} \\
&\quad + \omega^b{}_c \wedge d\omega^{ac} + \omega^b{}_c \wedge \omega^a{}_d \wedge \omega^{dc} \\
&= 0.
\end{aligned}$$

4. (a) Prove that

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\gamma\delta} = -2(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu), \ \ \epsilon^{\mu\nu\gamma\delta}\epsilon_{\mu\nu\gamma\delta} = -24 \quad \epsilon^{\alpha\beta\gamma\delta}\epsilon_{abc\delta} = -e^{\alpha\beta\gamma}_{abc}.$$

(b) Show that $\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}$.

Proof of (a) In order to prove $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\gamma\delta} = -2(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu)$ we notice that both sides are zero if $\alpha = \beta$ or $\mu = \nu$. Now assume that $\alpha \neq \beta$ and $\mu \neq \nu$. In the left hand side only summands with $\{\alpha, \beta, \gamma, \delta\} = \{\mu, \nu, \gamma, \delta\} = \{0, 1, 2, 3\}$ are non-zero. Thus, we may assume $\{\alpha, \beta\} = \{\mu, \nu\}$, that is, $\alpha = \mu, \beta = \nu$ or $\alpha = \nu, \beta = \mu$. In the first case $\{\gamma, \delta\} = \{u < v\}$ is fixed and

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\gamma\delta} = \epsilon^{\alpha\beta uv}\epsilon_{\alpha\beta uv} + \epsilon^{\alpha\beta vu}\epsilon_{\alpha\beta vu} = -2,$$

since $\epsilon^{0123} = 1, \epsilon_{0123} = -1$. The other case is handled similarly.

Second formula $\epsilon^{\mu\nu\gamma\delta}\epsilon_{\mu\nu\gamma\delta} = -24$ is proved similarly because in the left hand side we can assume that $\{\mu, \nu, \gamma, \delta\} = \{0, 1, 2, 3\}$, there are $4! = 24$ ways of enumerating $0, 1, 2, 3$ and the product $\epsilon^{\mu\nu\gamma\delta}\epsilon_{\mu\nu\gamma\delta}$ for each enumeration is -1 as above.

For the third formula we have

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{abc\delta} &= \epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha'\beta'\gamma'\delta}e^{\alpha'}{}_a e^{\beta'}{}_b e^{\gamma'}{}_c \\ &= (-\delta^\alpha_{\alpha'}\delta^\beta_{\beta'}\delta^\gamma_{\gamma'} \pm \text{permutations})e^{\alpha'}{}_a e^{\beta'}{}_b e^{\gamma'}{}_c \\ &= -e^{\alpha\beta\gamma}_{abc},\end{aligned}$$

where we observed that both sides are zero unless α, β, γ are distinct, in which case, the sum over δ is only the one with $\{\alpha, \beta, \gamma, \delta\} = \{0, 1, 2, 3\}$. Hence $\{\alpha, \beta, \gamma\} = \{\alpha', \beta', \gamma'\} = \{u < v < w\}$ and there are six permutations.

Proof of (b) We have

$$\begin{aligned}\eta_{\alpha\beta\gamma\delta} &= g_{\alpha\alpha'}g_{\beta\beta'}g_{\gamma\gamma'}g_{\delta\delta'}\eta^{\alpha'\beta'\gamma'\delta'} \\ &= g_{\alpha\alpha'}g_{\beta\beta'}g_{\gamma\gamma'}g_{\delta\delta'}\frac{\epsilon^{\alpha'\beta'\gamma'\delta'}}{\sqrt{-g}} \\ &= -g\,\epsilon_{\alpha\beta\gamma\delta}\frac{1}{\sqrt{-g}} \\ &= \sqrt{-g}\,\epsilon_{\alpha\beta\gamma\delta}.\end{aligned}$$

5. Show that the Euler-Lagrange equation $\epsilon_{abcd}\left[R^{ab} - \frac{\Lambda}{3}e^a \wedge e^b\right] \wedge e^c = 0$ is equivalent to the Einstein equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$.

Proof We have

$$
\begin{aligned}
0 &= \epsilon_{abcd}\left(R^{ab} - \frac{\Lambda}{3} e^a \wedge e^b\right) \wedge e^c \wedge dx^\beta \\
&= \epsilon_{abcd}\left(\frac{1}{2} R_{\mu\nu}{}^{ab} - \frac{\Lambda}{3} e_\mu{}^a e_\nu{}^b\right) e_\alpha{}^c \, dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta \\
&= \epsilon_{abcd}\left(\frac{1}{2} R_{\mu\nu}{}^{ab} - \frac{\Lambda}{3} e_\mu{}^a e_\nu{}^b\right) e_\alpha{}^c \epsilon^{\mu\nu\alpha\beta} \, d^4x \\
&= -e^{\mu\nu\beta}_{abd}\left(\frac{1}{2} R^{ab}_{\mu\nu} - \frac{\Lambda}{3} e_\mu{}^a e_\nu{}^b\right) d^4x,
\end{aligned}
$$

or

$$
\begin{aligned}
0 &= -\frac{1}{2}\left(R_{\mu\nu}{}^{\mu\nu} e^\beta{}_d + R_{\mu\nu}{}^{\nu\beta} e^\mu{}_d + R_{\mu\nu}{}^{\beta\mu} e^\nu{}_d - R_{\mu\nu}{}^{\nu\mu} e^\beta{}_d - R_{\mu\nu}{}^{\beta\nu} e^\mu{}_d - R_{\mu\nu}{}^{\mu\beta} e^\nu\right) d \\
&\quad + \frac{\Lambda}{3}\left(e^\mu{}_a e^\nu{}_b e^\beta{}_d + e^\nu{}_a e^\beta{}_b e^\mu{}_d + e^\beta{}_a e^\mu{}_b e^\nu{}_d - e^\nu{}_a e^\mu{}_b e^\beta{}_d - e^\beta{}_a e^\nu{}_b e^\mu{}_d - e^\mu{}_a e^\beta{}_b e^\nu{}_d\right) e_\mu{}^a e_\nu{}^b \\
&= -\frac{1}{2}(R e^\beta{}_d - R_d{}^\beta - R_d{}^\beta + R e^\beta{}_d - R_d{}^\beta - R_d{}^\beta) + \frac{\Lambda}{3}(16 e^\beta{}_d + e^\beta{}_d + e^\beta{}_d - 4e^\beta{}_d - 4e^\beta{}_d - 4e^\beta{}_d) \\
&= 2\left(R_d{}^\beta - \frac{1}{2} R e^\beta{}_d + \Lambda e^\beta{}_d\right) \\
&= 2\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}\right) e^\mu{}_d g^{\beta\nu},
\end{aligned}
$$

from which the equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ follows.

6. (a) Show that $d\,\mathrm{CS}(A) = \langle\langle F, F\rangle\rangle$, where $F = dA + A \wedge A$ is the curvature of the connection A.
(b) Prove that $\delta \mathrm{CS}(A) = d\,\langle\langle \delta A, A\rangle\rangle + 2\langle\langle \delta A, F\rangle\rangle$ and that the Euler-Lagrange equation of the Chern-Simons action is $F = 0$.

Proof of (a) Using properties (i)-(iv), we have

$$
\begin{aligned}
d\,\mathrm{CS}(A) &= d\left(\langle\langle dA, A\rangle\rangle + \frac{2}{3}\langle\langle A \wedge A, A\rangle\rangle\right) \\
&= \langle\langle d^2 A, A\rangle\rangle + \langle\langle dA, dA\rangle\rangle + \frac{2}{3}\langle\langle dA \wedge A, A\rangle\rangle - \frac{2}{3}\langle\langle A \wedge dA, A\rangle\rangle + \frac{2}{3}\langle\langle A \wedge A, dA\rangle\rangle \\
&= \langle\langle dA, dA\rangle\rangle + \langle\langle dA, A \wedge A\rangle\rangle + \langle\langle A \wedge A, dA\rangle\rangle \\
&= \langle\langle dA + A \wedge A,\; dA + A \wedge A\rangle\rangle \\
&= \langle\langle F, F\rangle\rangle,
\end{aligned}
$$

where we used the fact that our manifold has dimension 3, hence $A \wedge A \wedge A \wedge A = 0$.

Proof of (b) Using the same properties, we get

$$
\begin{aligned}
\delta\,\mathrm{CS}(A) &= \delta\left(\langle\langle dA, A\rangle\rangle + \frac{2}{3}\langle\langle A \wedge A, A\rangle\rangle\right) \\
&= \langle\langle d\delta A, A\rangle\rangle + \langle\langle dA, \delta A\rangle\rangle + \frac{2}{3}\langle\langle \delta A \wedge A, A\rangle\rangle + \frac{2}{3}\langle\langle A \wedge \delta A, A\rangle\rangle + \frac{2}{3}\langle\langle A \wedge A, \delta A\rangle\rangle \\
&= d\,\langle\langle \delta A, A\rangle\rangle + 2\langle\langle \delta A,\; dA + A \wedge A\rangle\rangle.
\end{aligned}
$$

Since M is closed and the pairing is non-degenerate, from

$$0 = \delta S_{\mathrm{CS}} = 2 \int_M \langle\langle \delta A,\, F \rangle\rangle$$

we get the Euler-Lagrange equation $F = 0$.

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